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Stability and stabilizability of special classes of discrete-time positive switched systems

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Abstract— **In this paper we consider discrete-time positive switched systems, switching among autonomous subsystems, characterized either by cyclic monomial matrices or by circulant matrices. For these two classes of systems, some interesting necessary and sufficient conditions for stability and stabilizability are provided. Such conditions lead to simple algorithms that allow to easily detect whether a given positive switched system is not stabilizable.**

I. Introduction

A discrete-time positive switched system (DPSS) consists of a family of positive state-space models and a switching law, specifying when and how the switching among the various subsystems takes place. This class of systems has interesting practical applications. They have been adopted for describing networks employing TCP and other congestion control applications [15], for modeling consensus and synchronization problems [8], and, quite recently, to describe the viral mutation dynamics under drug treatment [7].

Despite stability and stabilizability of positive switched systems have been investigated in depth for the continuous-time case by resorting to linear copositive and to quadratic Lyapunov functions [6], [9], [11], [12], [16], no computationally effective necessary and sufficient condition is presently available. For discrete-time systems the situation is even more unsatisfactory, as the results based on linear copositive functions find a straightforward extension from the continuous-time case (for instance, most the characterizations obtained in [3] and [9] have a discrete-time equivalent, and we will make us of this fact within the paper), but this is not true when dealing with quadratic stability and stabilizability. In any case, the existence of special classes of Lyapunov functions [5], [10] for checking stability and stabilizability of DPSS, at the present status, provides only sufficient conditions, and no criterion is available which allows to decide about these properties by simply inspecting the state transition matrices of the various subsystems.

Interestingly enough, this is the case for DPSS whose state transition matrices are either cyclic monomial (namely they correspond to a graph that consists of a single cycle, passing through all vertices) or (left/right) positive circulant. Stability and stabilizability characterizations obtained for these classes of DPSS are quite neat and they represent a first step toward the most ambitious goal of providing computationally tractable criteria that exploit the positivity and the structure of the system matrices. In addition, the results obtained for these classes provide, as a byproduct, some interesting conditions, that allow to easily detect when a generic DPSS is not stabilizable.

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Before proceeding, we introduce some notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). In the sequel, the (ℓ, j)th entry of a matrix A is denoted by $[A]_{\ell j}$, while the ℓth entry of a vector $\mathbf{v}$ is $[\mathbf{v}]_\ell$. A square symmetric matrix P is *positive definite* ($\succ 0$) if for every nonzero vector $\mathbf{x}$, of compatible dimension, $\mathbf{x}^\top P\mathbf{x} > 0$, and negative definite ($\prec 0$) if $-P$ is positive definite.

A square positive matrix endowed with the following structure

$$A = \begin{bmatrix} 0 & a_{12} & 0 & \dots & 0 \\ 0 & 0 & a_{23} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n} \\ a_{n1} & 0 & 0 & \dots & 0 \end{bmatrix}$$

is an $n \times n$ *cyclic monomial matrix*.

A real square matrix A is *Metzler* if its off-diagonal entries are nonnegative, *Schur* if all its eigenvalues lie within the unit circle (equivalently, its spectral radius, $\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$, is smaller than one), and *Hurwitz* if they all lie in the open left halfplane.

Given a matrix $A \in \mathbb{R}_+^{n\times n}$, we associate with it [1] a *digraph* $\mathcal{D}(A)$, with vertices $1, \dots, n$. There is an arc (j, ℓ) from j to ℓ if and only if $[A]_{\ell j} > 0$. If so, $[A]_{\ell j}$ represents the *weight* of the arc. A sequence $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_k \rightarrow j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), \dots, (j_k, j_{k+1})$ are arcs of $\mathcal{D}(A)$. A closed path is called a *cycle*. In particular, a cycle with no repeated vertices is called *elementary,* and its length coincides with the number of (distinct) vertices appearing in it.

Finally, we need some definitions borrowed from the algebra of non-commutative polynomials [14]. Given the *alphabet* $\Xi = \{\xi_1, \xi_2, \dots, \xi_p\}$, the free monoid Ξ^* with base Ξ is the set of all *words* $w = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}, k \in \mathbb{N}, \xi_{i_h} \in \Xi$. The integer k is called the *length* of w and is denoted by $|w|$, while $|w|_i$ represents the number of occurencies of ξ_i in w. If $\tilde{w} = \xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}$ is another element of Ξ^*, the product is defined by concatenation $w\tilde{w} = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}\xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}$. This produces a monoid with $\varepsilon = \emptyset$, the empty word, as unit element. Clearly, $|w\tilde{w}| = |w| + |\tilde{w}|$ and $|\varepsilon| = 0$.

