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On the linear control of the quad-rotor system

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Abstract— **In this article, a robust linear output feedback control scheme is proposed for the efficient regulation, and trajectory tracking tasks, in the nonlinear, multivariable, quad-rotor system model. The proposed linear feedback scheme is based on the use of a classical linear feedback controllers and suitably extended, high gain, linear Generalized Proportional Integral (GPI) observers; aiding the linear feedback controllers in two important tasks: 1) accurate estimation of the input-output system model nonlinearities, 2) accurate estimation of the unmeasured phase variables associated with the flat, or linearizing, output variables. These two key pieces of information are used in the proposed feedback controller in a) approximate, yet close, cancelation, as a lumped unstructured time-varying term, of the influence of the highly coupled nonlinearities and b) devising proper linear output feedback control laws based on the approximate estimates of the string of phase variables associated with the flat outputs simultaneously provided by the disturbance observers.**

I. INTRODUCTION

Asymptotic estimation of external, unstructured, perturbation inputs, with the aim of exactly, or approximately, canceling their influences at the controller stage, has been treated in the existing literature under several headings. The outstanding work of professor C.D. Johnson in this respect, under the name of *Disturbance Accommodation Control* (DAC), dates from the nineteen seventies (see [11]). Ever since, the theory and practical aspects of DAC theory have been actively evolving, as evidenced by the survey paper by Johnson [13]. The theory enjoys an interesting and useful extension to discrete-time systems, as demonstrated in the book chapter [12]. In a recent article, by Parker and Johnson [17], an application of DAC is made to the problem of decoupling two nonlinearly coupled linear systems.

A closely related vein to DAC is represented by the sustained efforts of the late Professor Jingqing Han, summarized in the posthumous paper, Han [9], and known as: *Active Disturbance Estimation and Rejection* (ADER). The numerous and original developments of Prof. Han, with many laboratory and industrial applications, have not been translated into English and his seminal contributions remain written in Chinese (see the references in [9]). Although the main idea of observer-based disturbance estimation, and subsequent cancelation via the control law, is similar to that advocated in DAC, the emphasis in ADER lies, mainly, on *nonlinear* observer based disturbance estimation, with necessary developments related to: efficient time derivative computation, practical relative degree computation and nonlinear PID control extensions. The work, and inspiration, of Professor Han has found interesting developments and applications in the work of Professor Z. Gao and his colleagues (see [7], [8], also, in the work by Sun and Gao [20] and in the article by Sun [21]). In a recent article, a closely related idea, proposed by Prof. M. Fliess and C. Join in [6], is at the core of *Intelligent PID Control* (IPIDC). The mainstream of the IPIDC developments makes use of the Algebraic Method and it implies to resort to first order, or at most second order, *non-phenomenological* plant models. The interesting aspect of this method resides in using suitable algebraic manipulations to locally deprive the system description of the effects of nonlinear uncertain additive terms and, via further special algebraic manipulations, to efficiently identify time-varying control gains as piece-wise constant control input gains (see [5]). An entirely algebraic approach for the control of a synchronous generator was presented in Fliess and Sira-Ramírez, [18].

In this article, we propose a robust observer-based linear output feedback control scheme for the trajectory tracking tasks in the quad-rotor system model. The linear observer-based controller design approach, presented here, is most suitable for the ubiquitous class of *differentially flat* systems (see Fliess *et al.*[3] for the original introduction of the flatness concept, the book by Sira-Ramírez and Agrawal [19], and the recent book by Lévine [15] for interesting real life examples). The proposed control approach, called Generalized Proportional Integral (GPI) observer-based control, rests on using highly simplified models of the inputs differential parameterizations, provided by the flatness property. In this simplification, only the order of integration of the subsystems and the control inputs, along with their associated matrix gains, are retained in full detail. All the additive nonlinearities, including their state couplings and complexities, are regarded as, unstructured, time-varying signals that need to be on-line estimated, and canceled, at the controller specification. This simplifying procedure, thus, produces a *non-phenomenological model* of the input-to-flat output behavior, in which the order of integration of each of the subsystems, and the matrix input gain, are truly significant for state estimation and control purposes. After input gain matrix cancelation, the resulting system is constituted by pure integration (linear) perturbed systems with time-varying additive disturbances. A set of linear extended observers, here called GPI observers, are subsequently produced which internally model the state dependent additive nonlinearities as time-polynomials of reasonable low orders. The observers state estimation errors are shown to satisfy a set of decoupled,

