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Adaptation Along Prescribed Directions

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Abstract **– In this paper, we revisit the state feedback model reference adaptive control problem. Specifically, we show that adaptive laws can be constructed such that the adaptation process proceeds along the desired directions. In order to substantiate our design, we formulate sufficient conditions to guarantee global asymptotic tracking of a bounded time-varying external command. We argue that the reported results provide an alternative control-theoretic methodology to design adaptive command tracking controllers for multi-input-multi-output dynamical systems with matched uncertainties.**

I. Introduction

In our recent paper[1], a constructive design was presented to compose Model Reference Adaptive Control (MRAC) systems in an output feedback setting. This method utilized asymptotic properties of Algebraic Riccati Equation (ARE), whose matrix weights were selected in a special form. The paper ended with a remark regarding utilization of the developed adaptive output feedback when all states were available.

Continuing this line of thoughts, in this paper we show how to design state feedback adaptive laws with their dynamics evolving along selected output directions in the system state space. This feature constitutes the novelty of our design.

The rest of the paper is organized as follows. Section II defines system dynamics of interest. Control problem formulation is given in Section III. Our *main results*, Theorems 4.1 and 5.1, are stated in Sections IV and V, respectively. A simulation example is given in Section VI. The paper ends with conclusions and a brief discussion of future research directions.

II. Preliminaries and System Dynamics

Throughout the paper, R^n denotes the Euclidean n – dimensional space, and $R^{n\times m}$ denotes the space of all *n*-by-*m* matrices, where n and m are integers. For a vector $x \in R^n$, we write $\|x\|$ for the Euclidean vector norm of x. For a matrix $A \in R^{n\times m}$, $\|A\|$ denotes the induced matrix norm. For a time-dependent function $f(t)$, we write $f \in L_\infty$ to claim uniform boundedness of f.

Based on the discussions and the resulting problem formulation from [1], we consider n – dimensional nonlinear MIMO uncertain dynamical systems in the form,

$$\dot{x} = A_{ref}\, x + B\,\Lambda\left(u + \Theta^T \Phi(x)\right) + B_{ref}\, z_{cmd} \tag{2.1}$$

with m control inputs $u \in R^m$ and m regulated outputs

$$z = C_z\, x \tag{2.2}$$

The system state is $x \in R^n$, and $z_{cmd} \in R^m$ is the external command for the regulated output z to follow. In (2.1), $d(x) = \Theta^T \Phi(x)$ represents a nonlinear state-dependent matched parametric uncertainty, $\Theta \in R^{N\times m}$ is the matrix of unknown constant "true" parameters, and $\Phi(x) \in R^N$ is the known *N*–dimensional regressor vector, whose components are locally Lipschitz-continuous in x. The dynamics (2.1) contain known matrices $A_{ref} \in R^{n\times n}$, $B \in R^{n\times m}$, $B_{ref} \in R^{n\times m}$, and $C_z \in R^{m\times n}$, while $\Lambda \in R^{m\times m}$ represents a constant diagonal unknown matrix with strictly positive diagonal elements. We assume that A_{ref} is Hurwitz, the pair (A_{ref}, B) is controllable, and the entire state vector is accessible.

Given the matrix pair (A_{ref}, B), it is not difficult to find an output matrix $C \in R^{m\times n}$ such that (A_{ref}, C) is observable and:

$$\begin{aligned} &\det\left[C\,B\right] \neq 0 \\ &\det\left[C\left(s\, I_{n\times n} - A_{ref}\right)^{-1} B\right] \neq 0, \quad \forall \operatorname{Re} s > 0 \end{aligned} \tag{2.3}$$

This is the "squaring-up" problem[5,6]. If the triplet (A_{ref}, B, C) represents a minimum realization of Linear-Time-Invariant (LTI) dynamics then the two conditions in (2.3) imply that the LTI system has a non-zero high frequency gain and no finite transmission zeros in the right half plane. Numerically efficient algorithms for solving the squaring-up problems can be found in [5, 6]. Throughout the paper, the two relations in (2.3) will be called the "*squaring-up conditions*".