$\mathbb{R}\langle\xi_1,\xi_2,\dots,\xi_p\rangle$ is the algebra of polynomials in the non-commuting indeterminates $\xi_1,\xi_2,\dots,\xi_p$. For every family of p matrices in $\mathbb{R}^{n\times n}$, $\mathcal{A}:=\{A_1,A_2,\dots,A_p\}$, the map ψ defined on $\{\varepsilon,\xi_1,\xi_2,\dots,\xi_p\}$ by the assignments $\psi(\varepsilon)=I_n$ and $\psi(\xi_i)=A_i$, $i=1,2,\dots,p$, uniquely extends to an algebra morphism of $\mathbb{R}\langle\xi_1,\xi_2,\dots,\xi_p\rangle$ into $\mathbb{R}^{n\times n}$ (as an example, $\psi(\xi_1\xi_2)=A_1A_2\in\mathbb{R}^{n\times n}$). If w is a word in Ξ^* (i.e. a monic monomial in $\mathbb{R}\langle\xi_1,\xi_2,\dots,\xi_p\rangle$), the ψ-image of w is denoted by $w(A_1,A_2,\dots,A_p)$.

II. Stability of systems with cyclic matrices

A *discrete-time positive switched system* is described by the following equation

$$\mathbf{x}(t+1)=A_{\sigma(t)}\mathbf{x}(t),\qquad t\in\mathbb{Z}_+,\tag{1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, σ is an arbitrary switching sequence, taking values in the set $[1,p]:=\{1,2,\dots,p\}$, and for each $i\in[1,p]$ the matrix A_i is the system matrix of a discrete-time positive system, which means that A_i is an $n\times n$ positive matrix. For this class of systems we introduce the concept of asymptotic stability, later on referred to simply as stability.

Definition 1: The discrete-time positive switched system (1) is *(asymptotically) stable* if, for every positive initial state $\mathbf{x}(0)$ and every switching sequence $\sigma:\mathbb{Z}_+\to[1,p]$, the state trajectory $\mathbf{x}(t), t\in\mathbb{Z}_+$, asymptotically converges to zero.

In this section we focus on the class of discrete-time positive switched systems whose matrices $A_i, i\in[1,p]$, are cyclic monomial matrices:

$$A_i=\begin{bmatrix}0 & a_{12}^{(i)} & 0 & \dots & 0\\ 0 & 0 & a_{23}^{(i)} & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_{n-1,n}^{(i)}\\ a_{n1}^{(i)} & 0 & 0 & \dots & 0\end{bmatrix},\tag{2}$$

$$a_{n1}^{(i)}>0,\qquad a_{j,j+1}^{(i)}>0,\forall\ j\in[1,n-1].$$

It is easily seen that

$$\Delta_{A_i}(z)=z^n-a_{12}^{(i)}a_{23}^{(i)}\cdots a_{n-1,n}^{(i)}a_{n1}^{(i)},$$

and hence A_i is a Schur matrix if and only if

$$|a_{12}^{(i)}a_{23}^{(i)}\cdots a_{n-1,n}^{(i)}a_{n1}^{(i)}|<1.\tag{3}$$

We have the following complete characterization of the stability of system (1), under the assumption that all matrices $A_i, i\in[1,p]$, are cyclic monomial.

Proposition 1: Given a discrete-time positive switched system (1), with cyclic monomial matrices $A_i\in\mathbb{R}_+^{n\times n}, i\in[1,p]$, the following facts are equivalent ones:

i) the system is stable;
ii) for every $w\in\Xi^*$, the matrix product $w(A_1,A_2,\dots,A_p)$ is a positive Schur matrix;
iii) for every $w\in\Xi^*$, with $|w|=n$, the matrix product $w(A_1,A_2,\dots,A_p)$ is a positive Schur matrix;
iv) for every choice of the indices $i_1,i_2,\dots,i_n\in[1,p]$
$$|a_{12}^{(i_1)}a_{23}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-1})}a_{n1}^{(i_n)}|<1;\tag{4}$$
v) the matrices A_i's admit a common linear copositive function, namely there exists $\mathbf{v}\gg 0$ such that $\mathbf{v}^\top A_i\ll\mathbf{v}^\top$, for every $i\in[1,p]$;
vi) the matrices A_i's admit a common diagonal Lyapunov function, namely there exists $D=\mathrm{diag}\{d_1,d_2,\dots,d_n\}$, $d_j>0,\forall\ j\in[1,n]$, such that $A_i^\top DA_i-D\prec 0$, for every $i\in[1,p]$;
vii) the matrices A_i's admit a common quadratic copositive function of rank 1, namely there exists $P=P^\top$ of rank 1 such that for every $\mathbf{x}>0$ one finds $\mathbf{x}^TP\mathbf{x}>0$ and $\mathbf{x}^T[A_i^\top PA_i-P]\mathbf{x}<0$, for every $i\in[1,p]$;
viii) the matrix A^*, whose (ℓ,j)th entry is
$$[A^*]_{\ell,j}:=\max_{i\in[1,p]}[A_i]_{\ell,j},$$
is Schur;
ix) for every $\mathcal{S}\subset\Xi^*$, with $|\mathcal{S}|<+\infty$, and every choice of the coefficients $\alpha_w>0$, with $\sum_{w\in\mathcal{S}}\alpha_w=1$, the matrix $\sum_{w\in\mathcal{S}}\alpha_w w(A_1,A_2,\dots,A_p)$ is Schur.