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perturbed, linear differential equations with assignable constant coefficients. Under the assumption that the exogenous time-varying perturbation inputs are uniformly absolutely bounded, the designed observers estimate each individual flat output's associated string of phase variables as well as the time-varying perturbation, or disturbance, input components. The state and perturbation estimation relies on a high gain observer design. The quad-rotor system is shown to be a differentially flat system. The flatness property allows one to obtain a meaningful input-to-highest derivative of flat outputs relation The proposed linear feedback scheme is based on the use of a classical linear feedback controller and a suitably extended high gain linear observer; aiding the linear feedback controller, in two important tasks: 1) accurate estimation of the input-output system model nonlinearities, 2) accurate estimation of the unmeasured phase variables associated with each of the linearizing output variables. These two key pieces of information are used in the proposed feedback controller to a) cancel, as a lumped unstructured time-varying term, the influence of the nonlinearities and b) devise a proper linear output feedback based on the approximate estimates of the flat outputs associated phase variables.

This article is organized as follows: Section II presents the quad-rotor model and its flatness property. It formulates the problem and presents the main results, Section III is devoted to some computer simulations depicting the performance of the proposed GPI observer-based linear controllers on the quad-rotor system. We do so under the assumptions of known nonlinear input gain matrix and also removing this crucial assumption. Section IV contains the conclusions and suggestions for further work.

II. Problem formulation and main results

A. The quad-rotor model and its flatness property

Consider the following model of the quad-rotor, as derived through the Euler-Lagrange formalism in the book by Castillo *et al.* [1]

$$\begin{aligned} m\ddot{x} &= -u\sin\theta, \quad m\ddot{y} = u\cos\theta\sin\phi \\ m\ddot{z} &= u\cos\theta\cos\phi - mg \\ \ddot{\psi} &= \tau_\psi, \quad \ddot{\theta} = \tau_\theta, \quad \ddot{\phi} = \tau_\phi \end{aligned} \tag{1}$$

The system is differentially flat, with flat outputs given by the four-vector:

$$F = (x, y, z, \psi)^T \tag{2}$$

Indeed, all system variables are differentially parameterizable in terms of F (i.e., they are functions of F and of a finite number of derivatives of its components). It is easy to verify that the state variables, ϕ and θ, are differentially parameterized by,

$$\begin{aligned} \phi &= \arctan\left(\frac{\ddot{y}}{\ddot{z}+g}\right) \\ \theta &= -\arctan\left(\frac{\ddot{x}}{\sqrt{\ddot{y}^2+(\ddot{z}+g)^2}}\right) \end{aligned} \tag{3}$$

while the control inputs, u, τ_ψ and τ_ϕ are obtained, after long but straightforward computations, as:

$$u = m\sqrt{\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2}, \quad \tau_\psi = \ddot{\psi} \tag{4}$$

$$\begin{aligned} \tau_\phi &= \frac{y^{(4)}(\ddot{z}+g)-\ddot{y}z^{(4)}}{\ddot{y}^2+(\ddot{z}+g)^2} \\ &\quad -2\frac{(y^{(3)}(\ddot{z}+g)-\ddot{y}z^{(3)})(\ddot{y}y^{(3)}+(\ddot{z}+g)z^{(3)})}{(\ddot{y}^2+(\ddot{z}+g)^2)^2} \end{aligned} \tag{5}$$

$$\begin{aligned} \tau_\theta &= -\left\{\frac{\eta(\ddot{x},\ddot{y},\ddot{z},x^{(3)},y^{(3)},z^{(3)}x^{(4)},y^{(4)},z^{(4)})}{(\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2)(\ddot{y}^2+(\ddot{z}+g)^2)^{\frac{3}{2}}}\right\} \\ &+\left\{\frac{(x^{(3)}(\ddot{y}^2+(\ddot{z}+g)^2)-\ddot{x}(y^{(3)}+(\ddot{z}+g)z^{(3)}))}{(\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2)^2(\ddot{y}^2+(\ddot{z}+g)^2)^3}\right\}\times \\ &\Big[2(\ddot{x}x^{(3)}+\ddot{y}y^{(3)}+(\ddot{z}+g)z^{(3)})(\ddot{y}^2+(\ddot{z}+g)^2)^{\frac{3}{2}} \\ &+6(\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2)(\ddot{y}^2+(\ddot{z}+g)^2)^{\frac{1}{2}}(y^{(3)}+(\ddot{z}+g)z^{(3)})\Big] \end{aligned} \tag{6}$$

