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A new Design Method for Mismatch-based Anti-Windup Compensators: Achieving Local Performance and Global Stability in the SISO Case

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Abstract— **This paper proposes a new design method for full-order mismatch-based anti-windup compensators applicable to SISO systems. It is based on multiplier theory and path following. In contrast to available approaches the new method allows synthesizing globally stabilizing mismatch-based compensators with enhanced local performance based on small-signal L_2 gains. A hydraulic actuator serves as an example to demonstrate the effectiveness of the proposed technique.**

I. Introduction

The problem of actuator saturation in control has been tackled from various different perspectives. Three principal directions of research are the synthesis of (nonlinear) controllers which directly account for saturation constraints, model predictive control strategies and anti-windup techniques which are subject of this paper.

The philosophy of anti-windup is to separate the controller into a nominal linear controller and an anti-windup compensator (aw compensator). The former is synthesized to achieve stability and a good performance in case the input saturation of the plant would not exist. The latter handles the so-called "windup-effects" caused by actuator constraints. For a recent survey see [1].

In [2] the goal of anti-windup was properly mathematically formalized for the first time. Roughly speaking, the task of an aw compensator is to ensure a swift return of the plant output to the nominal plant output, i.e. the output in the absence of saturation. Since then, many systematic anti-windup approaches have been proposed, e.g. [3], [4], [5] and the references therein. One of them is the mismatch-based aw compensator introduced in [6] and further developed in [7], [8], [9]. Due to its special structure, it can directly account for the above mentioned goal. Furthermore, the design consists of a one-step convex optimization problem with a simple and transparent structure, sureley contributing to the success of this approach.

Unfortunately, the available design methods may either guarantee global stability and a sluggish performance or an improved performance and a restricted region of stability as discussed in [8]. To improve this situation, we propose a new design procedure for SISO systems combining the advantages of both methods: global stability and enhanced local performance. Global stability is established using an multiplier approach similar to [10], [11], whereas the local performance is based on ideas in [8].

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The paper is organized as follows. In Section II the problem is introduced and available design methods are reviewed. Section III presents new stability conditions based on multiplier theory. In Section IV the new design method is described. Its effectiveness is demonstrated by an example in Section V before some conclusions are drawn.

Notation. $\mathbf{A} \succ \mathbf{0} (\mathbf{A} \prec \mathbf{0})$ indicates that the matrix $\mathbf{A}$ is symmetric and positive (negative) definite. The symbol $\star$ is short for the term required to make an expression symmetric and $\sim$ is used to denote equivalence of a transfer function and a state-space representation. The L_2 norm of a square integrable signal $y(t)$ is defined as $\|y\|_{L_2} = (\int_0^\infty y^2(t)dt)^{1/2}$. The phase response of a transfer function $W(s)$ is denoted as $\arg W(j\omega)$. The symbol $\lfloor \kappa \rfloor$ denotes the largest integer not greater than κ.

II. Problem Statement

The linear plant is assumed to be stable and completely controllable. It is described by

$$G(s) = \mathbf{c}^\mathsf{T}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \sim \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}^\mathsf{T}\mathbf{x} \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}$ is the control input to the plant and $y \in \mathbb{R}$ is the plant output.

The linear controller is given by

$$u_c(s) = \mathbf{K}(s) \begin{bmatrix} r(s) \\ y_c(s) \end{bmatrix} = \begin{bmatrix} K_r(s) & K_y(s) \end{bmatrix} \begin{bmatrix} r(s) \\ y_c(s) \end{bmatrix} \tag{2}$$

where $u_c \in \mathbb{R}$ is the controller output, $r \in \mathbb{R}$ the reference signal and $y_c \in \mathbb{R}$ the controller input. The interconnection of $\mathbf{K}(s)$ and $G(s)$ with $u = u_{\text{lin}} = u_c$ and $y = y_{\text{lin}} = y_c$ is assumed to be stable and depicted in Fig. 1. It is referred to as the linear closed-loop.

If the actuator saturates, the relation between u_c and u is defined by the saturation function

$$u = \text{sat}(u_c) = \text{sgn}(u_c)\min(u_0, |u_c|) \ , \tag{3}$$

limiting the absolute value of the plant input u to u_0. The arising system is called the saturated closed-loop.

The negative effects of saturation are handled by the mismatch-based aw compensator. The following discussion is based on [6], [8].

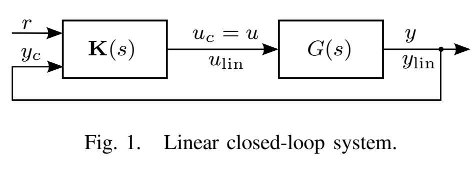

Fig. 1. Linear closed-loop system.

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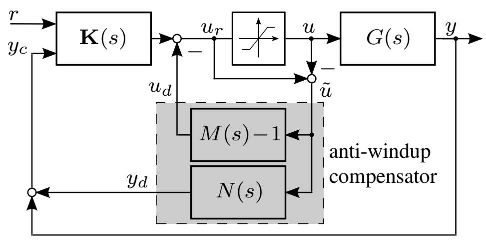

Fig. 2. Anti-windup closed-loop system.

