2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Functional Series Expansions For Continuous-Time Switched Systems

Luis A. Duffaut Espinosa [a] W. Steven Gray [b]

***Abstract*— The main objective of this paper is to describe a class of functional series expansions, known as Fliess operators, which admit inputs from a ball in an L_p space as well as Poisson random processes. It is shown that a continuous-time switched input-affine nonlinear system with a Poisson switching signal can be represented as a Fliess operator, and that the underlying combinatorics can be used to obtain, for certain cases, a closed-form solution in terms of Poisson integrals.**

I. Introduction

Fliess operators provide a general framework under which analytic nonlinear input-output systems can be studied [7], [8], [11], [12], [18]. In the classical setting, they are described by an infinite summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For a measurable function $u : [a, b] \to \mathbb{R}^m$ define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : 1 \le i \le m\}$, where $\|u_i\|_{L_p}$ is the usual L_p-norm for a measurable real-valued component function u_i. Define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\eta'}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$. For convenience assume $t_0 = 0$. The input-output operator corresponding to c is then

$$F_c[u](t) \triangleq \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. The most general results regarding the convergence of Fliess operators were presented in [12]. There it was shown that if the generating series c is *globally convergent*, i.e., satisfies the growth condition

$$|(c, \eta)| \le KM^{|\eta|}, \ \ \forall \eta \in X^*,$$

where $|\eta|$ denotes the number of symbols in η and $K, M > 0$, then $F_c[u]$ converges absolutely on $[0, \infty)$ for $u \in L_{p,e}^m(0)$. On the other hand, if the generating series c is *locally convergent*, i.e., satisfies the growth condition

$$|(c, \eta)| \le KM^{|\eta|}|\eta|!, \ \ \forall \eta \in X^*, \tag{1}$$

then $F_c[u]$ converges absolutely on $[0, T]$ for $u \in B_p^m(R)[0, T] \triangleq \{u \in L_p^m[0, T] : \|u\|_{L_p} \le R\}$ if T and R sufficiently small. More recently in [4]–[6], it was shown that the notion of a Fliess operator can be generalized to admit a class of L_2-Itô stochastic processes. Specifically, such operators were defined as an infinite summation of Lebesgue and Stratonovich iterated integrals, and conditions for their absolute convergence were given. This class of input-output systems, however, is still too limited for many engineering applications.

A number of systems encountered in engineering involve the stochastic coupling of several subsystems. It is well known, for example, that the flight control computers on board fly-by-wire aircraft are subject to faults induced by lightning and atmospheric neutrons [10], [19]. In turn, these faults can induce system-level errors by corrupting the control law computations. Once the system detects a fault, it switches from a nominal mode, which models the aircraft under ideal conditions, to a recovery mode, which models the effect of the fault and the recovery mechanism used to restore the system back to the nominal mode. Such dynamics can be modeled as a *switched input-affine nonlinear system*

$$\begin{aligned} \dot z &= f_v(z) + g_v(z)u, \quad z(0) = z_0 \\ y &= h(z), \end{aligned} \tag{2}$$

where $u \in L_p[0, T]$; $v : [0, \infty) \to \{0, 1\}$ is a switching signal; and $f_0, f_1,\ g_0, g_1$ and h are analytic functions on some neighborhood of $z_0 \in \mathbb{R}^n$ [15]. Equivalently,

$$\begin{aligned} \dot z &= f_0(z) + g_0(z)\, u + (f_1(z) - f_0(z))\, v \\ &\quad + (g_1(z) - g_0(z))\, u\, v \\ y &= h(z). \end{aligned}$$

When the integral process induced by v is a Poisson process, say N, then

$$\begin{aligned} z(t) &= z_0 + \int_0^t f_0(z(s)) + g_0(z(s))\, u(s)\, ds \\ &\quad + \int_0^t f_1(z(s)) - f_0(z(s))\, dN(s) \\ &\quad + \int_0^t (g_1(z(s)) - g_0(z(s)))\, u(s)\, dN(s), \end{aligned} \tag{3}$$

where $\int \cdot dN$ denotes a stochastic integral with respect to N. Observe that for each $t \in [0, \infty)$, $v(t)$ is actually representing $\Delta N(t) \triangleq N(t) - N(t-)$, where $N(t-) = \lim_{s\to t, s<t} N(s)$ is the left continuous version of N.

Poisson processes fall into the class of *jump processes* or *Lévy processes* [16], which are distinct from the class of processes being considered in [1], [4]–[6]. It is, however, possible to describe (3) in terms of a Fliess operator if a more general type of stochastic integral is used, namely an integral with respect to a semimartingale. One challenge of allowing jumps in the integral is the loss of the chain rule, which cannot be recovered as is done for the Itô integral by using the Stratonovich integral [16]. In addition, the underlying

[a] Department of Electrical and Computer Engineering, The Johns Hopkins University, Baltimore, Maryland 21218-2680, USA, lduffaut@jhu.edu
[b] Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, Virginia 23529-0246, USA, sgray@odu.edu

algebraic structure is no longer the shuffle algebra since the integration by parts formula admits extra terms [14]. So in this paper, the necessary extension of the theory is fully developed. As a result, it will be possible to give a series solution for (2) and express the map $u \mapsto y$ as a Fliess operator.