where,

$$\begin{aligned} &\eta(\ddot{x},\ddot{y},\ddot{z},x^{(3)},y^{(3)},z^{(3)},x^{(4)},y^{(4)},z^{(4)},y^{(4)}) = \\ &\quad x^{(4)}(\ddot{y}+\ddot{z}+g)+2x^{(3)}y^{(3)}y^{(4)} \\ &\quad +2x^{(3)}(\ddot{z}+g)z^{(3)}-x^{(3)}(y^{(3)}+(\ddot{z}+g)z^{(3)}) \\ &\quad -\ddot{x}(y^{(4)}+(z^{(3)})^2+(\ddot{z}+g)z^{(4)}) \end{aligned} \tag{7}$$

The lack of invertibility of the relation between the control input vector and the flat outputs highest derivative reveals that the forward velocity variable, u, which also acts as a control input, needs to be extended twice. We obtain,

$$\begin{aligned} \ddot{u} &= \frac{(x^{(3)})^2+\ddot{x}x^{(4)}+(y^{(3)})^2+\ddot{y}y^{(4)}+(z^{(3)})^2+(\ddot{z}+g)z^{(4)}}{\sqrt{\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2}} \\ &\quad -\frac{(\ddot{x}x^{(3)}+\ddot{y}y^{(3)}+(\ddot{z}+g)z^{(3)})^2}{(\ddot{x}^2+\ddot{y}^2+(\ddot{z}+g)^2)^{\frac{3}{2}}} \end{aligned} \tag{8}$$

It is evident, from the above expressions, that the highly non-linear, coupled, nature of the system precludes any practical on-line implementation of an exactly linearizing feedback control approach. This fact motivates us to adopt a *linear* disturbance and linear state estimation approach, aimed at cancelation of additive nonlinearities and imposition of decoupled closed loop linearity, via linear phase variables feedback control, on a simplified perturbed dynamical model. This results in effectively controlling, in a fundamentally *linear* manner, such a complex nonlinear multi-variable system. In order to achieve this, we first produce a simplified non-phenomenological model of the multi-variable plant, devise a set of GPI observers for simultaneous disturbance and state estimation and proceed to formulate a set of canceling linear feedback controllers, based on the estimates of the phase variables associated with the components of the flat output vector.

We formulate the problem as follows:

Given a flat output vector of reference trajectories, $F^*(t)$, devise a linear multi-input output feedback controller that suitably cancels, even if in an approximate manner, the vector of coupling nonlinearities and forces the flat output tracking error vector dynamics to exhibit a closed loop, predominantly linear, asymptotically stable convergent behavior so that the tracking error trajectories are ultimately confined to a small as desired neighborhood of the origin of the tracking error phase space.

B. A GPI observer based linear controller with disturbance estimation-rejection

One obtains, from equations (4)-(8) the following input-to-flat outputs highest derivatives simplified relation,

$$\begin{bmatrix} mx^{(4)} \\ my^{(4)} \\ mz^{(4)} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} -s_\theta & -uc_\theta & 0 & 0 \\ c_\theta s_\phi & -us_\theta s_\phi & uc_\theta c_\phi & 0 \\ c_\theta c_\phi & -us_\theta c_\phi & -uc_\theta s_\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \tau_\theta \\ \tau_\phi \\ \tau_\psi \end{bmatrix} + \varphi(t) \tag{9}$$

where $\varphi(t)$ summarizes all the nonlinearities affecting the system behavior here regarded as unknown disturbance inputs and c_χ, s_χ stand, respectively for $\cos(\chi)$, $\sin(\chi)$. Here we have used the equations in (3) to obtain an input gain matrix explicitly depending on the angular displacements θ and ϕ. We henceforth assume that these two angles can be measured. At the end of the article, we lift this assumption and carry out an illustrative example where linear estimates of these angles are used in the linear feedback law.

The previous expressions have been substantially simplified by resorting to the original system representation and acknowledging the flatness based structural findings: The outputs: x, y, and z are, each, relative degree four with a second order extension of the input u and, secondly, the output, ψ, is only relative degree two, devoid of additive nonlinearities, and totally decoupled of the rest of the dynamics.