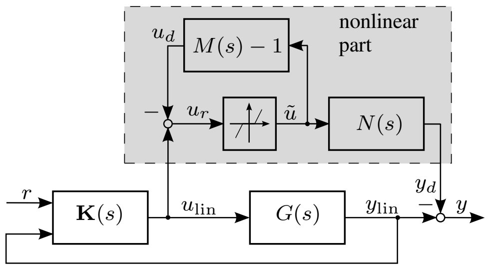

Fig. 3. Mismatch system: Equivalent representation of the anti-windup closed-loop system. This structure visualizes the achievable decoupling of the linear closed-loop and the saturation through a mismatch-based anti-windup compensator.

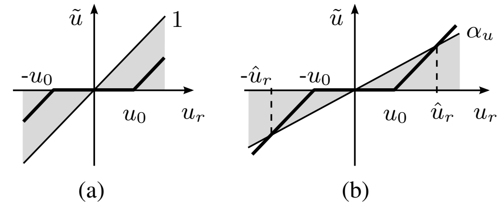

Fig. 4. (a) Deadzone nonlinearity in the sector $[0, 1]$. (b) Local sector bounds of the deadzone $[0, \alpha_u]$.

The mismatch-based aw compensator feeds additional signals y_d and u_d in the controller input and output respectively. It is parameterized by the transfer function $M(s)$ as follows

$$u_d(s) = \big(M(s) - 1\big)\tilde{u}(s)\,, \tag{4a}$$
$$y_d(s) = G(s)M(s)\tilde{u}(s) = N(s)\tilde{u}(s)\,, \tag{4b}$$

where $\tilde{u} = u_r - \text{sat}(u_r)$. Fig. 2 shows the arising closed-loop referred to as the anti-windup closed-loop system. The transfer functions $M(s), N(s)$ are chosen as

$$M(s) = \mathbf{f}^{\mathsf{T}}\left(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{f}^{\mathsf{T}}\right)\mathbf{b} + 1\,, \tag{5a}$$
$$N(s) = \mathbf{c}^{\mathsf{T}}\left(s\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{f}^{\mathsf{T}}\right)\mathbf{b}\,, \tag{5b}$$

where $\mathbf{A}, \mathbf{b}, \mathbf{c}$ are plant matrices and $\mathbf{f}$ is the design parameter of the aw compensator ensuring that $M(s)$, $N(s)$ are Hurwitz.

Fig. 2 can be re-drawn as Fig. 3. Following [1], the arising closed-loop is called mismatch system, consisting of the linear closed-loop and a nonlinear part highlighted in gray. The nonlinear part contains the deadzone

$$\tilde{u} = \text{dzn}(u_r) = u_r - \text{sat}(u_r) \tag{6}$$

and has the following state-space realization

$$\dot{\mathbf{z}} = \left(\mathbf{A} + \mathbf{b}\mathbf{f}^{\mathsf{T}}\right)\mathbf{z} + \mathbf{b}\,\text{dzn}\left(u_{\text{lin}} - \mathbf{f}^{\mathsf{T}}\mathbf{z}\right)\,, \tag{7a}$$
$$y_d = \mathbf{c}^{\mathsf{T}}\mathbf{z}\,, \tag{7b}$$

where $\mathbf{z} \in \mathbb{R}^{n_p}$ is the state vector, $u_{\text{lin}} \in \mathbb{R}$ is the input and $y_d \in \mathbb{R}$ is the output.

A valuable property of the mismatch system is that the mismatch between y_{lin} and y, which the aw compensator aims to keep small, is characterized by the output y_d of the nonlinear part. Consequently, the parameter $\mathbf{f}$ of the compensator has to be chosen in such a way that an appropriate gain of system (7) from input u_{lin} to output y_d is small. Then the aw compensator is successful at keeping performance close to the desired linear performance. Choosing the L_2 gain[1] γ^* of (7) as a performance criterion leads to the following theorem proven in [8].

Theorem 1: There exists an aw compensator described by (5) which guarantees global stability of the anti-windup closed-loop system in Fig. 2 if a matrix $\mathbf{Q}_{11} \succ \mathbf{0}$, a vector $\mathbf{v}$ and positive scalars γ, μ can be found such that for $\alpha = 1$ the following LMI is satisfied

$$\begin{bmatrix} \mathbf{Q}_{11}\mathbf{A}^{\mathsf{T}} + \mathbf{v}\mathbf{b}^{\mathsf{T}} + \star & \mathbf{b}\mu - \mathbf{v}\alpha & \mathbf{0} & \mathbf{Q}_{11}\mathbf{c} \\ \star & -2\mu & \alpha & 0 \\ \star & \star & -\gamma & 0 \\ \star & \star & \star & -\gamma \end{bmatrix} \prec \mathbf{0}\,. \tag{8}$$

The parameter $\mathbf{f}$ of the compensator is given by $\mathbf{f} = \mathbf{Q}_{11}^{-1}\mathbf{v}$ and ensures that (7) has an L_2 gain $\gamma^* \leq \gamma$.

Theorem 1 allows to cast the design of an aw compensator into a convex optimization problem:

$$\begin{aligned} &\text{Minimize } \gamma\,, \text{ subject to} \\ &\mathbf{Q}_{11} \succ \mathbf{0}\,,\ \mu > 0\,,\ \alpha = 1\,, (8)\,. \end{aligned} \tag{9}$$

For easy reference, this method will be called "global approach". The drawback is that γ may be unacceptably large for some practical applications [8]. This is due to a rather conservative assumption: The deadzone is approximated as a sector nonlinearity in the sector $[0, 1]$ as shown in Fig. 4(a).