Proof: i) $\Rightarrow$ ii) If the system is stable, then all periodic switching sequences must ensure convergence. This amounts to saying that for every $\mathbf{x}(0)>0$ and every $w\in\Xi^*$, with $m:=|w|$, the state vector $\mathbf{x}(km)=w(A_1,A_2,\dots,A_p)^k\mathbf{x}(0)$ converges to zero as k goes to $+\infty$. But this implies that $w(A_1,A_2,\dots,A_p)$ is Schur.
ii) $\Rightarrow$ iii) Obvious.
iii) $\Rightarrow$ iv) For every $w\in\Xi^*$, with $|w|=n$, the matrix product $w(A_1,A_2,\dots,A_p)$ is a diagonal matrix whose diagonal elements take the form $a_{12}^{(i_1)}a_{23}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-1})}a_{n1}^{(i_n)}$ for suitable indices $i_1,i_2,\dots,i_n\in[1,p]$, (which appear in permuted form when moving along the diagonal). So, the Schur property of every such diagonal matrix requires that all diagonal entries have moduli smaller than 1. As w freely varies within the words in Ξ^* of length n, the indices $i_1,i_2,\dots,i_n$ freely vary within $[1,p]$, thus implying condition iv).
iv) $\Leftrightarrow$ v) Notice, first, that A is a positive Schur matrix if and only if $A-I_n$ is a Metzler Hurwitz matrix. So, condition v) can be restated by saying that there exists $\mathbf{v}\gg 0$ such that $\mathbf{v}^\top(A_i-I_n)\ll 0,\forall\ i\in[1,p]$., i.e. A_i-I_n are Metzler Hurwitz having a common linear copositive function. This condition, in turn, is equivalent (see [3], [4], [9]) to the fact that for every map $\pi:[1,n]\to[1,p]$, the square matrix

$$A_\pi-I_n:=[\,\mathrm{col}_1(A_{\pi(1)})\ \mathrm{col}_2(A_{\pi(2)})\ \ \dots\ \mathrm{col}_n(A_{\pi(n)})\,]-I_n$$

is a (Metzler) Hurwitz matrix or, equivalently, every A_π is a positive Schur matrix. Since any such matrix A_π takes the following form:

$$A_\pi=\begin{bmatrix}0 & a_{12}^{(\pi(2))} & 0 & \dots & 0\\ 0 & 0 & a_{23}^{(\pi(3))} & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a_{n-1,n}^{(\pi(n))}\\ a_{n1}^{(\pi(1))} & 0 & 0 & \dots & 0\end{bmatrix},\tag{5}$$

A_π is Schur for every map π if and only if (4) holds true.
iv) $\Rightarrow$ vi) Corresponding to the positive diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \dots, d_n\}, d_j > 0, j \in [1, n]$, we get

$$A_i^\top D A_i - D = \begin{bmatrix} \left(a_{n1}^{(i)}\right)^2 d_n - d_1 & & & \\ & \left(a_{12}^{(i)}\right)^2 d_1 - d_2 & & \\ & & \ddots & \\ & & & \left(a_{n-1,n}^{(i)}\right)^2 d_{n-1} - d_n \end{bmatrix},$$

$\forall\, i \in [1, p]$. We want to show that if (4) holds true for every choice of $i_1, i_2, \dots, i_n \in [1, p]$, then there exist positive numbers $d_j, j \in [1, n]$, such that for every $i \in [1, p]$

$$\begin{aligned} &\left(a_{jj+1}^{(i)}\right)^2 d_j - d_{j+1} < 0, \quad j \in [1, n-1], \\ &\left(a_{n1}^{(i)}\right)^2 d_n - d_1 < 0. \end{aligned} \tag{6}$$

Set

$$\begin{aligned} a_j^2 &:= \max\nolimits_{i \in [1,p]} \left(a_{jj+1}^{(i)}\right)^2, \ j \in [1, n-1], \\ a_n^2 &:= \max\nolimits_{i \in [1,p]} \left(a_{n1}^{(i)}\right)^2. \end{aligned}$$

It is clear that for every choice of $\varepsilon > 0$, by setting $\frac{d_{j+1}}{d_j} = a_j^2 + \varepsilon$, $j \in [1, n-1]$, we satisfy the first $n-1$ inequalities in (6). To conclude, it is enough to prove that there exists a choice of $\varepsilon > 0$ for which also the last inequality in (6) holds. We notice that $\frac{d_1}{d_n}$ is uniquely determined by the other ratios as

$$\frac{d_1}{d_n} = \frac{d_1}{d_2} \cdot \frac{d_2}{d_3} \cdots \frac{d_{n-1}}{d_n} = \prod_{j=1}^{n-1} \frac{1}{a_j^2 + \varepsilon}.$$