The paper is organized as follows. Section II presents the main results of the paper. In Section III, the analysis tools from stochastic integration of semimartingales are introduced. In particular, the Poisson integral and its properties are summarized. Then in Section IV the proofs of the main results are given. Finally, Section V provides the conclusions and suggestions for future work.

II. Main Results

To model switched systems with more than two modes, the idea of "thinning" a Poisson process is useful [17]. Consider a Poisson process N with intensity λ. The events are classified into k disjoint types: type 1, type 2,. . ., type k. Let p_j denote the probability that a given event is of type j, and let N_j denote the process counting the events of type j. Then N_j is a Poisson process with intensity $\lambda_j = p_j\lambda$. Moreover, for any set of positive numbers $t_1, t_2, \ldots, t_k$ the random variables $N_1(t_1), N_2(t_2), \ldots, N_k(t_k)$ are independent. It is also important to observe that for every $t \geq 0$ and $j_1 \neq j_2$ the probability $P(N_{j_1}(t) + N_{j_2}(t) \geq 2) = 0$. A switching signal of this type will be called a *Poisson switching signal of k-types with probabilities* p_j, $j = 1, \ldots, k$.

To introduce Poisson processes into the Fliess operator formalism, consider the following alphabets: $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0^{(1)}, \ldots, y_m^{(1)}, \ldots, y_0^{(k)}, \ldots, y_m^{(k)}\}$ and $XY = X \cup Y$. For each $\eta \in XY^*$, define recursively a Poisson-Lebesgue iterated integral E_η by first setting $E_\emptyset = 1$ and then, for $x_i \in X$ and $y_i^{(j)} \in Y$, letting

$$E_{x_i\eta'}[w](t) \triangleq \int_0^t u_i(s) E_{\eta'}[w](s)\, ds \tag{4}$$

$$E_{y_i^{(j)}\eta'}[w](t) \triangleq \int_0^t u_i(s-) E_{\eta'}[w](s-)\, dN_j(s), \tag{5}$$

where $\eta' \in XY^*$, $w = (u, \bar{v})$, $u \in B_p^m(R)[0, T]$, $u_0 = 1$, and $\bar{v} = (v_1, \ldots, v_k) = (\Delta N_1, \ldots, \Delta N_k)$. The process $\bar{v}$ will be called the *decomposition* of a Poisson process N of k-types. A Fliess operator over $B_p^m(R)[0, T]$ with Poisson jumps is defined as follows.

Definition 1: A causal m-input, ℓ-output *Fliess operator* F_c, $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$, driven by $u \in B_p^m(R)[0, T]$ and a Poisson process N of k-type with probabilities p_i, $i = 1, \ldots, k$ is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c, \eta)\, E_\eta[w](t), \tag{6}$$

where each E_η is given in (4)-(5).

Theorem 1: Suppose $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ satisfies the growth condition (1). Then there exist $R, T > 0$ such that for each $u \in B_1^m(R)[t_0, t_0 + T]$ and Poisson process of k-type with probabilities p_j, $j = 1, \ldots, k$, the series (6) converges in the mean absolutely on $[0, T]$.

Theorem 2: A switched input-affine nonlinear system with $k+1$ modes and driven by an input from $B_p^m(R)[0, T]$ and a Poisson switching signal of k-types with probabilities p_j, $j = 1, \ldots, k$ can be written as a Fliess operator F_c for some $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$.

Example 1: Consider the n-dimensional switched system

$$\dot{z} = A_v z + B_v z\, u, \quad z(0) = z_0, \quad y = Cz, \tag{7}$$

where $v : [0, \infty) \to \{0, 1, \ldots, k\}$ is a switching signal, $u \in B_1(R)[0, T]$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times 1}$, $i = 0, 1, \ldots, k$, $C \in \mathbb{R}^{\ell\times n}$ and $z_0 \in \mathbb{R}^{n\times 1}$. For each $t \in [0, \infty)$, let

$$v(t) = \sum_{j=1}^{k} j\, v_j(t), \tag{8}$$

where $v_j = \Delta N_j$, $j = 0, \ldots, k$. Thus, (7) can be expressed as

$$\begin{aligned} \dot{z} &= (A_0 z + B_0 z\, u)\left(1 - \sum_{j=1}^{k} v_j\right) + \sum_{j=1}^{k} (A_j z + B_j z\, u)\, v_j \\ &= M_{0,0} z + M_{0,1} z\, u + \sum_{j=1}^{k} (M_{j,0} z + M_{j,1} z\, u)\, v_j, \end{aligned}$$

where $M_{0,0} = A_0$, $M_{0,1} = B_0$, $M_{j,0} = A_j - A_0$, $M_{j,1} = B_j - B_0$ for $j = 1, \ldots, k$, and v is characterized as in (8) by a Poisson switching signal of k-types. It will be shown in Section IV that $y = F_c[w]$, where $w = (u, \bar{v})$ and $(c, \eta) = CM_\eta z_0$ with $M_{x_i\eta} = M_{0,i}M_\eta$ and $M_{y_i^{(j)}\eta} = M_{j,i}M_\eta$ for $\eta \in XY^*$. Moreover, if the $M_{j,i}$'s commute then

$$\begin{aligned} y(t) = C \exp\left(M_{0,0}\, t + M_{0,1} \int_0^t u(s)\, ds\right)\cdot \\ \prod_{j=1}^{k} \exp\left(\int_0^t \ln\left(1 + M_{j,0} + M_{j,1}\, u(s)\right) dN_j(s)\right) z_0. \end{aligned}$$

□

III. Stochastic Setting

Now a brief summary is given of the concepts needed from the theory of stochastic integration to prove the main results. The treatment is based on [16] and the references therein.