A solution to this problem is given in [8] by relaxing the sector constraints. Locally, the deadzone is contained in a narrower sector $[0, \alpha_u], \alpha_u < 1$ as shown in Fig 4(b). Choosing $\alpha = \alpha_u$ in Theorem 1 leads to a compensator guaranteeing local stability and a local performance level γ, i.e. a small-signal L_2 gain $\gamma_s^* \leq \gamma$ of (7), as long as $|u_r| < \hat{u}_r = u_0(1 - \alpha_u)^{-1}$. This "local approach" can significantly improve the performance. The drawbacks are that global stability is abandoned and the resulting region of attraction for the closed-loop system is connected to α_u in a rather complex way as discussed in [8].

To sum it up, the global and the local approach are not completely satisfying and an aw compensator combining global stability and a local performance level γ would be highly desirable. A first step in this direction is done in the next section by stating a less conservative stability criterion.

[1]System (7) with $\mathbf{z}(0) = \mathbf{0}$ is said to have an L_2 gain $\gamma^* \leq \gamma$, if $\sup(\|y_d\|_{L_2}/\|u_{\text{lin}}\|_{L_2}) \leq \gamma$ and $\|u_{\text{lin}}\|_{L_2} \neq 0$. The following statements will be used synonymously from now on: 1.) System (7) has an L_2 gain less or equal than γ. 2.) The aw compensator has a performance level of γ.

III. New Conditions for Anti-windup Compensator Design

A. Multiplier Based Stability Results

The stability of the nonlinear part of the mismatch system in Fig. 3 with $u_{\mathrm{lin}} = 0$ is solely determined by the stability of the following closed-loop system

$$u_d(s) = \big(M(s) - 1\big)\tilde{u}(s)\,, \tag{10a}$$

$$\tilde{u} = \mathrm{dzn}\,(u_r) = \mathrm{dzn}\,(-u_d)\ . \tag{10b}$$

Note that the linear part $M(s) - 1$ is Hurwitz by design and the deadzone $\tilde{u} = \mathrm{dzn}(u_r)$ is a static, time-invariant and monotone nonlinearity, i.e. it satisfies

$$\tilde{u}\,(\tilde{u} - u_r) \leq 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\,u_r}\mathrm{dzn}(u_r) \geq 0\ . \tag{11}$$

The latter condition is a slope restriction which is not taken into account in Theorem 1. Therefore, Theorem 1 guarantees stability for a broader class of nonlinearities than necessary at the expense of restricting $M(s)$. A less conservative stability lemma regarding $M(s)$ can be stated if the class of nonlinearities is constrained to satisfy (11). To show this, the notion of positive realness is recalled first.

Definition 1 (Positive realness, [12]): A transfer function $W(s)$ is positive real (p.r.) if all poles are in $\mathrm{Re}(s) < 0$ and $\mathrm{Re}\,\{W(j\omega)\} \geq 0\ \forall\omega \in \mathbb{R}$. It is called strictly positive real (s.p.r.) if $W(s-\epsilon)$ is positive real for some $\epsilon > 0$. Furthermore, two subsets of strictly positive real systems are introduced, as e.g. in [13].

Definition 2: The set $\mathcal{W}_{\mathrm{RL}}$ consists of all linear systems

$$W(s) = \prod_{j=1}^{\nu} \frac{s+\theta_j}{s+\eta_j} \sim \begin{cases} \dot{\mathbf{x}}_w = \mathbf{A}_w\mathbf{x}_w + \mathbf{b}_w u_w \\ y_w \;= \mathbf{c}_w^{\mathsf{T}}\mathbf{x}_w + 1\cdot u_w \end{cases} \tag{12}$$

with $\epsilon < \theta_1 < \eta_1 < \ldots < \theta_\nu < \eta_\nu$ for an arbitrary small $\epsilon > 0$. The set $\mathcal{W}_{\mathrm{RC}}$ consists of all linear systems (12) with $\epsilon < \eta_1 < \theta_1 \ldots < \eta_\nu < \theta_\nu$ for an arbitrary small $\epsilon > 0$.

Note, that if $W(s) \in \mathcal{W}_{\mathrm{RC}}$, then $-90^\circ < \arg W(j\omega) \leq 0\ \forall\omega \geq 0$. If $W(s) \in \mathcal{W}_{\mathrm{RL}}$, then $0 \leq \arg W(j\omega) < 90^\circ\ \forall\omega \geq 0$. Now, a well-known stability result for sector- and slope-restricted nonlinearities satisfying (11) can be restated.

Lemma 1: The nonlinear loop (10) has a globally asymptotically stable equilibrium point at the origin if there exists a transfer function $W(s) \in \mathcal{W} = \mathcal{W}_{\mathrm{RL}} \cup \mathcal{W}_{\mathrm{RC}}$ such that $M(s)W(s)$ is strictly positive real.

Proof: Follows directly from [13, Crit. 2a]. ■

Remark 1: $W(s)$ is called stability multiplier. As described in [14] the multiplier of Theorem 1 is static, i.e. $W(s) = W_c \in \mathbb{R}$ and stability is guaranteed if $M(j\omega)W_c$ is s.p.r., i.e. $|\arg M(j\omega)W_c| < 90^\circ$. This condition is satisfied if and only if $|\arg M(j\omega)| < 90^\circ$. Constraining the nonlinearities to satisfy (11) offers the possibility to use all multipliers contained in $\mathcal{W}$. For example, if $\arg M(j\omega)$ exceeds 90° in a certain frequency range a multiplier $W(s) \in \mathcal{W}_{\mathrm{RC}}$ can compensate this with an appropriate negative phase. Hence, more freedom is gained in the design of $M(s)$.