Since the selection of C is never unique, such a matrix can be chosen to represent the desired (for control) output channels or their linear combinations. We are going to exploit this property in the design of MRAC.

III. Reference Model and Control Problem

The reference model dynamics are chosen in the form of a state observer[1],

$$\dot{x}_{ref} = A_{ref}\, x_{ref} + B_{ref}\, z_{cmd} + L_v\, C\left(x - x_{ref}\right) \tag{3.1}$$

where $L_v \in R^{n\times m}$ is a constant parameter-dependent matrix that will be determined later. The reference model output is

$$z_{ref} = C_z\, x_{ref} \tag{3.2}$$

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Readers who are familiar with the conventional MRAC architecture[2,3,4] may have noticed the not-so-subtle alteration introduced into the reference model (3.1). It is the addition of the "innovation term" $L_v C(x - x_{ref})$. The thought here is exactly the same as in the design of linear state observers for LTI systems. These observers consist of the system dynamics and output innovation terms, with the latter representing a mismatch between the system output and its observation. As the estimated output approaches its observed reference, the observer dynamics recover the original system. Similarly in our case, if the system state x is forced to track its reference x_{ref} from (3.1) then the innovation term becomes small and the reference dynamics (3.1) become "almost ideal".

$$\dot{x}_{ref} = A_{ref}\, x_{ref} + B_{ref}\, z_{cmd} + o(1), \quad \text{as } t \to \infty \tag{3.3}$$

We assume that the ideal dynamics matrices (A_{ref}, B_{ref}) are chosen such that z_{ref} adequately tracks bounded time-varying commands z_{cmd} with bounded errors.

The *control objective* of interest is bounded command tracking in the presence of parametric uncertainties $\{\Lambda, \Theta\}$. Essentially, we are going to design u such that the system state x globally asymptotically tracks the state of the reference model x_{ref}. Then as a consequence, the system regulated output z will track the reference model output z_{ref}, globally and asymptotically, and hence the system regulated output z will approach z_{cmd}, while keeping the rest of the signals in the closed-loop system bounded. This is our design strategy. The design details are given in the next section.

IV. Adaptive Control Design

In this section, we derive and analyze an adaptive command tracking controller in the form,

$$u = -\hat{\Theta}^T \Phi(x) \tag{4.1}$$

where $\hat{\Theta}(t) \in R^{N\times m}$ is the vector of adaptive parameters whose dynamics (the adaptive law) will be determined using Lyapunov-based arguments. Substituting (4.1) into (2.1), gives the closed-loop system,

$$\dot{x} = A_{ref}\, x - B\,\Lambda \underbrace{\left(\hat{\Theta} - \Theta\right)}_{\Delta\Theta}^T \Phi(x) + B_{ref}\, z_{cmd} \tag{4.2}$$

with $\Delta\Theta(t) = \hat{\Theta}(t) - \Theta$ denoting the parameter estimation errors. Let,

$$e = x - x_{ref} \tag{4.3}$$

represent the state tracking error. Subtracting (2.1) from (4.2), yields the tracking error dynamics:

$$\dot{e} = \left(A_{ref} - L_v\, C\right) e - B\,\Lambda\,\Delta\Theta^T\,\Phi(x) \tag{4.4}$$

We are going to choose the observer gain matrix L_v to represent a steady-state Kalman filter gain, such as

$$L_v = P_v\, C^T\, R_v^{-1} \tag{4.5}$$

where $P_v = P_v^T > 0$ is the unique solution of the v-dependent Algebraic Riccati Equation (ARE),

$$\begin{aligned} &P_v \left(A_{ref} + \eta\, I_{n\times n}\right)^T + \left(A_{ref} + \eta\, I_{n\times n}\right) P_v \\ &- P_v\, C^T\, R_v^{-1}\, C\, P_v + Q_v = 0 \end{aligned} \tag{4.6}$$

and $\eta > 0$ is a positive constant, (enforces prescribed degree of stability). Also, (Q_v, R_v) are symmetric positive definite parameter-dependent weight matrices in the form,