A. Semimartingales

Assume that $(\Omega, \mathcal{F}, \mathbf{F}, P)$ is a complete filtered probability space, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and $\mathbf{F}$ is right continuous. Denote by $\mathbb{D}$ the set of adapted processes with *càdlàg* (right continuous and left limits) and $\mathbb{L}$ the set of adapted processes with *càglàd* (left continuous and right limits).

Definition 2: A process H is said to be *simple predictable* if H has a representation

$$H(t) = H(0)\mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t),$$

where $0 = \tau_1 \leq \cdots \leq \tau_{n+1} < \infty$ is a finite sequence of stopping times, $\mathbb{1}_A$ denotes the indicator function of the set A, and $H_i \in \mathcal{F}_{\tau_i}$ with $|H_i| < \infty$ a.s. for $0 < i < n$. The collection of simple predictable processes is denoted by S, and by S_{ucp} when S is endowed with the topology of uniform convergence on compacts in probability (ucp convergence). Here a sequence $\{H^n\}_{n\geq 0}$ of jointly measurable stochastic processes converges in the ucp sense to a process H when for each $t > 0$ and any $\epsilon > 0$

$$\lim_{n\to\infty} P\left(\sup_{0\leq s\leq t} |H^n(s)-H(s)| \geq \epsilon\right) = 0.$$

Under upc convergence, the set $\mathbb{D}$ is complete and the set S is dense in $\mathbb{L}$.

Let L^0 be the set of random variables endowed with the topology of convergence in probability. Let $X \in S$ be a stochastic process and define the linear mapping $I_X : S \to L^0$ induced by X as

$$I_X(H) = H(0)X(0) + \sum_{i=1}^{n} H_i \left(X_{t_{i+1}} - X_{t_i}\right).$$

The continuity of this mapping is considered next under the upc topology.

Definition 3: A process X is called a *semimartingale* if it is adapted, has càdlàg paths, and the mapping I_X is continuous on any bounded interval $[0,t]$.

Definition 4: The *pure jump process* induced by a semimartingale X at $t \geq 0$ is defined as $\Delta X(t) = X(t) - X(t-)$.

In the same way that I_X maps processes in S to random variables in L^0, an operator induced by X can map processes to processes.

Definition 5: Let $H \in S$ and $X \in \mathbb{D}$. The *stochastic integral of the simple predictable process H with respect to X* is defined by the linear mapping $J_X(H)_t : S \to \mathbb{D}$ as

$$J_X(H)_t = H(0)X(0) + \sum_{i=1}^{n} H_i \left(X(t\wedge\tau_{i+1}) - X(t\wedge\tau_{t_i})\right),$$

where $t \wedge \tau \triangleq \min(t,\tau)$.

Theorem 3: Let X be a semimartingale. Then the mapping $J_X : S_{upc} \to D_{upc}$ is continuous and linear.

Definition 6: Let X be a semimartingale. The continuous linear mapping $J_X(H)_t : \mathbb{L}_{upc} \to \mathbb{D}_{upc}$ obtained as the unique extension of $J_X : S \to \mathbb{D}$ is called the *stochastic integral of H with respect to X* and is written as

$$J_X(H)_t = \int_0^t H(s-)\, dX(s). \qquad (9)$$

Theorem 4: Let $H \in \mathbb{L}_{upc}$ and X be a semimartingale. Then:

- $i.$ $J_X(H)_t$ has no dependence on times exceeding t because $X(t\wedge\tau_{i+1}) = X(t\wedge\tau_i)$ for $\tau_{i+1} \geq \tau_i \geq t$.
- $ii.$ $J_X(H)_t$ is consistent with the Itô integral definition because H is calculated at the left end of (τ_i, τ_{i+1}).
- $iii.$ If X is cádlág and H is cáglád, then $J_X(H)_t$ is a semimartingale.
- $iv.$ The jumps in the integral occur at jump points of X, i.e., $\Delta\left(\int_0^t H(s)\, dX(s)\right) = H(t)\,\Delta X(t)$.

The following concepts will be useful in the next subsection.

Definition 7: A stochastic process X is called *increasing* if it is adapted, $X(0) = 0$, and its sample paths are nondecreasing and a.s. right continuous.

Theorem 5: Let X be an increasing stochastic process such that $\mathbf{E}[X(t)] < \infty$. Then there exists a unique increasing process $\widetilde{X}$ such that

$$\mathbf{E}\left[\int_0^t Y(s)\, dX(s)\right] = \mathbf{E}\left[\int_0^t Y(s)\, d\widetilde{X}(s)\right]$$

for all t and each non-negative predictable process Y. The process $\widetilde{X}$ is called the *dual predictable projection* of X. An important characterization of the dual predictable projection is given next in terms of Martingales.

Theorem 6: Let X be an increasing process so that $\mathbf{E}[X(t)] < \infty$. Then the dual predictable projection of X is the only predictable increasing process $\widetilde{X}$ such that the process $X - \widetilde{X}$ is a Martingale.