Remark 2: Note that [10], [11] consider more general multipliers. However, the advantage of (12) is that it has a simple structure and can intuitively be chosen by the designer.

To state an LMI-based stability condition for (10), the following lemma is required.

Lemma 2 (strictly positive real lemma, e.g. [15]): Assume that $\mathcal{G}(s) = \mathcal{C}^{\mathsf{T}}(s\mathbf{I} - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$, where $\mathcal{A}$ is Hurwitz. Then $\mathcal{G}(s)$ is strictly positive real if, and only if, there exists $\mathbf{P} \succ \mathbf{0}$ such that

$$\begin{bmatrix} \mathcal{A}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathcal{A} & \mathbf{P}\mathcal{B} - \mathcal{C} \\ \star & -2\mathcal{D} \end{bmatrix} \prec \mathbf{0}\,. \tag{13}$$

Considering (5) and (12), a minimal state-space realization of $M(s)W(s)$ is given by

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}^{\mathsf{T}} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} \mathbf{A} + \mathbf{b}\mathbf{f}^{\mathsf{T}} & \mathbf{0} & \mathbf{b} \\ \mathbf{b}_w\mathbf{f}^{\mathsf{T}} & \mathbf{A}_w & \mathbf{b}_w \\ \hline \mathbf{f}^{\mathsf{T}} & \mathbf{c}_w^{\mathsf{T}} & 1 \end{array}\right] \tag{14}$$

where $\mathcal{A}$ is Hurwitz. This leads to the desired result.

Lemma 3 (stability lemma): The nonlinear loop (10) has a globally asymptotically stable equilibrium point at the origin if there exists a symmetric matrix $\mathbf{Q} \succ \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{Q}\mathcal{A}^{\mathsf{T}} + \mathcal{A}\mathbf{Q} & \mathcal{B} - \mathbf{Q}\mathcal{C} \\ \star & -2\mathcal{D} \end{bmatrix} \prec \mathbf{0} \tag{15}$$

is satisfied with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ according to (14).

Proof: Inequality (15) equals (13) after applying a congruence transformation $\mathrm{diag}(\mathbf{Q}^{-1}, \mathbf{I})$ and substituting $\mathbf{Q}^{-1} = \mathbf{P}$. It ensures strict positive realness of $M(s)W(s)$ and thus global stability according to Lemma 1. ■

B. Global Stability and Local Performance

A combination of the stability results from the last section and the local version of Theorem 1 leads to globally stabilizing aw compensators with a performance based on local L_2 gains. The following theorem can be stated.

Theorem 2: If there exist matrices

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^{\mathsf{T}} & \mathbf{Q}_{22} \end{bmatrix} \succ \mathbf{0}\,, \tag{16}$$

a vector $\mathbf{f}$ as well as positive scalars μ, γ and a multiplier (12), such that for a given $\alpha \in (0, 1]$ the matrix inequalities

$$\begin{bmatrix} \boldsymbol{Y} & \mathbf{b}\mu - \mathbf{Q}_{11}\mathbf{f}\alpha & \mathbf{0} & \mathbf{Q}_{11}\mathbf{c} \\ \star & -2\mu & \alpha & \mathbf{0} \\ \star & \star & -\gamma & 0 \\ \star & \star & \star & -\gamma \end{bmatrix} \prec \mathbf{0}\,, \tag{17}$$

$$\mathbf{X} + \mathbf{L}\mathbf{F}\boldsymbol{\Lambda}^{\mathsf{T}} + \boldsymbol{\Lambda}\mathbf{F}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} \prec \mathbf{0} \tag{18}$$

with

$$\boldsymbol{Y} = \mathbf{Q}_{11}\mathbf{A}^{\mathsf{T}} + \mathbf{A}\mathbf{Q}_{11} + \mathbf{b}\mathbf{f}^{\mathsf{T}}\mathbf{Q}_{11} + \mathbf{Q}_{11}\mathbf{f}\mathbf{b}^{\mathsf{T}}\,,$$

$$\mathbf{L} = \mathrm{diag}\left(\mathbf{Q}_{11}, \mathbf{Q}_{12}^{\mathsf{T}}, 0\right)\,, \mathbf{F}^{\mathsf{T}} = \begin{bmatrix} \mathbf{f}^{\mathsf{T}} & \mathbf{f}^{\mathsf{T}} & 0 \end{bmatrix}\,,$$

$$\boldsymbol{\Lambda}^{\mathsf{T}} = \begin{bmatrix} \mathbf{b}^{\mathsf{T}} & \mathbf{b}_w^{\mathsf{T}} & -1 \end{bmatrix}\,,$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{Q}\tilde{\mathbf{A}}^{\mathsf{T}} + \tilde{\mathbf{A}}\mathbf{Q} & \mathcal{B} - \mathbf{Q}\tilde{\mathbf{C}} \\ \star & -2 \end{bmatrix}\,,$$

$$\tilde{\mathbf{A}} = \mathrm{diag}\,(\mathbf{A}, \mathbf{A}_w)\,, \quad \tilde{\mathbf{C}}^{\mathsf{T}} = \begin{bmatrix} \mathbf{0} & \mathbf{c}_w^{\mathsf{T}} \end{bmatrix}$$

are satisfied, then there exists a globally stabilizing aw compensator (5). If $\alpha = 1$ system (7) has an L_2 gain $\gamma^* \leq \gamma$. If $\alpha < 1$ it has a small-signal L_2 gain $\gamma_s^* \leq \gamma$.