If for every choice of $\varepsilon > 0$ it would be

$$a_n^2 \geq \prod_{j=1}^{n-1} \frac{1}{a_j^2 + \varepsilon},$$

then it should be

$$a_n^2 \geq \prod_{j=1}^{n-1} \frac{1}{a_j^2},$$

and hence

$$\prod_{j=1}^{n} \frac{1}{a_j^2} = \left(\prod_{j=1}^{n} \frac{1}{a_j}\right)^2 \geq 1,$$

thus contradicting iv).
vi) $\Rightarrow$ i) If the matrices A_i have a common Lyapunov function, then the switched system (1) is obviously stable.
v) $\Leftrightarrow$ vii) The proof follows the same lines of the proof of b3) $\Leftrightarrow$ b4) of Proposition 2 in [5].
iv) $\Rightarrow$ viii) Condition iv) implies

$$\left|\max_{i \in [1,p]} a_{12}^{(i)} \cdot \max_{i \in [1,p]} a_{23}^{(i)} \cdots \max_{i \in [1,p]} a_{n-1,n}^{(i)} \cdot \max_{i \in [1,p]} a_{n1}^{(i)}\right| < 1.$$

On the other hand,

$$A^* = \begin{bmatrix} 0 & \max_{i \in [1,p]} a_{12}^{(i)} & 0 & \dots & 0 \\ 0 & 0 & \max_{i \in [1,p]} a_{23}^{(i)} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \max_{i \in [1,p]} a_{n-1,n}^{(i)} \\ \max_{i \in [1,p]} a_{n1}^{(i)} & 0 & 0 & \dots & 0 \end{bmatrix}$$

so the result immediately follows.
viii) $\Rightarrow$ ix) For every $\mathcal{S} \subset \Xi^*$, with $|\mathcal{S}| < +\infty$, and every choice of the coefficients $\alpha_w > 0$, with $\sum_{w \in \mathcal{S}} \alpha_w = 1$, one finds

$$0 \leq \sum_{w \in \mathcal{S}} \alpha_w w(A_1, A_2, \dots, A_p) \leq \sum_{w \in \mathcal{S}} \alpha_w (A^*)^{|w|}.$$

Since A^* is Schur, it is easily seen that the matrix on the right-hand side is Schur, too. So,

$$\rho\Big(\sum_{w \in \mathcal{S}} \alpha_w w(A_1, A_2, \dots, A_p)\Big) \leq \rho\Big(\sum_{w \in \mathcal{S}} \alpha_w (A^*)^{|w|}\Big) < 1,$$

which proves that the convex combination is Schur, too.
ix) $\Rightarrow$ ii) Obvious. ∎

Remark 1: The results of Proposition 1 have been stated under the assumption that all entries $a_{j,j+1}^{(i)}, j \in [1, n-1]$, and $a_{n1}^{(i)}$, are positive for every $i \in [1, p]$, but it is easily seen that they immediately extend to the nonnegative case, namely when some of these entries are zero. Also, from a computational viewpoint, condition viii) is undoubtedly the easiest condition to check to ascertain the stability of the switched system (1), with cyclic monomial matrices.

III. Stabilizability of systems with cyclic matrices

Definition 2: The discrete-time positive switched system (1) is said to be *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence (possibly depending on $\mathbf{x}(0)$), $\sigma : \mathbb{Z}_+ \to [1, p]$, such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

As it has been shown in [5], if a discrete-time positive switched system (1) is stabilizable, then it can be stabilized by means of a periodic switching sequence, which amounts to saying that some word $w \in \Xi^*$ can be found, such that $w(A_1, A_2, \dots, A_p)$ is a Schur matrix. So, to characterize stability we will resort to this result. In addition, it is easily seen that if at least one of the matrices, say A_ℓ, is Schur, then the system is stabilizable by means of the constant switching sequence $\sigma(t) = \ell$. We will show that for the DPSS (1), which switch among cyclic monomial matrices, this is the only case when stabilizability is possible.

Proposition 2: Given a discrete-time positive switched system (1), with cyclic monomial matrices $A_i \in \mathbb{R}_+^{n \times n}, i \in [1, p]$, the following facts are equivalent:

i) the system is stabilizable;
ii) one of the matrices $A_i, i \in [1, p]$, is Schur.