B. The Poisson Integral

Definition 8: Let $\{\tau_i\}_{i\geq 0}$ be an increasing sequence of stopping times. A process $N(t) \triangleq \sum_{i\geq 1} \mathbb{1}_{\{t\geq\tau_i\}}$ taking values in $\mathbb{N}$ is called a *Poisson process* with intensity λ if it satisfies:

- $i.$ For any $0 \leq s < t < \infty$, $N(t) - N(s)$ is independent of $\mathcal{F}_s$.
- $ii.$ For any $0 \leq s_1 < t_1 < \infty$ and $0 \leq s_2 < t_2 < \infty$ such that $t_1 - s_1 = t_2 - s_2$, the distribution of $N(t_1) - N(s_1)$ is the same as that of $N(t_2) - N(s_2)$.

From this definition, it can be inferred that $N(t)$ is adapted, it has a Poisson distribution with intensity λ, and $N(t) = \sum_{0\leq s\leq t} \Delta N(s)$. Using Definition 5, the Poisson integral is defined next.

Definition 9: Let H be a stochastic process and N be a Poisson process. The *Poisson integral of H* is defined as

$$\begin{aligned} J_N(H)_t &= \int_0^t H(s-)\, dN(s) \\ &= \sum_{k=1}^{N(t)} H(t-)[N(\tau_k\wedge t) - N(\tau_{k-1}\wedge t)]. \end{aligned}$$

Observe that the integral $\int_0^t N(s)\, dN(s) = \sum_{\tau_i\leq t} N(\tau_i)$ is a well-defined Stieltjes integral since $N(t)$ is an increasing process of finite first variation. But it is not a stochastic integral because $N(t)$ is not predictable. On the other hand, $\int_0^t N(s-)\, dN(s)$ is a stochastic integral with the characteristic that it is indistinguishable from the re-defined Stieltjes integral $\int_0^t N(s)\, dN(s) = \sum_{\tau_i\leq t} N(\tau_{i-1})$. The advantages of the Stratonovich integral with respect to Wiener processes are well-known. In particular, its relationship with the Itô stochastic integral has been widely used in the literature [1], [9]. However, such advantages are not available for stochastic integrals with respect to jump processes. For example, the Poisson-Itô integral gives extra terms that cannot be removed by using the Poisson-Stratonovich integral.

Definition 10: Let X and Y be two semimartingales. The *Stratonovich integral* of Y with respect to X is

$$\oint_0^t Y(s-)dX(s) = \int_0^t Y(s-)dX(s) + \frac{1}{2}[X,Y]_t^c,$$

where $[X,Y]_t^c$ is the continuous part of the quadratic covariation of X and Y defined as

$$[X,Y]_t = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right),$$

where $\|\Pi\| = \max_{i=1,\ldots,n} (t_i - t_{i-1})$ is the measure of the partition Π of $[0,t]$.

Theorem 7: Let $F \in \mathcal{C}^2$. The Itô formula for semimartingales is

$$F(X(t)) - F(X(0)) = \int_{0+}^{t} f'(X(s-))\, dX(s) + \frac{1}{2}\int_{0+}^{t} f''(X(s-))\, d[X,X]^c(s) + \sum_{0<s\leq t} \left[(f(X(s)) - f(X(s))) - f'(X(s-))\Delta X(s)\right]. \quad (10)$$

Observe that all the integrals are well-defined since $f'(X(s-))$ and $f''(X(s-))$ are càglàd, and $X(s)$ and $[X,X]^c(s)$ are càdlàg. From (10), the relationship between the stochastic integral (9) and the Stratonovich integral is

$$\oint_0^t F'(X(s-))\, dX(s) = \int_0^t F'(X(s-))\, dX(s) + \frac{1}{2}\int_0^t f''(X(s-))\, d[X,X]^c(s).$$

Example 2: For a Poisson process N, observe

$$\oint_0^t F'(N(s-))\, dN(s) = \int_0^t F'(N(s-))\, dN(s).$$

This shows that the Itô and Stratonovich integrals for N coincide since $[N,N]_t^c = 0$ and $[N,N]_t = N(t)$. □

A very useful identity for stochastic iterated integrals with respect to Poisson processes is presented next.

Theorem 8: Let N be a Poisson process. Then

$$N^{\{n\}}(t) \triangleq \int_0^t N^{\{n-1\}}(s-)\, dN(s) = \sum_{0\leq t_1\leq t_2\cdots t_n\leq t} \Delta N_{t_1}\cdots\Delta N_{t_n} = \mathbb{1}_{N\geq n}\binom{N}{n}$$

with $N^{\{0\}} = 1$.

In general, this identity is valid for all pure jump processes.

Lemma 1: Let X and Y be two semimartingales. It follows that

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s-)\, dY(s) + \int_0^t Y(s-)\, dX(s) + [X,Y]_t. \quad (11)$$

From Theorem 8 and Lemma 1, the following useful identity can be obtained.