Proof: Matrix inequality (17) equals (8) in Theorem 1 if $\mathbf{v} = \mathbf{Q}_{11}\mathbf{f}$ is substituted. It ensures the performance level γ. Matrix inequality (18) equals (15) after expanding the products of $\mathbf{L}, \mathbf{F}, \mathbf{\Lambda}$. It ensures global stability due to Lemma 3. ■

The matrix inequalities in Theorem 2 are not linear in the decision variables[2] and thus the design of a globally stabilizing compensator with local performance level can not be cast directly into a convex optimization problem. Therefore, a two-step approach is proposed. First we find an initial aw compensator by tightening the constraints of Theorem 2. This aw compensator is then optimized by an iterative optimization algorithm based on path following [16].

To compute an initial solution, the following corollary of Theorem 2 is required.

Corollary 1: If there exist matrices

$$\mathbf{Q} = \operatorname{diag}(\mathbf{Q}_{11}, \mathbf{Q}_{22}) \succ \mathbf{0}\,, \tag{19}$$

a vector $\mathbf{v}$ as well as positive scalars μ, γ and a multiplier (12), such that for a given $\alpha \in (0, 1]$ the linear matrix inequalities (8) and

$$\mathbf{X} + \mathbf{V}\mathbf{\Lambda}^{\mathsf{T}} + \mathbf{\Lambda}\mathbf{V}^{\mathsf{T}} \prec \mathbf{0} \tag{20}$$

are satisfied with $\mathbf{\Lambda}, \mathbf{X}$ as in Theorem 2 and

$$\mathbf{V}^{\mathsf{T}} = \begin{bmatrix}\mathbf{v}^{\mathsf{T}} & \mathbf{0} & 0\end{bmatrix}\,, \tag{21}$$

then there exists a globally stabilizing aw compensator (5) with $\mathbf{f} = \mathbf{Q}_{11}^{-1}\mathbf{v}$. If $\alpha = 1$ system (7) has an L_2 gain $\gamma^* \leq \gamma$. If $\alpha < 1$ it has a small-signal L_2 gain $\gamma_s^* \leq \gamma$.

Proof: Follows directly from Theorem 2 by choosing $\mathbf{Q}_{12} \equiv \mathbf{0}$ and substituting $\mathbf{v} = \mathbf{Q}_{11}\mathbf{f}$ in (17) and (18). Then (17) is equal to (8) and (18) is equal to (20). ■

IV. Anti-Windup Compensator Design

A. Computing an Initial Solution

The computation of an initial solution via Corollary 1 requires a multiplier $W(s)$ enabling less conservative designs (see Remark 1). Inspired by the phase design aid, which states that the dynamic behaviour of the closed-loop (10) can be improved if the phase of $M(j\omega)$ is allowed to stay within the range of $\pm 125°$ (see discussion in [5]), we choose the multiplier as follows.

Step 1: Compute a globally stabilizing aw compensator (5) with performance level γ by solving optimization problem (9). The phase of the resulting $M(j\omega)$ will be restricted to $\pm 90°$, as shown for example in Fig. 5.

Step 2: Locate the global extremum Φ_e of $\arg M(j\omega)$ at frequency ω_e. If $\Phi_e > 0°$, select $W(s) \in \mathcal{W}_{\mathrm{RC}}$. If $\Phi_e < 0°$, select $W(s) \in \mathcal{W}_{\mathrm{RL}}$.

Step 3: Parameterize the multiplier in such a way that in a crucial frequency intervall $[\omega_l, \omega_h]$ the phase $\arg W(j\omega)$ is

[2] The inequalities (17) and (18) contain products of $\mathbf{Q}_{11}$ and $\mathbf{f}$. Additionally, (18) contains products of $\mathbf{Q}_{12}$ and $\mathbf{f}$.

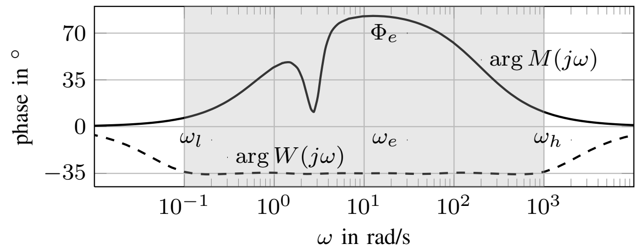

Fig. 5. Phase plots of $M(j\omega)$ (— solid) and the chosen multiplier $W(j\omega) \in \mathcal{W}_{\mathrm{RC}}$ (- - - dashed).

approximately $\Phi_d = -35°$ if $W(s) \in \mathcal{W}_{\mathrm{RC}}$ or $\Phi_d = 35°$ if $W(s) \in \mathcal{W}_{\mathrm{RL}}$. A precondition to keep $\arg W(j\omega)$ close to the target value Φ_d is a sufficiently high order ν of $W(s)$.