$$Q_v = Q_0 + \left(\frac{v+1}{v}\right) B\, B^T, \quad R_v = \frac{v}{v+1} R_0 \tag{4.7}$$

where Q_0 and R_0 are symmetric and positive definite weights. It is easy to verify that the ARE (4.6) possesses the unique symmetric positive definite solution P_v, for any positive parameter v. Furthermore, because of (4.6), the observer closed-loop matrix,

$$A_v = A_{ref} - L_v\, C = A_{ref} - P_v\, C^T\, R_v^{-1}\, C \tag{4.8}$$

satisfies,

$$\begin{aligned} &P_v \underbrace{\Big(A_{ref} - \underbrace{P_v\, C^T\, R_v^{-1}}_{L_v} C\Big)}_{A_v}^T + \underbrace{\Big(A_{ref} - \underbrace{P_v\, C^T\, R_v^{-1}}_{L_v} C\Big)}_{A_v} P_v \\ &+ P_v\, C^T\, R_v^{-1}\, C\, P_v + Q_v + 2\eta\, P_v = 0 \end{aligned} \tag{4.9}$$

or, equivalently

$$P_v\, A_v^T + A_v\, P_v = -P_v\, C^T\, R_v^{-1}\, C\, P_v - Q_v - 2\eta\, P_v < 0 \tag{4.10}$$

and therefore, A_v is Hurwitz, uniformly in $v > 0$. The resulting closed-loop error dynamics (4.4) can be written as:

$$\dot{e} = A_v\, e - B\,\Lambda\,\Delta\Theta^T\,\Phi(x) \tag{4.11}$$

Introducing matrix inverse $\tilde{P}_v = P_v^{-1}$ allows to rewrite (4.10) as:

$$A_v^T\, \tilde{P}_v + \tilde{P}_v\, A_v = -C^T\, R_v^{-1}\, C - \tilde{P}_v\, Q_v\, \tilde{P}_v - 2\eta\, \tilde{P}_v < 0 \tag{4.12}$$

In [1, Corollary 3.1], using the asymptotic expansion

$$P_v = P_0 + O(v), \quad \text{as } v \to 0 \tag{4.13}$$

we proved that

$$\tilde{P}_v\, B = C^T\, R_0^{-\frac{1}{2}}\, W + O(v), \quad \text{as } v \to 0 \tag{4.14}$$

where the $(m\times m)$ – unitary matrix

$$W = (U\, V)^T \tag{4.15}$$

was computed using the two unitary matrices U and V defined by the singular value decomposition,

$$B^T\, C^T\, R_0^{-\frac{1}{2}} = U\,\Omega\, V \tag{4.16}$$

with Ω representing the diagonal matrix of the corresponding singular values. We had also shown that P_v was invertible for any $v \geq 0$, and

$$\lim_{v \to 0} x^T P_v\, x \geq \lambda_{\min}\left(P_0\right) > 0 \quad (4.17)$$

where $\lambda_{\min}\left(P_0\right)$ denoted the minimum eigenvalue of $P_0 = P_0^T > 0$. The two relations, (4.12) and (4.14), are of key importance in that both will be utilized to design a state feedback adaptive control law with its dynamics evolving along pre-selected directions in the system state space.

Towards that end, we are going to choose adaptive laws for $\hat{\Theta}(t)$ so that x tracks x_{ref} globally, asymptotically, and in the presence of the system uncertainties. Our design is Lyapunov-based and as such, we start with a Lyapunov function candidate in the form,

$$V\left(e,\, \Delta\Theta\right) = e^T\, \tilde{P}_v\, e + \operatorname{trace}\left(\Lambda\, \Delta\Theta^T\, \Gamma_\Theta^{-1}\, \Delta\Theta\right) \quad (4.18)$$