Theorem 9: Let X and Y be semimartingales satisfying $X(0) = 0$, $Y(0) = 0$ and $[X,Y]_t = 0$. Then

$$(X+Y)^{\{n\}}(t) = \sum_{k=0}^{n} X^{\{i\}}(t)Y^{\{n-i\}}(t).$$

Proof: By induction and using the integration by parts formula (11), the statement follows directly. ■

IV. Proof of Main Results

In order to prove Theorem 1, upper bounds for the Poisson-Lebesgue iterated integral given in (4)-(5) are needed. The dual predictable projection of the Poisson process N will play a key role in the calculation. To find such a process, observe

$$\mathbf{E}[N(t) - N(s)\,|\,\mathcal{F}_s] = \mathbf{E}[N(t) - N(s)] = \lambda\,(t-s)$$

$$\mathbf{E}[N(t) - \lambda\, t\,|\,\mathcal{F}_s] = N(s) - \lambda s$$

for $0 \leq s < t$, which means that $N(t) - \lambda t$ is a Martingale. By Theorem 6, the dual predictable projection of N is the process λt. Now, let $|\eta|_A$ denote the number of letters in η that belongs to A for any $A \subset XY$, and define the language $X^{n_1}Y^{n_2} = \{\eta \in XY^*, |\eta|_X = n_1, |\eta|_Y = n_2\}$. The next lemma gives upper bounds for Poisson-Lebesgue iterated integrals.

Lemma 2: Let $\eta \in X^{n_1}Y^{n_2}$, $u \in B_p^m(R)[0,T]$ and N be a Poisson process of k types. An upper bound for the iterated Poisson-Lebesgue integral $E_\eta[w]$ at a fixed $t \in [0,T]$ is

$$\|E_\eta[w](t)\|_1 \leq \lambda^{n_2}\left(\prod_{j=1}^{k} p_j^{\beta^j}\right)\left(\prod_{i=1}^{m}\frac{U_i^{\alpha_i+\beta_i}(t)}{(\alpha_i+\beta_i)!}\right), \quad (12)$$

where $U_i(t) = \int_0^t |u_i(s)|\, ds$, $\alpha_i = |\eta|_{x_i}$, $\sum_{i=0}^m \alpha_i = n_1$, $\beta^j = \sum_{i=0}^m |\eta|_{y_i^{(j)}}$, $\beta_i = \sum_{j=0}^k |\eta|_{y_i^{(j)}}$ and $\sum_{j=1}^k \beta^j = \sum_{i=0}^m \beta_i = n_2$.

Proof: The inequality is proved by induction over the total number of $n_1 + n_2$ integrals. For $n_1 + n_2 = 0$, the claim is trivial. If $n_1 + n_2 = 1$, then there are two cases to consider. The case when $\eta = x_0$ is trivial. The second case is when $\eta = y_i^{(j)}$. Since a Poisson process is an increasing process, it follows from Theorem 5 that

$$\left\|E_{y_i^{(j)}}[w](t)\right\|_1 \leq \mathbf{E}\left[\int_0^t |u_i(s-)|\, dN_j(s)\right] = \lambda p_j\, U_i(t).$$

Now assume that (12) holds for every $\eta' \in X^{n_1}Y^{n_2}$ up to some fixed $n_1 + n_2 > 0$. If $\eta = y_i^{(j)}\eta'$ then

$$\begin{aligned}
\left\|E_{y_i^{(j)}\eta'}[w](t)\right\|_1 &= \mathbf{E}\left[\left|\int_0^t u_i(s)E_{\eta'}[w](s)\, dN_j(s)\right|\right] \\
&\leq \mathbf{E}\left[\int_0^t |u_i(s)|\,|E_{\eta'}[w](s)|\, dN_j(s)\right] \\
&\leq \lambda p_j \int_0^t \mathbf{E}[|u_i(s)|]\lambda^{n_2}\left(\prod_{l=1}^{k} p_l^{\beta^l}\right)\prod_{l=0}^{m}\frac{U_l^{\alpha_l+\beta_l}(s)}{(\alpha_l+\beta_l)!}ds \\
&\leq \lambda^{n_2+1}p_1^{\beta^1}\cdots p_j^{\beta^j+1}\cdots p_k^{\beta^k}\prod_{\substack{l=0\\ l\neq i}}^{m}\frac{U_l^{\alpha_l+\beta_l}(t)}{(\alpha_l+\beta_l)!}\cdot \\
&\quad \int_0^t \mathbf{E}[|u_i(s)|]\frac{U_i^{\alpha_i+\beta_i}(s)}{(\alpha_i+\beta_i)!}\, ds \\
&= \lambda^{n_2+1}p_1^{\beta^1}\cdots p_j^{\beta^j+1}\cdots p_k^{\beta^k}\cdot \\
&\quad \frac{U_0^{\alpha_0+\beta_0}(t)\cdots U_i^{\alpha_i+\beta_i+1}(t)\cdots U_m^{\alpha_m+\beta_m}(t)}{(\alpha_0+\beta_0)!\cdots(\alpha_i+\beta_i+1)!\cdots(\alpha_m+\beta_m)!}.
\end{aligned}$$

The inductive step for $\eta = x_i\eta'$ is done similarly. Hence, the proof is complete. ■

Proof of Theorem 1: Assume that the coefficients of c satisfy the growth condition (1) for some $K, M > 0$. Without loss of generality, it is assumed that $\ell = 1$ and $\lambda \geq 1$. Fix some $T > 0$. Pick any $u \in L_1^m(R)[0,T]$ and let $R = \max\{\|u\|_1, T\}$. For $(a_0,\ldots,a_m) \in \mathbb{N}^{m+1}$, define $a! = a_0!\cdots a_m!$. From Lemma 2 and since $p_j \leq 1$ for all j's, it follows for any