A reasonable choice for the intervall would be $[10^{-\kappa}\omega_e, 10^{\kappa}\omega_e]$ with $\kappa > 0$ and a desired parameterization can be found by choosing $\nu = 3 + \lfloor\kappa\rfloor$ and minimizing the squared error $(\arg W(j\omega) - \Phi_d)^2$ in this intervall.

Remark 3: The above mentioned method yielded good results in all analyzed examples. However, it is based on the empiric phase design aid [5] and therefore not optimal in any sense. The final choice of $W(s)$ is left to the designer.

Once a multiplier is chosen, the initial solution can be computed by solving the following convex optimization problem for an $\alpha \in (0, 1]$.

$$\begin{aligned}&\text{Minimize } \gamma\,, \text{ subject to}\\&\mathbf{Q} = \operatorname{diag}(\mathbf{Q}_{11}, \mathbf{Q}_{22}) \succ \mathbf{0}\,,\ \mu > 0\,,\ (8), (20)\,.\end{aligned} \tag{22}$$

If all the steps are succesfully completed a parameter set $\mathbf{Q}$, $\mathbf{f}$, μ, γ and a multiplier $W(s)$ satisfying (17) and (18) for a chosen α is found.

Remark 4: The parameter α does not affect the region of stability as in the local approach [8]. Thus for now, the choice of α is also left to the designer. Another possibility would be to incorporate α in the optimization process as discussed in Section IV-C.

B. Optimization via Path Following

The initial solution can be improved by an iterative optimization algorithm based on path following [16]. The main idea of this technique is to linearize nonlinear matrix inequalities around a solution[3] and solve the arising optimization problem. This leads to a four-step optimization algorithm.

Step 1: Linearize the matrix inequalities (17), (18) around the parameter set $\mathbf{Q}$, $\mathbf{f}$, μ, γ by means of perturbations $\mathbf{Q}_\Delta$, $\mathbf{f}_\Delta$, μ_Δ, γ_Δ. This leads to the linearized matrix inequalities shown on top of the next page. The perturbations are kept small by the constraint $\|\mathbf{Q}_\Delta\| < \beta\|\mathbf{Q}\|$ which can be expressed as

$$\begin{bmatrix}\beta\mathbf{Q} & \mathbf{Q}_\Delta\\ \star & \beta\mathbf{Q}\end{bmatrix} \succ \mathbf{0}\,. \tag{23}$$

[3] The linearization of matrix inequalities is demonstrated by the example $\mathbf{QP} + \star \succ \mathbf{0}$ where $\mathbf{Q}$, $\mathbf{P}$ are variables. Around a valid solution $\mathbf{Q}_0$, $\mathbf{P}_0$ we have $\mathbf{P} \approx \mathbf{P}_0 + \mathbf{P}_\Delta$ and $\mathbf{Q} \approx \mathbf{Q}_0 + \mathbf{Q}_\Delta$. A first order Taylor series expansion of the matrix inequality is $\mathbf{Q}_0\mathbf{P}_0 + \mathbf{Q}_\Delta\mathbf{P}_0 + \mathbf{Q}_0\mathbf{P}_\Delta + \star \succ \mathbf{0}$ where the perturbations $\mathbf{Q}_\Delta$ and $\mathbf{P}_\Delta$ are the new variables.

$$\begin{bmatrix} \mathbf{Y}_{11} & \mathbf{b}(\mu-\mu_\Delta)-\mathbf{Q}_{11\Delta}\mathbf{f}\alpha-\mathbf{Q}_{11}(\mathbf{f}\alpha+\mathbf{f}_\Delta\alpha) & \mathbf{0} & \mathbf{Q}_{11}\mathbf{c}-\mathbf{Q}_{11\Delta}\mathbf{c} \\ \star & -2(\mu+\mu_\Delta) & \alpha & \mathbf{0} \\ \star & \star & -\gamma-\gamma_\Delta & 0 \\ \star & \star & \star & -\gamma-\gamma_\Delta \end{bmatrix} \prec \mathbf{0} \tag{24}$$

with $\mathbf{Y}_{11} = \mathbf{Q}_{11}\left(\mathbf{A}+\mathbf{b}(\mathbf{f}+\mathbf{f}_\Delta)^\mathsf{T}\right)^\mathsf{T} + (\mathbf{A}+\mathbf{b}(\mathbf{f}+\mathbf{f}_\Delta)^\mathsf{T})\mathbf{Q}_{11} + \mathbf{Q}_{11\Delta}(\mathbf{A}+\mathbf{b}\mathbf{f}^\mathsf{T})^\mathsf{T} + (\mathbf{A}+\mathbf{b}\mathbf{f}^\mathsf{T})\mathbf{Q}_{11\Delta}$

$$\mathbf{X}+\mathbf{X}_\Delta+\mathbf{\Lambda}(\mathbf{F}^\mathsf{T}\mathbf{L}^\mathsf{T}+\mathbf{F}^\mathsf{T}\mathbf{L}_\Delta^\mathsf{T}+\mathbf{F}_\Delta^\mathsf{T}\mathbf{L}^\mathsf{T})+(\mathbf{L}\mathbf{F}+\mathbf{L}_\Delta\mathbf{F}+\mathbf{L}\mathbf{F}_\Delta)\mathbf{\Lambda}^\mathsf{T} \prec \mathbf{0} \tag{25}$$

with $\mathbf{L}_\Delta = \operatorname{diag}\left(\mathbf{Q}_{11\Delta}, \mathbf{Q}_{12\Delta}^\mathsf{T}, 0\right)$, $\mathbf{F}_\Delta^\mathsf{T} = \begin{bmatrix}\mathbf{f}_\Delta^\mathsf{T} & \mathbf{f}_\Delta^\mathsf{T} & 0\end{bmatrix}$,