Proof: i) ⇒ ii) According to the previous comments, assume that there exists $w \in \Xi^*$ such that $w(A_1, A_2, \dots, A_p)$ is a Schur matrix. If this is the case, $w(A_1, A_2, \dots, A_p)^n$ is is a Schur diagonal matrix, and consequently its diagonal entries are smaller than 1. But then, also the product of the diagonal entries is smaller than one, and this product is easily seen to coincide with

$$\det w(A_1, A_2, \dots, A_p)^n = \prod_{i=1}^{p} (a_{12}^{(i)} a_{23}^{(i)} \dots a_{n1}^{(i)})^{n \cdot |w|_i},$$

which implies $|a_{12}^{(i)} a_{23}^{(i)} \dots a_{n1}^{(i)}| < 1$ for at least one index $i \in [1, p]$. So one of the A_i's is Schur.
ii) ⇒ i) Obvious. ■

Remark 2: Also in this case, the stabilizability characterization obtained under the assumption that all entries $a_{j,j+1}^{(i)}, j \in [1, n-1]$, and $a_{n1}^{(i)}$, are positive for every $i \in [1, p]$, immediately extends to the nonnegative case, namely when some of these entries are zero. It must be remarked, however, that this becomes a trivial case, since a cyclic monomial matrix A for which $a_{12} a_{23} \dots a_{n1} = 0$ is nilpotent, and hence stable.

Clearly, since stabilizability can be achieved only when one of the matrices A_i's is Schur, the most natural way to ensure the asymptotic convergence of a state trajectory is simply by steadily remaining set on the asymptotically stable subsystem, without switching at all. It may be of interest, however, to know what happens when switching takes place, namely under what conditions stabilizability can still be ensured.

Proposition 3: Consider a discrete-time positive switched system (1), with cyclic monomial matrices $A_i \in \mathbb{R}_+^{n \times n}, i \in [1, p]$, and suppose that it is stabilizable. If $w(A_1, A_2, \dots, A_p)$ is a Schur matrix, $w \in \Xi^*$, then the following condition holds:

$$[\, |w|_1 \quad |w|_2 \quad \dots \quad |w|_n \,] \begin{bmatrix} \log \rho(A_1) \\ \log \rho(A_2) \\ \vdots \\ \log \rho(A_p) \end{bmatrix} < 0. \tag{7}$$

Proof: Notice, first, that $a_{12}^{(i)} a_{23}^{(i)} \dots a_{n1}^{(i)} \equiv \rho(A_i)^n$, for every index $i \in [1, p]$. As in the previous proof, if $w(A_1, A_2, \dots, A_p)$ is Schur, then also $w(A_1, A_2, \dots, A_p)^n$ is Schur, thus implying that

$$\prod_{i=1}^{p} (a_{12}^{(i)} a_{23}^{(i)} \dots a_{n1}^{(i)})^{n \cdot |w|_i} = \prod_{i=1}^{p} \rho(A_i)^{n^2 \cdot |w|_i} < 1.$$

By applying the logarithm, we get

$$n^2 \cdot \sum_{i=1}^{p} |w|_i \cdot \log \rho(A_i) < 0,$$

which immediately proves the result. ■

Remark 3: It is clear that since all the occurrences $|w|_i$ are nonnegative integers, condition (7) requires $\log \rho(A_i) < 0$ for at least one index $i \in [1, p]$, which shows again the necessity that one of the matrices A_i's is Schur.

IV. Stability and stabilizability: the case of positive circulant matrices

In this section we focus on the class of discrete-time positive switched systems (1), described by either left or right positive circulant matrices. Right circulant matrices, simply known as circulant matrices, are described as follows:

$$C_i = \begin{bmatrix} a_0^{(i)} & a_1^{(i)} & a_2^{(i)} & \dots & a_{n-1}^{(i)} \\ a_{n-1}^{(i)} & a_0^{(i)} & a_1^{(i)} & \ddots & a_{n-2}^{(i)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2^{(i)} & a_3^{(i)} & a_4^{(i)} & \ddots & a_1^{(i)} \\ a_1^{(i)} & a_2^{(i)} & a_3^{(i)} & \dots & a_0^{(i)} \end{bmatrix}. \tag{8}$$

Once we set

$$p^{(i)}(s) := a_0^{(i)} + a_1^{(i)} s + a_2^{(i)} s^2 + \dots + a_{n-1}^{(i)} s^{n-1},$$

it is well-known [2] that the eigenvalues of C_i are

$$p^{(i)}(1), p^{(i)}(\varepsilon_n), p^{(i)}(\varepsilon_n^2), \dots, p^{(i)}(\varepsilon_n^{n-1}),$$

where $\varepsilon_n = e^{j \frac{2\pi}{n}}$ is a primitive nth root of 1. If the circulant matrices are positive, i.e. $a_\ell^{(i)} \geq 0, \forall\, \ell \in [0, n-1]$, it is easily seen that the eigenvalue of maximal modulus among them is the first one, and hence

$$\rho(C_i) = p^{(i)}(1) = a_0^{(i)} + a_1^{(i)} + a_2^{(i)} + \dots + a_{n-1}^{(i)}.$$

This implies that C_i is Schur if and only if

$$\mathbf{1}_n^\top C_i = \mathbf{1}_n^\top \rho(C_i) \ll \mathbf{1}_n^\top, \tag{9}$$

where $\mathbf{1}_n^\top$ is the n-dimensional vector with all entries equal to 1, and hence the circulant matrix is strictly (column) sub-stochastic.