$\eta \in X^{n_1} Y^{n_2}$ and $t \in [0,T]$ that

$$\mathbf{E}\left[|(c,\eta)E_\eta[w](t)|\right] \leq KM^r r! \frac{\lambda^{n_2} R^r}{(\alpha+\beta)!}, \tag{13}$$

where $r = |\eta| = n_1 + n_2$. Next define $a_r(t) = \sum_{|\eta|=r} |(c,\eta)E_\eta[w](t)|$. Then from (13), observe

$$\begin{aligned} \mathbf{E}[a_r(t)] &= \sum_{|\eta|=r} \mathbf{E}[|(c,\eta)E_\eta[w](t)|] \\ &\leq KM^r \lambda^r R^r \sum_{|\eta|=r} \frac{r!}{(\alpha+\beta)!} \\ &= KM^r \lambda^r R^r \sum_{\substack{\alpha_0+\cdots+\alpha_m \\ +\beta_0+\cdots+\beta_m=r}} \frac{r!}{(\alpha+\beta)!} \cdot \frac{r!}{\alpha!\beta!} \\ &\leq KM^r \lambda^r R^r \left(\sum_{\substack{\alpha_0+\cdots+\alpha_m \\ +\beta_0+\cdots+\beta_m=r}} \frac{r!}{\alpha!\beta!} \right)^2 \\ &= KM^r \lambda^r R^r (2m+2)^{2r}, \end{aligned}$$

where the last step employs the multinomial theorem. It then follows that

$$\sum_{r=0}^{\infty} \mathbf{E}[a_r(t)] \leq \sum_{r=0}^{\infty} K(4M\lambda R(m+1)^2)^r.$$

This shows that if $R < 1/(4M\lambda(m+1)^2)$ then (6) converges in the mean absolutely on $[0,T]$. ■

In [4]–[6], the shuffle product was used to prove results analogous to Lemma 2 and Theorem 1. However, when Poisson integrals are present, the shuffle product is not applicable since the integration by parts formula differs from the classical case [14]. Instead, $E_\eta E_\xi$ can be expressed as

$$E_\eta E_\xi = E_{\eta \sqcup\!\sqcup\, \xi} + E_{\eta \diamondsuit \xi}, \tag{14}$$

where $\sqcup\!\sqcup$ denotes the usual shuffle product, and $\diamondsuit$ is defined recursively as follows. For $\eta', \xi' \in XY^*$, $q_k^{(i)}, q_l^{(j)} \in XY$ ($x_k^{(i)} \triangleq x_k$ for all i), $\eta = q_k^{(i)}\eta'$, and $\xi = q_l^{(j)}\xi'$, let

$$\eta \diamondsuit \xi = q_k^{(i)}(\eta' \diamondsuit \xi) + q_l^{(j)}(\eta \diamondsuit \xi') + \delta_{q_k^{(i)}, q_l^{(j)}} \left(\eta' \sqcup\!\sqcup\, \xi' + \eta' \diamondsuit \xi'\right),$$

where $q_k^{(i)} \diamondsuit \emptyset = \emptyset \diamondsuit q_k^{(i)} = 0$, and $\delta_{q_k^{(i)}, q_l^{(j)}} = 1$ if $q_k^{(i)} = y_k^{(i)}$, $q_l^{(j)} = y_l^{(j)}$ and $i = j$, otherwise $\delta_{q_k^{(i)}, q_l^{(j)}} = 0$. Identity (14) plays a fundamental role in the proof of Theorem 2.

Proof of Theorem 2 (outline): The objective is to write any switched input-affine nonlinear system with a Poisson switching signal as a Fliess operator. Observe that if the system has $k+1$ modes then

$$\begin{aligned} \dot{z} &= f_0(z) + \sum_{i=0}^{m} g_{0i}(z)u_i + \sum_{j=1}^{k} (f_j(z) - f_0(z))v_j \\ &\quad + \sum_{i=0,j=1}^{m,k} (g_{ji}(z) - g_{0i}(z))u_i\, v_j, \end{aligned}$$

where the integrals of the v_j's come from a Poisson switching signal, N, of k-types with probabilities p_j for $j = 1,\ldots,k$. That is, for each $t \in [0,T]$ either all v_j's are zero or just one $v_j = 1$. It is sufficient to show that one can write the following switched system as a Fliess operator

$$\dot{z} = \sum_{i=0}^{m} f_i(z)\, u_i + \sum_{i=0,j=1}^{m,k} g_{ji}(z)\, u_i\, v_j, \tag{15}$$

where $u \in B_p^m(R)[0,T]$, and f_i, g_{ji} are analytic functions on some neighborhood of $z_0 \in \mathbb{R}^n$. Assume for brevity that $m = k = 1$ and $f_0 = g_{10} = 0$. Note that $v = \Delta N$, so abusing the notation, let $v = dN$ and $z_t = z(t)$. In integral form, (15) becomes after dropping the subscripts

$$z_t = \int_0^t f(z_s)\, u_s\, ds + \int_0^t g(z_s)\, u_s\, dN(s).$$

Given a differentiable function F, the semimartingale Poisson chain rule is

$$\begin{aligned} F(z_t) &= F(z_0) + \int_0^t \left(f(z_s)\frac{\partial}{\partial z}F(z_s) \right) u_s\, ds \\ &\quad + \int_0^t \left(F(z_s) - F(z_{s-})\right)\, u_{s-}\, dN(s), \end{aligned} \tag{16}$$