$\mathbf{X}_\Delta = \begin{bmatrix} \mathbf{Q}_\Delta\tilde{\mathbf{A}}^\mathsf{T}+\tilde{\mathbf{A}}\mathbf{Q}_\Delta & -\mathbf{Q}_\Delta\tilde{\mathbf{C}} \\ \star & 0 \end{bmatrix}$ and $\mathbf{L}, \mathbf{F}, \mathbf{\Lambda}, \mathbf{X}, \tilde{\mathbf{A}}, \tilde{\mathbf{C}}$ as in Theorem 2

The parameter β plays the role of a step size and is adapted during the optimization. Now, the conditions (17), (18) of Theorem 2 are locally approximated by (24), (25) and can be cast into the following convex optimization problem[4].

$$\begin{aligned} &\text{Minimize } \gamma_\Delta\,, \text{ subject to} \\ &\mathbf{Q}+\mathbf{Q}_\Delta \succ \mathbf{0}\,,\ \mathbf{Q}_{11}+\mathbf{Q}_{11\Delta} \succ \mathbf{0}\,,\ \mu+\mu_\Delta > 0\,, \\ &\gamma_\Delta \le 0\,,\ \gamma+\gamma_\Delta > 0 \text{ and (23), (24), (25).} \end{aligned} \tag{26}$$

Step 2: Update the decision variables. The new values are $\mathbf{f} := \mathbf{f}+\mathbf{f}_\Delta$, $\mu := \mu+\mu_\Delta$, $\gamma := \gamma+\gamma_\Delta$.

Step 3: Validation. With the updated values, solve the following feasibility problem in the variable $\mathbf{Q}$.

$$\begin{aligned} &\text{Find } \mathbf{Q}, \text{ subject to} \\ &\mathbf{Q} \succ \mathbf{0}\,,\ \mathbf{Q}_{11} \succ \mathbf{0} \quad \text{and (17), (18).} \end{aligned} \tag{27}$$

If this problem is feasible, a new, improved parameter set $\mathbf{Q}$, $\mathbf{f}$, μ, γ has been found. Let $\beta := 1.02 \cdot \beta$ and go to step 4. If this problem is not feasible, let $\beta := \beta/2$, restore the old values of $\mathbf{Q}$, $\mathbf{f}$, μ, γ prior to step 2 and go to step 4.

Step 4: Termination conditions. Stop the algorithm after a preset number of iterations or if the decrease of γ is smaller than the desired accuracy γ_Δ. Otherwise go to step 1.

C. Possible Extensions and Refinements of the Method

The parameter α and the multiplier $W(s)$ can be incorporated in the optimization process. This enables the design algorithm to modify α and the multiplier autonomously in order to improve the solution. To this end, the matrix inequalities (17), (18) have to be linearized around the extended parameter set $\mathbf{Q}$, $\mathbf{f}$, μ, γ, α, $\mathbf{A}_w$, $\mathbf{b}_w$, $\mathbf{c}_w$. Furthermore, the constraints $\alpha+\alpha_\Delta-\alpha_l > 0$ and $\alpha+\alpha_\Delta-1 < 0$ have to be added to the optimization problem (26) to ensure that a valid $\alpha \in (\alpha_l, 1)$ with $\alpha_l > 0$ is chosen.

Incorporating the multiplier into the optimization problem requires a state-space description of (12) given by

$$\dot{\mathbf{x}}_w = \operatorname{diag}(-\boldsymbol{\eta})\mathbf{x}_w + \boldsymbol{\eta}u_w\,, \tag{28a}$$

$$y_w = [k_1 \dots k_\nu]\mathbf{x}_w + u_w\,, \tag{28b}$$

where $\boldsymbol{\eta} = [\eta_1\ \dots\ \eta_\nu]^\mathsf{T} \in \mathbb{R}^\nu$. The additional constraints

$$0 < \epsilon < \eta_1 < \ldots < \eta_\nu\,, \tag{29}$$

$$k_i > 0\ \forall\ i = 1, \ldots, \nu\,, \tag{30}$$

have to be added to (26) if $W(s) \in \mathcal{W}_{\mathrm{RC}}$. If $W(s) \in \mathcal{W}_{\mathrm{RL}}$ the additional constraints are (29) and

$$k_i < 0 \quad \forall\ i = 1, \ldots, \nu \quad \text{and} \quad \sum_{i=1}^{\nu} k_i \ge -1\,. \tag{31}$$

This extension may lead to better results if an initial solution is hard to find. Then α and the multiplier can be adjusted such that (22) is feasible and are optimized by the algorithm.

V. EXAMPLE

The hydraulic actuator depicted in Fig. 6 is considered. A state-space model of the stable plant is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -10 & -1.167 & 25 \\ 0 & 0 & -0.8 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 2.4 \end{bmatrix}u\,, \tag{32a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}\,, \tag{32b}$$

where the states are the position of the load mass $x_1 = \sigma$ in cm, the velocity of the mass $x_2 = \dot{\sigma}$ in $\mathrm{cm/s}$ and the pressure $x_3 = p_1$ in $\mathrm{N/cm}^2$ [17]. The plant input is limited by the saturation function (3) with $u_0 = 10{,}5$ V.