As the product of two circulant matrices, C_i and C_j, is in turn a circulant matrix, when they are both positive we get

$$\rho(C_i C_j) = \textstyle\sum_{k=1}^{n} [C_i C_j]_{k1}$$
$$= \mathbf{1}_n^\top C_i \begin{bmatrix} [C_j]_{11} \\ [C_j]_{21} \\ \vdots \\ [C_j]_{n1} \end{bmatrix} = \mathbf{1}_n^\top \rho(C_i) \begin{bmatrix} [C_j]_{11} \\ [C_j]_{21} \\ \vdots \\ [C_j]_{n1} \end{bmatrix} = \rho(C_i) \cdot \rho(C_j).$$

Once we have realized these facts, the characterization of stability and stabilizability of positive switched systems whose matrices A_i are circulant becomes rather easy.

Proposition 4: Given a discrete-time positive switched system (1), with positive circulant matrices $A_i = C_i \in \mathbb{R}_+^{n \times n}, i \in [1, p]$, the following facts are equivalent ones:

i) the system is stable;
ii) for every $w \in \Xi^*$, the matrix product $w(C_1, C_2, \dots, C_p)$ is a positive Schur matrix;
iii) all matrices $C_1, C_2, \dots, C_p$ are Schur;
iv) the matrices C_i's admit a common linear copositive function, namely there exists $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top C_i \ll \mathbf{v}^\top$, for every $i \in [1, p]$;
v) for every choice of $\alpha_i \geq 0$, with $\sum_{i=1}^{p} \alpha_i = 1$, $\sum_{i=1}^{p} \alpha_i C_i$ is a positive (circulant) Schur matrix.

Proof: i) $\Rightarrow$ ii) $\Rightarrow$ iii), iv) $\Rightarrow$ i) and v) $\Rightarrow$ iii) are obvious. iii) $\Rightarrow$ iv) follows from the fact that if all the C_i's are asymptotically stable, then they all satisfy (9). This also proves that iii) $\Rightarrow$ v) since

$$\mathbf{1}_n^\top \left(\sum_{i=1}^p \alpha_i C_i\right) \ll \mathbf{1}_n^\top \sum_{i=1}^p \alpha_i = \mathbf{1}_n^\top.$$

■

As far as stabilizability is concerned, also for this class of systems, like for DPSS with cyclic monomial matrices, this property imposes the stability of at least one subsystem.

Proposition 5: Given a discrete-time positive switched system (1), with positive circulant matrices $A_i = C_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, the following facts are equivalent ones:

i) the system is stabilizable;
ii) one of the matrices $C_1, C_2, \ldots, C_p$ is Schur.

Proof: ii) $\Rightarrow$ i) is obvious. i) $\Rightarrow$ ii) follows from the fact that if the system is stabilizable then there exists $w \in \Xi^*$ such that $w(C_1, C_2, \ldots, C_p)$ is a Schur matrix. But

$$1 > \rho\left(w(C_1, C_2, \ldots, C_p)\right) = \rho(C_1)^{|w|_1} \cdot \rho(C_2)^{|w|_2} \cdots \rho(C_p)^{|w|_p}$$

implies that $\rho(C_i) < 1$ for at least one index $i \in [1,p]$. ■

We consider now positive left circulant matrices, which are described as:

$$C_i^L = \begin{bmatrix} a_0^{(i)} & a_1^{(i)} & a_2^{(i)} & \ldots & a_{n-1}^{(i)} \\ a_1^{(i)} & a_2^{(i)} & \ldots & a_{n-1}^{(i)} & a_0^{(i)} \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ a_{n-2}^{(i)} & a_{n-1}^{(i)} & \iddots & a_{n-4}^{(i)} & a_{n-3}^{(i)} \\ a_{n-1}^{(i)} & a_0^{(i)} & \ldots & a_{n-3}^{(i)} & a_{n-2}^{(i)} \end{bmatrix}, \tag{10}$$

with $a_\ell^{(i)} \geq 0, \ell \in [0, n-1]$. If we compare a left circulant matrix with a right circulant matrix having the same entries in the first line, it can be shown [13] that they can be reduced (by means of the same similarity transformation T, which only depends on n and not on C_i) to the following forms:

$$T^{-1}C_iT = \begin{bmatrix} p^{(i)}(1) & & & & \\ & p^{(i)}(\varepsilon_n) & & & \\ & & p^{(i)}(\varepsilon_n^2) & & \\ & & & \ddots & \\ & & & & p^{(i)}(\varepsilon_n^{n-1}) \end{bmatrix}$$

$$T^{-1}C_i^LT = \begin{bmatrix} p^{(i)}(1) & & & & \\ & & & & p^{(i)}(\varepsilon_n) \\ & & & p^{(i)}(\varepsilon_n^2) & \\ & & \iddots & & \\ & p^{(i)}(\varepsilon_n^{n-1}) & & & \end{bmatrix}$$