where $F(z_t) = F(z_{t-} - g(z_{t-}))$. Using this equation, one can identify the operators $L_f F(z) \triangleq f(z)\frac{\partial F(z)}{\partial z}$ and $\Delta_g F(z) \triangleq F(z+g(z)) - F(z)$. Now, let $F(z)$ in (16) be replaced by either $f(z)$ or $g(z)$, and substitute $f(z)$ and $g(z)$ into (15). This yields

$$z_t = z_0 + f(z_0)\int_0^t u_s\, ds + g(z_0)\int_0^t u_{s-}\, dN(s) + R_1(z_t),$$

where $R_1(z_t)$ contains all the iterated integrals of order 2 whose integrands do not depend on z_0. In light of (4)-(5), define $X = \{x_1\}$, $Y = \{y_1^{(1)}\}$ and the iterated operators $L_{x_1\eta} = L_\eta L_{x_1}$ and $L_{y_1^{(1)}\eta} = L_\eta L_{y_1^{(1)}}$, where $L_{x_1} = L_f$, $L_{y_1^{(1)}} = \Delta_g$, and $\eta \in XY^*$. Repeating this procedure iteratively yields the Peano-Baker formula for equation (15)

$$z_t = F_{c_z}[w](t) = \sum_{\eta \in XY^*} L_\eta(id(z_0))\, E_\eta[w](t), \tag{17}$$

where id denotes the identity map. Thus (f, g, id, z_0) realizes the operator F_{c_z} driven by u and a Poisson process N of 1 type when $(c_z, \eta) = L_{g_\eta}(id(z_0))$, $\forall \eta \in XY^*$, is locally convergent. Note now that if (17) is the solution of (15) then

$$\begin{aligned} dz_t &= \sum_{\eta \in XY^*} L_\eta f(z_0)\, E_\eta[w](t)\, u_t\, dt \\ &\quad + \sum_{\eta \in XY^*} L_\eta g(z_0)\, E_\eta[w](t-)\, u_{t-}\, dN(t). \end{aligned}$$

Considering that the product of Poisson-Lebesgue iterated integrals satisfies (14), and that the product rule for the operator Δ_g and any $\psi_1, \psi_2 \in \mathcal{C}^\omega$ is

$$\Delta_g(\psi_1\psi_2) = \psi_1\Delta_g(\psi_2) + \Delta_g(\psi_1)\psi_2 + \Delta_g(\psi_1)\Delta_g(\psi_2),$$

the Fliess pre-lemma ([7, Proposition III.1], [18, Lemma 3.4.1]) still holds, and therefore,

$$\dot{z}_t = f(z_t)\, u_t + g(z_t)\, u_t\, v_t,$$

which is the simplified version of (15). Furthermore, for any analytic output function h such that $y_t = h(z_t)$, the Fliess pre-lemma also gives

$$y(z_t) = F_c[w](t) = \sum_{\eta \in XY^*} L_\eta h(z_0)\, E_\eta[w](t),$$

where $(c,\eta) = L_\eta h(z_0)$. Hence, the proof is complete. ■

Example 3: Reconsider the switched system presented in Example 1. Let $z_0 = (1,\ldots,1)^T \in \mathbb{R}^{n\times 1}$ and $XY = \{x_0, x_1, y_0^{(1)}, \ldots, y_0^{(k)}, y_1^{(1)}, \ldots, y_1^{(k)}\}$. Then from (17), $y(t) = F_c[w](t)$, where $w = (u, \bar{v})$, $u \in B_1(R)[0,T]$, $\bar{v} = (\Delta N_1, \ldots, \Delta N_k)$,

$$(c,\eta) = CL_\eta(id(z_0)) = CM_\eta z_0,$$

$M_{x_i\eta} = M_{0,i}M_\eta$ and $M_{y_i^{(j)}\eta} = M_{j,i}M_\eta$ for $\eta \in XY^*$. Observe that c satisfies (1). So from Theorem 1, $y(t) = F_c[w](t)$ at least converges to a well-defined output process for some $R, T > 0$. Thus,

$$c = \sum_{l=0}^{\infty} C\left(\sum_{i=0}^{1} M_{0,i}x_i + \sum_{i=0,j=1}^{1,k} M_{j,i}\, y_i^{(j)}\right)^l z_0.$$