A GPI observer based controller is devised as follows:

$$\begin{bmatrix} \ddot{u} \\ \tau_\theta \\ \tau_\phi \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} -s_\theta & c_\theta s_\phi & c_\theta c_\phi & 0 \\ -\frac{1}{u}c_\theta & -\frac{1}{u}s_\theta s_\phi & -\frac{1}{u}s_\theta c_\phi & 0 \\ 0 & \dfrac{c_\phi}{uc_\theta} & -\dfrac{s_\phi}{uc_\theta} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ v_\psi \end{bmatrix} \tag{10}$$

with

$$\begin{aligned} v_x &= m\left[-\zeta_1^x + [x^*(t)]^{(4)} - \sum_{i=0}^{i=3} k_i^x(\widehat{x^{(i)}} - [x^*(t)]^{(i)})\right] \\ v_y &= m\left[-\zeta_1^y + [y^*(t)]^{(4)} - \sum_{i=0}^{i=3} k_i^y(\widehat{y^{(i)}} - [y^*(t)]^{(i)})\right] \\ v_z &= m\left[-\zeta_1^z + [z^*(t)]^{(4)} - \sum_{i=0}^{i=3} k_i^z(\widehat{z^{(i)}} - [z^*(t)]^{(i)})\right] \end{aligned}$$

$$v_\psi = [\ddot{\psi}^*(t)] - \sum_{i=0}^{i=1} k_i^\psi(\widehat{\psi^{(i)}} - [\psi^*(t)]^{(i)}) \tag{11}$$

The design coefficients k_i^x, k_i^y, k_i^z and k_i^ψ are chosen so that the dominant characteristic polynomials:

$$p_j(s) = s^4 + k_3^j s^3 + k_2^j s^2 + k_1^j s + k_0^j, \quad j = x,y,z \tag{12}$$

and

$$p_\psi(s) = s^2 + k_1^\psi s + k_0^\psi \tag{13}$$

are Hurwitz.

The quantities, $\widehat{x^{(j)}} = x_j$, $j = 0,1,2,3$ and ζ_1^x are generated by,

$$\begin{aligned} \dot{x}_0 &= x_1 + \lambda_7^x(x - x_0) \\ \dot{x}_1 &= x_2 + \lambda_6^x(x - x_0) \\ \dot{x}_2 &= x_3 + \lambda_5^x(x - x_0) \\ \dot{x}_3 &= \frac{1}{m}[-s_\theta \ddot{u} - uc_\theta \tau_\theta] + \zeta_1^x + \lambda_4^x(x - x_0) \\ \dot{\zeta}_1^x &= \zeta_2^x + \lambda_3^x(x - x_0) \\ \dot{\zeta}_2^x &= \zeta_3^x + \lambda_2^x(x - x_0) \\ \dot{\zeta}_3^x &= \zeta_4^x + \lambda_1^x(x - x_0) \\ \dot{\zeta}_4^x &= \lambda_0^x(x - x_0) \end{aligned} \tag{14}$$

The design coefficients λ_k^x, $k = 0,1,..,7$ are chosen so that the reconstruction error dynamics dominant characteristic polynomial is an 8-th degree Hurwitz polynomial, i.e.,

$$p_{xo}(s) = s^8 + \lambda_7^x s^7 + \lambda_6^x s^6 + \cdots + \lambda_1^x s + \lambda_0^x \in \text{Hurwitz}_8(s) \tag{15}$$

Similarly, the quantities: $\widehat{y^{(j)}} = y_j$, $j = 0,1,2,3$, and ζ_1^y are generated by,

$$\begin{aligned} \dot{y}_0 &= y_1 + \lambda_7^y(y - y_0) \\ \dot{y}_1 &= y_2 + \lambda_6^y(y - y_0) \\ \dot{y}_2 &= y_3 + \lambda_5^y(y - y_0) \\ \dot{y}_3 &= \frac{1}{m}\left[c_\theta s_\phi \ddot{u} - us_\theta s_\phi \tau_\theta + uc_\theta c_\phi \tau_\phi\right] + \zeta_1^y + \lambda_4^y(y - y_0) \\ \dot{\zeta}_1^y &= \zeta_2^y + \lambda_3^y(y - y_0) \\ \dot{\zeta}_2^y &= \zeta_3^y + \lambda_2^y(y - y_0) \\ \dot{\zeta}_3^y &= \zeta_4^y + \lambda_1^y(y - y_0) \\ \dot{\zeta}_4^y &= \lambda_0^y(y - y_0) \end{aligned} \tag{16}$$

where the design coefficients λ_k^y, $k = 0,1,..,7$ are chosen so that the reconstruction error dynamics dominant characteristic polynomial is Hurwitz, i.e.,

$$p_{yo}(s) = s^8 + \lambda_7^y s^7 + \lambda_6^y s^6 + \cdots + \lambda_1^y s + \lambda_0^y \in \text{Hurwitz}_8(s) \tag{17}$$