This implies, in particular, that $\rho(C_i^L) = \rho(C_i) = p^{(i)}(1) = \sum_{k=0}^{n-1} a_k^{(i)}$. So, also in this case C_i^L is Schur if and only if

$$\mathbf{1}_n^\top C_i^L = \mathbf{1}_n^\top \rho(C_i^L) \ll \mathbf{1}_n^\top,$$

and hence the left circulant matrix is strictly (column) substochastic. Notice, however, that the product of two left circulant matrices is a right circulant matrix, having as spectral radius the product of the spectral radii of the two matrices. In general, given a family $\{A_1, A_2, \ldots, A_p\}$ that includes both left and right circulant matrices, for every $w \in \Xi^*$, the matrix product $w(A_1, A_2, \ldots, A_p)$ is a circulant matrix, left if the number of left circulant matrices involved in the matrix product is odd and right if it is even. In both cases, $\rho\left(w(A_1, A_2, \ldots, A_p)\right) = \rho(A_1)^{|w|_1}\rho(A_2)^{|w|_2}\cdots\rho(A_p)^{|w|_p}$. So, the previous results about stability and stabilizability can be generalized as follows.

Proposition 6: Given a discrete-time positive switched system (1), with positive (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, the following facts are equivalent ones:

i) the system is stable;
ii) for every $w \in \Xi^*$, the matrix product $w(A_1, A_2, \ldots, A_p)$ is a positive Schur matrix;
iii) all matrices $A_1, A_2, \ldots, A_p$ are Schur;
iv) the matrices A_i's admit a common linear copositive function;
v) for every choice of $\alpha_i \geq 0$, with $\sum_{i=1}^p \alpha_i = 1$, $\sum_{i=1}^p \alpha_i A_i$ is a positive (not necessarily circulant) Schur matrix.

Example 1: Consider a discrete-time positive switched system (1), with $p = 3$ and

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.3 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.5 & 0 & 0.2 \\ 0 & 0.2 & 0.5 \\ 0.2 & 0.5 & 0 \end{bmatrix}.$$

Notice that A_1 is right circulant, while A_2 and A_3 are left circulant. It is easily seen that $\mathbf{1}_3^\top A_i \ll \mathbf{1}_3^\top$ for every $i \in [1,3]$. This ensures that the switched system is stable.

Proposition 7: Given a discrete-time positive switched system (1), with positive (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, the following facts are equivalent ones:

i) the system is stabilizable;
ii) one of the matrices $A_1, A_2, \ldots, A_p$ is Schur.

It is worthwhile to notice that, under these strong assumptions, the result given in Proposition 3 becomes a necessary and sufficient condition. Indeed, we have:

Proposition 8: Consider a discrete-time positive switched system (1), with positive (either left or right) circulant matrices $A_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, and suppose that it is stabilizable. Given some word $w \in \Xi^*$, the corresponding matrix product $w(A_1, A_2, \ldots, A_p)$ is Schur if and only if condition (7) holds.

V. Stabilizability criteria for general positive switched systems

The stabilizability results obtained in the previous sections provide very simple sufficient conditions for checking the lack of stabilizability of a generic DPSS (1).

Corollary 1: Consider a discrete-time positive switched system (1), with $A_i, i \in [1,p]$, arbitrary positive matrices. If there exist matrices $\tilde{A}_i \in \mathbb{R}_+^{n\times n}, i \in [1,p]$, that are either all cyclic monomial or all positive circulant, and such that, for every $i \in [1,p]$,

(a) $A_i \geq \tilde{A}_i$;
(b) $\tilde{A}_i$ is not Schur,

then the positive switched system is not stabilizable.

Proof: Under assumption (a) we have that for every $w \in \Xi^*$, $w(A_1, A_2, \ldots, A_p) \geq w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)$, but this implies $\rho\left(w(A_1, A_2, \ldots, A_p)\right) \geq \rho\left(w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)\right)$. If all matrices A_i are either cyclic monomial or positive circulant, then under assumption (b), $\rho\left(w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)\right) \geq 1$ for every $w \in \Xi^*$. So, the (original) positive switched system cannot be stabilizable. ∎

As a matter of fact, the previous result can be easily extended, as shown in the following corollaries. We first consider the case when the submatrices are cyclic monomial.

Corollary 2: Consider a discrete-time positive switched system (1), with $A_i, i \in [1,p]$, arbitrary positive matrices. Suppose that there exists an elementary cycle γ, of length $k \leq n$, say $j_1 \to j_2 \to j_3 \to \ldots \to j_k \to j_1$, appearing in every digraph $\mathcal{D}(A_i)$. If the product of the weights of its edges in every digraph $\mathcal{D}(A_i)$, namely $[A_i]_{j_1 j_2} \cdot [A_i]_{j_2 j_3} \ldots [A_i]_{j_k j_1}$, is greater than or equal to 1, then the positive switched system is not stabilizable.