From Theorem 9, and assuming that the $M_{j,i}$'s commute, then

$$\begin{aligned}
&F_c[w](t)\\
&= \sum_{l=0}^{\infty} C\left(E_{\left(\sum_{i=0}^{1} M_{0,i}x_i\right)}[w](t) + E_{\left(\sum_{i=0,j=1}^{1,k} M_{j,i}y_i^{(j)}\right)}[w](t)\right)^{\{l\}} z_0\\
&= C\sum_{l=0}^{\infty} E_{\left(\sum_{i=0}^{1} M_{0,i}x_i\right)^l}[w](t)\cdot\\
&\qquad \prod_{j=1}^{k}\prod_{s\le t}\left(1+\Delta\left(E_{\left(\sum_{i=0}^{1} M_{j,i}y_i^{(j)}\right)}[w](s)\right)\right) z_0\\
&= C\exp\left(M_{0,0}\,t + M_{0,1}\int_0^t u(s)\,ds\right)\cdot\\
&\qquad \prod_{j=1}^{k}\prod_{s\le t}\left(1+(M_{j,0}+M_{j,1}\,u(s-))\,\Delta N_j(s)\right) z_0\\
&= C\exp\left(M_{0,0}\,t + M_{0,1}\int_0^t u(s)\,ds\right)\cdot\\
&\qquad \prod_{j=1}^{k}\exp\left(\sum_{s\le t}\ln\left(1+M_{j,0}+M_{j,1}\,u(s-)\right)\Delta N_j(s)\right) z_0\\
&= C\exp\left(M_{0,0}\,t + M_{0,1}\int_0^t u(s)\,ds\right)\cdot\\
&\qquad \prod_{j=1}^{k}\exp\left(\int_0^t \ln\left(1+M_{j,0}+M_{j,1}\,u(s-)\right)dN_j(s)\right) z_0.
\end{aligned}$$

□

It is worth pointing out that explicit solutions for the previous example can also be obtained in terms of exponentials when the vector fields are not commutative. The expression for the logarithm of this exponential (known as the Magnus expansion or the Chen-Strichartz formula) has been developed in terms of iterated Lie brackets [2], [3], [13], [14].

V. Conclusions and Future Work

This paper described a class of convergent Fliess operators admitting L_p and Poisson process inputs. It was then shown how Poisson switched input-affine nonlinear systems have an input-output map that can be described in terms of such Fliess operators. It is conjectured that such an approach can also be applied to Markov switched systems, i.e., where the interarrival times are not necessarily exponentially distributed, but the independence of the increments still holds.

Acknowledgements

The authors want to thank Kurusch Ebrahimi-Fard for the valuable discussions concerning the algebraic setting. They also wish to thank Kurusch Ebrahimi-Fard, Matthias Kawski and David Martín de Diego for the invitation to attend the 2010 Trimester in Combinatorics and Control in Madrid, where this project was first conceived. Travel support was provided by the National Science Foundation grant DMS 0960589.

References

[1] G. B. Arous, "Flots et séries de Taylor stochastiques," *Prob. Theory Related Fields*, vol. 81, pp. 29–77, 1989.

[2] F. Baudoin, *An Introduction to the Geometry of Stochastic Flows*, Imperial College Press, London, 2004.

[3] S. Blanes, F. Casas, J. Oteo, and J. Ros, "The Magnus expansion and some of its applications", *Phys. Rep.*, vol. 470, pp. 151–238, 2009.

[4] L. A. Duffaut Espinosa, *Interconnections of Nonlinear Systems Driven by L_2-Itô Stochastic Processes*, Ph.D. dissertation, Old Dominion University, 2009.

[5] L. A. Duffaut Espinosa, W. S. Gray, and O. R. González, "On Fliess operators driven by L_2-Itô random processes," *Proc. 48th IEEE Conf. on Decision and Control*, Shanghai, China, 2009, pp. 7478-7484.

[6] ——, "On the local convergence of Fliess operators driven by L_2-Itô processes," *Proc. 49th IEEE Conf. on Decision and Control*, Atlanta, Georgia, 2010, pp. 2631-2637.

[7] M. Fliess, "Fonctionnelles causales non linéaires et indéterminées non commutatives," *Bull. Soc. Math. France*, vol. 109, pp. 3–40, 1981.

[8] ——, "Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives," *Invent. Math.*, vol. 71, pp. 521–537, 1983.

[9] J. Gaines, "The algebra of iterated stochastic integrals," *Stoch. Stoch. Rep.*, vol. 49, pp. 169–179, 1994.

[10] W. S. Gray, O. R. González, and M. Doğan, "Stability analysis of digital linear flight controllers subject to electromagnetic disturbances," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 36, pp. 1204–1218, 2000.

[11] W. S. Gray and Y. Li, "Generating series for interconnected analytic nonlinear systems," *SIAM J. Contr. Optim.*, vol. 44, pp. 646–672, 2005.

[12] W. S. Gray and Y. Wang, "Fliess operators on L_p spaces: Convergence and continuity," *Systems Control Lett.*, vol. 46, pp. 67–74, 2002.

[13] H. Kunita "Asymptotic self-similarity and short time asymptotics of stochastic flows," *J. Math. Sci. Univ. Tokyo*, vol. 4, pp. 595–619, 1997.

[14] C. W. Li and X. Q. Liu, "Algebraic structure of multiple stochastic integrals with respect to Brownian motions and Poisson processes," *Stoch. Stoch. Rep.*, vol. 61, pp. 107–120, 1997.

[15] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, 2003.

[16] P. E. Protter, *Stochastic Integration and Differential Equations*, 2nd ed, Springer, Berlin, 2005.

[17] S. M. Ross, *Introduction to Probability Models*, 6th ed, Academic Press, San Diego, 1997.

[18] Y. Wang, *Algebraic Differential Equations and Nonlinear Control Systems*, Ph.D. dissertation, Rutgers University, 1990.

[19] H. Zhang, W. S. Gray, O. R. González, and A. V. Lakdawala, "Tracking performance of a recoverable flight control system in neutron environments," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, pp. 321–335, 2009.