The quantities: $\widehat{z^{(j)}} = z_j$, $j = 0,1,2,3$, and ζ_1^z, are generated by,

$$\begin{aligned}
\dot{z}_0 &= z_1 + \lambda_7^z (z - z_0) \\
\dot{z}_1 &= z_2 + \lambda_6^z (z - z_0) \\
\dot{z}_2 &= z_3 + \lambda_5^z (z - z_0) \\
\dot{z}_3 &= \frac{1}{m}\left[c_\theta c_\phi \ddot{u} - u s_\theta c_\phi \tau_\theta - u c_\theta s_\phi \tau_\phi\right] + \zeta_1^z + \lambda_4^z (z - z_0) \\
\dot{\zeta}_1^z &= \zeta_2^z + \lambda_3^z (z - z_0) \\
\dot{\zeta}_2^z &= \zeta_3^z + \lambda_2^z (z - z_0) \\
\dot{\zeta}_3^z &= \zeta_4^z + \lambda_1^z (z - z_0) \\
\dot{\zeta}_4^z &= \lambda_0^z (z - z_0)
\end{aligned} \tag{18}$$

As before, The design coefficients λ_k^z, $k = 0, 1, .., 7$ are chosen so that the reconstruction error dynamics dominant characteristic polynomial is Hurwitz, i.e.,

$$p_{zo}(s) = s^8 + \lambda_7^z s^7 + \lambda_6^z s^6 + \cdots + \lambda_1^z s + \lambda_0^z \in \text{Hurwitz}_8(s) \tag{19}$$

Finally, the quantities: $\widehat{\psi^{(j)}} = \psi_j$, $j = 0, 1$, are generated by,

$$\begin{aligned}
\dot{\psi}_0 &= \psi_1 + \lambda_1^\psi (\psi - \psi_0) \\
\dot{\psi}_1 &= \frac{1}{m}\tau_\psi + \lambda_0^\psi (\psi - \psi_0)
\end{aligned} \tag{20}$$

The coefficients λ_k^ψ, $k = 0, 1$. are chosen so that,

$$p_{\psi o} = s^2 + \lambda_1^\psi s + \lambda_0^\psi \in \text{Hurwitz}_2(s) \tag{21}$$

We have the following result.

Theorem 1: Given a smooth vector of desired reference trajectories for the components of the flat output vector, $F^*(t) = (x^*(t), y^*(t), z^*(t), \psi^*(t))^T$, and provided the observers and the controllers constant gains appearing in (12), (13), (15), (17), (19), (21), are chosen so that the roots of the corresponding closed loop characteristic polynomials are chosen deep into the left half of the complex plane, then the GPI observer based linear feedback controllers given by equations: (10), (11), (14), (16), (18), (20), produce a set of perturbed closed loop flat outputs tracking error dynamics whose trajectories converge, in an asymptotically exponentially dominated manner, to a small as desired neighborhood of the origins of the flat outputs tracking error phase spaces. Moreover, the flat output phase variables estimation errors satisfy linear perturbed dynamics whose trajectories also dominantly converge in an asymptotically exponentially dominated manner to small as desired neighborhoods of the origins of the reconstruction errors phase spaces. As a result, the disturbance vector components of $\varphi(t)$ are closely estimated with an error bounded by a small as desired neighborhood of zero. As the location of the roots of the dominating characteristic polynomials are further pushed into the left half of the complex plane, the tighter around the origin are all these tracking, or estimation, bounding neighborhoods.