Proof: It entails no loss of generality assuming that γ is the elementary cycle: $1 \to k \to k-1 \to \ldots \to 2 \to 1$. So, if we keep of the matrices A_i only the entries that represent the weights of these arcs, we get the p not Schur matrices:

$$\tilde{A}_i = \left[\begin{array}{ccccc|c} 0 & a_{12} & 0 & \ldots & 0 & \\ 0 & 0 & a_{23} & \ddots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & a_{k-1k} & \\ a_{k1} & 0 & 0 & \ldots & 0 & \\ \hline & & & & & 0_{(n-k)\times(n-k)} \end{array}\right].$$

Clearly, for every $w \in \Xi^*$, $w(A_1, A_2, \ldots, A_p) \geq w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)$, but this implies

$$\rho\left(w(A_1, A_2, \ldots, A_p)\right) \geq \rho\left(w(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)\right) \geq 1.$$

So, since none of matrix products $w(A_1, A_2, \ldots, A_p)$ is Schur, the switched system is not stabilizable. ∎

Example 2: Consider a discrete-time positive switched system (1), with $p = 2$ and

$$A_1 = \begin{bmatrix} 0.1 & 0.2 & 2 \\ 0.5 & 0.2 & 0.5 \\ 1 & 0.5 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.01 & 0.1 & 1 \\ 0 & 0 & 0.1 \\ 3 & 0 & 0.2 \end{bmatrix}.$$

$\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ have in common an elementary cycle including vertices 1 and 3. It is easily seen that

$$A_1 > \tilde{A}_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_2 > \tilde{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Since, neither $\tilde{A}_1$ nor $\tilde{A}_2$ is Schur, the switched system is not stabilizable.

As a further extension of Corollary 1 we get the following result, whose proof follows the same lines as the previous one and hence it is omitted.

Corollary 3: Consider a discrete-time positive switched system (1), with $A_i, i \in [1,p]$, arbitrary positive matrices. Suppose that there exist a permutation matrix P, a positive integer $k \leq n$ and positive (left or right) circulant matrices $\tilde{A}_i \in \mathbb{R}_+^{k\times k}, i \in [1,p]$, such that, for every index i,

(a) $P^T A_i P \geq \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{bmatrix}$;
(b) $\tilde{A}_i$ is not Schur.

Then the positive switched system is not stabilizable.

References

[1] R.A. Brualdi and H.J. Ryser. *Combinatorial matrix theory*. Cambridge Univ.Press, Cambridge (GB), 1991.

[2] P.J. Davis. *Circulant matrices*. J.Wiley & Sons, New York, 1979.

[3] E. Fornasini and M.E. Valcher. Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Trans. Aut. Contr.*, 55 (8):1933–1937, 2010.

[4] E. Fornasini and M.E. Valcher. On the stability of continuous-time positive switched systems. In *Proceedings of the 2010 American Control Conference*, pages 6225–6230, Baltimore, 2010.

[5] E. Fornasini and M.E. Valcher. Stabilizability of discrete-time positive switched systems. In *Proceedings of the 49th IEEE Conf. on Decision and Control*, pages 432–437, Atlanta, Georgia, 2010.

[6] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52 (6):1099–1103, 2007.

[7] E. Hernandez-Varga, R. Middleton, P. Colaneri, and F. Blanchini. Discrete-time control for switched positive systems with application to mitigating viral escape. *International Journal of Robust and Nonlinear Control*, DOI: 10.1002/rnc.1628, 2010.

[8] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Aut. Contr.*, 48 (6):988–1001, 2003.

[9] F. Knorn, O. Mason, and R.N. Shorten. On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45, Issue 8:1943–1947, August 2009.

[10] O. Mason and R.N. Shorten. On common quadratic Lyapunov functions for stable discrete-time LTI systems. *IMA J. of Applied Mathematics*, 69:271–283, 2004.

[11] O. Mason and R.N. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52, no. 7:1346 – 1349, 2007.

[12] O. Mason and R.N. Shorten. Quadratic and copostive Lyapunov functions and the stability of positive switched linear systems. In *Proceedings of the American Control Conference (ACC 2007)*, pages 657–662, New York, 2007.

[13] O. Rojo and R.L. Soto. Guo perturbation for symmetric nonnegative circulant matrices. *Lin. Alg. Appl.*, 431:594–607, 2009.

[14] A. Salomaa and M. Soittola. *Automata theoretic aspects of formal power series*. Springer-Verlag, 1978.

[15] R. Shorten, F. Wirth, and D. Leith. A positive systems model of TCP-like congestion control: asymptotic results. *IEEE/ACM Transactions on Networking*, 14 (3):616–629, 2006.

[16] A. Zappavigna, P. Colaneri, J.C. Jeromel, and R. Middleton. Stabilization of continuous-time switched linear positive systems. In *Proceedings of the 2010 American Control Conference*, pages 3275–3280, Baltimore, MD, 2010.