A PID controller with additional phase lead controls the position of the load mass. Its transfer matrix is given by $\mathbf{K}(s) = [K_c(s)\ \ -K_c(s)]$ where

$$K_c(s) = \frac{17.764(s^2+1.167s+10)}{s(s+100)}\frac{(s+0.731)}{(s+5)}\,. \tag{33}$$

This linear controller guarantees a fast step response of the linear-closed loop with a negligible overshoot of 1% but causes enormous windup effects in the saturated closed-loop for large reference values. To recover the linear performance, an aw compensator is designed according to Section IV. The multiplier $W(s) \in \mathcal{W}_{\mathrm{RC}}$ is of order 7 and given by

$$W(s) = \frac{(s+0.16)(s+0.594)(s+2.977)}{(s+0.063)(s+0.363)(s+1.559)} \cdot \frac{(s+13.22)(s+64.14)(s+275.5)(s+1595)}{(s+7.564)(s+33.59)(s+168.5)(s+627.1)}\,, \tag{34}$$

[4]This optimization problem is always feasible, since $\mathbf{Q}_\Delta = \mathbf{0}$, $\mathbf{f}_\Delta = \mathbf{0}$, $\mu_\Delta = 0$ and $\gamma_\Delta = 0$ is a valid solution.

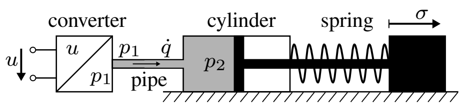

Fig. 6. Hydraulic actuator: The converter transforms the input voltage u in a pressure p_1. This pressure induces a fluid current $\dot{q}$ in the cylinder causing a movement of the piston which is connected to a load mass. The position of the load mass σ needs to be controlled.

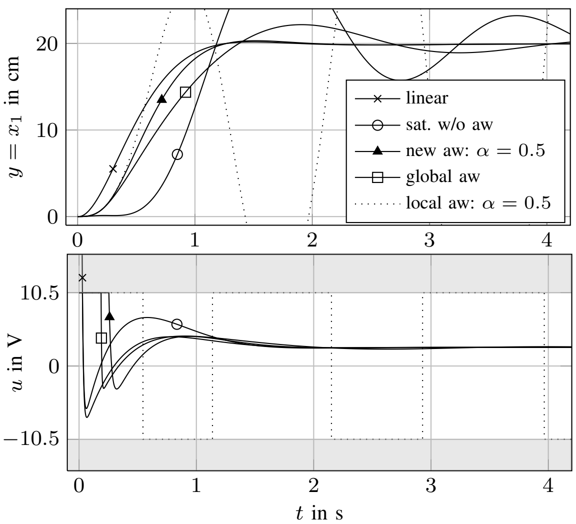

Fig. 7. Step responses and related input signals for the linear system (linear), the saturated closed-loop without aw (sat w/o aw) and the aw closed-loop with a compensator designed by the new method (new aw), the global approach (global aw) and the local approach (local aw).

with a constant phase of $\Phi_d = -35°$ in the frequency range $[0.1, 1000]$. Fig 5 shows the phase plots of the multiplier and of the solution $\arg M(j\omega)$ of optimization problem (9).

Solving the convex optimization problem[5] (22) with $\alpha = 0.5$ leads to an initial solution which is further optimized by the iterative algorithm from Section IV-B with $\beta = 0.01$ and $\gamma_\Delta = 10^{-4}$. The resulting aw compensator is given by (5) where $\mathbf{f} = \mathbf{f}_{\text{new}} = [-2.98 \;\; -3.35 \;\; -21.06]^\mathsf{T}$. The global approach results in $\mathbf{f}_{\text{global}} = [13.01 \;\; -0.158 \;\; -81.19]^\mathsf{T}$ and the local approach with $\alpha = 0.5$ yields the vector $\mathbf{f}_{\text{local}} = [-2 \cdot 10^4 - 516.4 - 158.4]^\mathsf{T}$.

Fig. 7 shows simulation results for a reference value of 20cm. The new method almost recovers the performance of the linear system. Choosing $\alpha = 0.25$ or $\alpha = 0.75$ does not change this behaviour significantly. The global approach guarantees a stable closed-loop but shows an unsatisfying performance. The local approach exhibits limit cycles for reference values larger than 2.25cm if $\alpha = 0.5$. Increasing α to 0.9995 leads to a performance comparable to the one of the new method for this particular reference value but without a global stability guarantee. Furthermore, this aw compensator has a very fast pole at $s = -9.3 \cdot 10^6$ which may cause problems when implementing the control strategy.

[5]All convex optimization problems are solved with [18], [19].

VI. Conclusions

This paper proposed a new design method for full-order mismatch-based aw compensators based on less conservative stability conditions and an iterative path following algorithm. In contrast to available approaches it allows to synthesize globally stabilizing compensators with enhanced local performance based on small-signal L_2 gains. First, an initial solution is obtained by solving a convex optimization problem. Then, the initial solution is improved using a path following technique which sequentially optimizes a local linearization of the problem. By designing an aw compensator for a hydraulic actuator, the considerable improvement compared to standard approaches was shown.

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