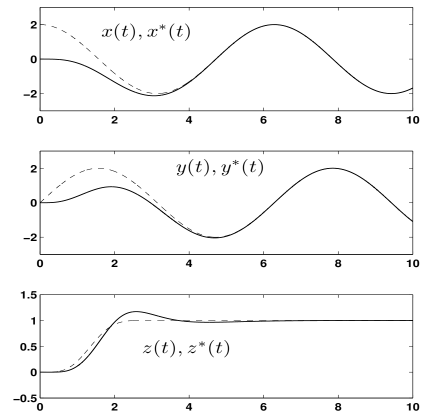

Fig. 1. Position variables, $x(t)$, $y(t)$, $z(t)$ and desired trajectories $x^*(t)$, $y^*(t)$, $z^*(t)$

III. SIMULATION RESULTS

It is desired to track a circular trajectory, of radius $R = 2$, around the origin of the plane xy, at the height of 1 [m], in a counterclockwise direction with angular speed, w, of 1 [rad/s]. It is desired that the quad-rotor advances along this trajectory continuously changing its orientation in a tangential direction to the circle. At this point, no restrictions are placed on the collective sustaining force nor on the torques applied to each orientation parameter. The desired z coordinate is specified by means of a 10-th degree Bézier polynomial, denoted by $\text{Bezier}_{10}(t, t_0, t_f)$, smoothly rising from ground level: $z = 0$, to the desired height, $z = 1$ [m], in 3 seconds (from $t_0 = 0$, to $t_f = 3$ [s]).

$$\begin{aligned}
x^*(t) &= R\cos(wt), \; y^*(t) = R\sin(wt), \\
z^*(t) &= \text{Bezier}_{10}(t, 0, 3)
\end{aligned}$$

Figures 1 to 4 depict the performance of the linear, GPI observer based, controller with exact cancelation of the nonlinear input gain matrix and on-line estimation of the nonlinearities.

A. Further simulation results

It is desired to track a trajectory represented by a "penta-folium" inscribed in a plane parallel to the xy plane at a height of 1 [mt]. In this case, it is not enforced that the quad-rotor advances along this trajectory continuously changing its orientation in a tangential direction to the path. The desired z coordinate is specified by means of a 10-th degree Bézier polynomial smoothly rising from ground level: $z = 0$, to the desired height, $z = 1$ [m], in 3 seconds. The fundamental difference is that, in these simulations, the gain input matrix is no longer assumed to be perfectly known and its elements are now conformed from the estimated flat outputs phase variables, ($\hat{x}, \hat{y}$ and $\hat{z}$), using the observers variables and the parameterizations found in expression (3).

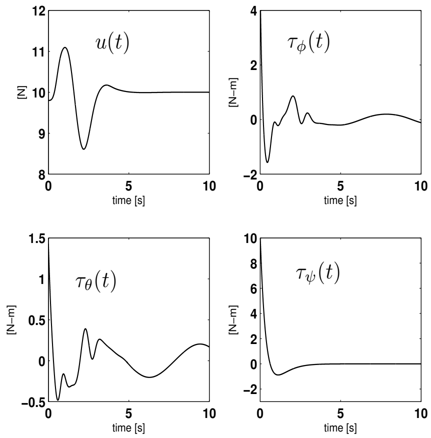

Fig. 2. Collective input force $u(t)$ and input torques $\tau_\theta(t)$, $\tau_\phi(t)$ and τ_ψ

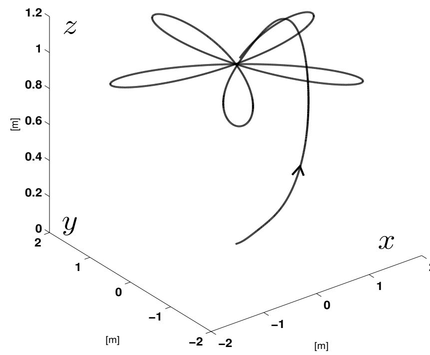

Fig. 5. Evolution of the quad-rotor center of mass trajectory following a "penta-folium"

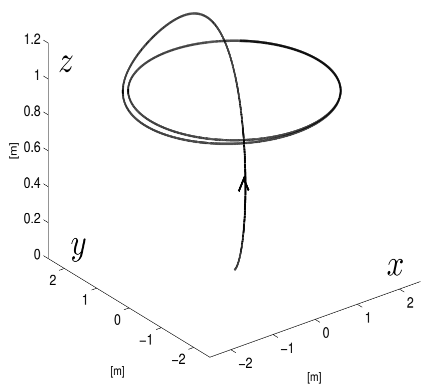

Fig. 3. Tracking of circular trajectory in 3-D space

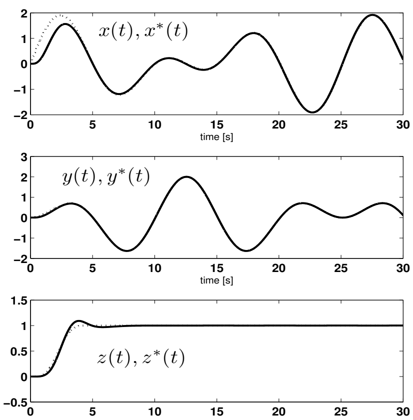

Fig. 6. Position trajectories, $x(t)$, $y(t)$, $z(t)$ and desired reference trajectories, $x^*(t)$, $y^*(t)$, $z^*(t)$

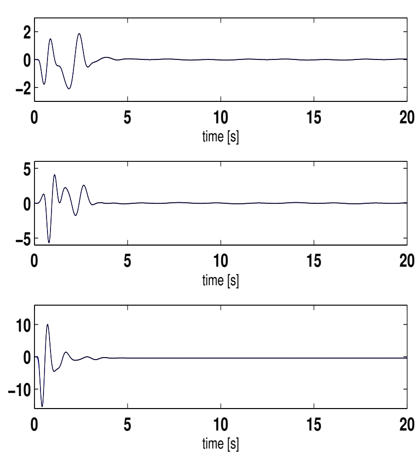

Fig. 4. Comparison between state dependent estimated disturbances and their actual values

The "penta-folium", trajectory is given, in parametric coordinates, by the formula:

$$x^*(t) \;\; = \;\; a\sin t + \sin bt, \quad y^*(t) = a\cos t - \cos bt \tag{22}$$

with $a = 0.5$, $b = 0.75$.

Figures 5 to 8 depict the performance of the linear, GPI observer based, controller with asymptotic, approximate, cancelation of the nonlinear input gain matrix and on-line estimation of the nonlinearities.

IV. Conclusions

A. Conclusions

In this article, we have explored, within the context of the trajectory tracking problem in the highly nonlinear, multivariable, quad-rotor model, the use of approximate, yet accurate, simultaneous state-dependent disturbance estimation and state estimation via linear Generalized Proportional Integral (GPI) observers. These observers aid linear output

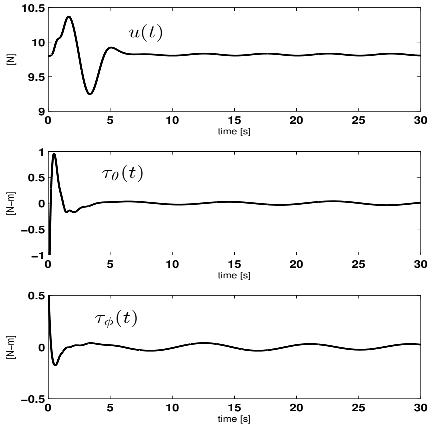

Fig. 7. Collective input force $u(t)$ and input torques $\tau_\theta(t)$, $\tau_\phi(t)$

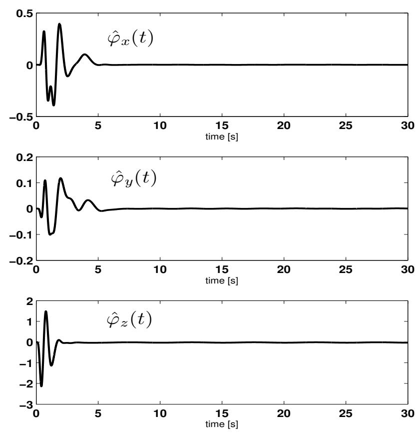

Fig. 8. State-dependent estimated disturbances

feedback controllers in the perturbation canceling task and the conformation of a linear feedback scheme for each flat output evolution. The overall observer-based control scheme is, however, approximate; since only small as desired reconstruction and reference trajectory tracking errors are guaranteed at the expense of high, noise-sensitive, observer-controller gains. Digital computer simulations were provided where the efficiency of the proposed control method is assessed.

B. Future Works

GPI observer-based linear control of nonlinear systems is naturally fit for differentially flat systems, provided the flat output vector components are available for measurements. The fundamental restriction of unavailable flat outputs remains to be fully explored. In this respect, the minimum-phase restriction seems to be natural. These topics, and other related limitations, need to be explored and resolved in the future.

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