where $\tilde{P}_v = P_v^{-1}$, $P_v = P_v^T > 0$ is the unique solution of the ARE (4.6) with the matrix weights from (4.7), and $\Gamma_\Theta = \Gamma_\Theta^T > 0$ is a constant matrix of adaptation rates. Computing the time derivative of $V\left(e,\, \Delta\Theta\right)$, along the trajectories of the error dynamics (4.11), while utilizing (4.12), gives:

$$\begin{aligned} \dot{V}\left(e,\, \Delta\Theta\right) &= -e^T\left(C^T\, R_v^{-1}\, C + \tilde{P}_v\, Q_v\, \tilde{P}_v + 2\eta\, \tilde{P}_v\right)e \\ &\quad -2\, e^T\, \tilde{P}_v\, B\, \Lambda\, \Delta\Theta^T\, \Phi(x) + 2\operatorname{trace}\left(\Lambda\, \Delta\Theta^T\, \Gamma_\Theta^{-1}\, \dot{\hat{\Theta}}\right) \\ &= -e^T\left(C^T\, R_v^{-1}\, C + \tilde{P}_v\, Q_v\, \tilde{P}_v + 2\eta\, \tilde{P}_v\right)e \\ &\quad +2\operatorname{trace}\left(\Lambda\, \Delta\Theta^T\left\{\Gamma_\Theta^{-1}\, \dot{\hat{\Theta}} - \Phi(x)\, e^T\, \tilde{P}_v\, B\right\}\right) \end{aligned} \quad (4.19)$$

In an effort to make $\dot{V}\left(e,\, \Delta\Theta\right) \leq 0$, we introduce the following *adaptive laws*,

$$\dot{\hat{\Theta}} = \Gamma_\Theta\, \operatorname{Proj}\left(\hat{\Theta},\; \Phi(x)\, e^T\, \tilde{P}_v\, B\right) \quad (4.20)$$

with the Projection Operator[7] enforcing uniform bounds on the estimated parameters,

$$\hat{\Theta}(t) \in \Omega_\Theta = \left\{\Theta \in R^{N \times m} :\; \left\|\Theta\right\| \leq \Theta_{\max}\right\} \quad (4.21)$$

uniformly in t. In (4.21), $\Theta_{\max}$ is the known maximum-allowable induced norm upper bound for $\hat{\Theta}(t)$. Substituting (4.20) into (4.19) and using convex properties of the Projection Operator, results in

$$\dot{V}\left(e,\, \Delta\Theta\right) \leq -e^T\left(C^T\, R_v^{-1}\, C + \tilde{P}_v\, Q_v\, \tilde{P}_v + 2\eta\, \tilde{P}_v\right)e \leq 0 \quad (4.22)$$

which immediately proves uniform boundedness of $e(t)$ and $\Delta\Theta(t)$, i.e., $\left(e_x,\, \Delta\Theta\right) \in L_\infty$. Since A_{ref} is Hurwitz and $\left(e,\, z_{cmd}\right) \in L_\infty$ then (3.1) implies $x_{ref} \in L_\infty$. Consequently, $\left(x,\, \hat{\Theta}\right) \in L_\infty$. Since the regressor vector $\Phi(x)$ is Lipchitz-continuous and $x \in L_\infty$ then $\Phi(x) \in L_\infty$, and consequently $u \in L_\infty$. Therefore, $\left(\dot{x},\, \dot{x}_{ref}\right) \in L_\infty$ and so $\dot{e} \in L_\infty$. Integrating both sides of (4.22), proves that $e \in L_2$. These two conditions, $e \in L_2$ and $\dot{e} \in L_\infty$, imply (Barbalat's Lemma[8]),

$$\lim_{t \to \infty}\left\|e(t)\right\| = \lim_{t \to \infty}\left\|x(t) - x_{ref}(t)\right\| = 0 \quad (4.23)$$

which immediately proves global asymptotical stability (GAS) of the tracking error dynamics (4.11).

We have shown that the system state $x(t)$ tracks the state of the reference model $x_{ref}(t)$, globally and asymptotically. Consequently, the system regulated output $z(t) = C_z\, x(t)$ will globally asymptotically track the reference model regulated output $z_{ref}(t) = C_z\, x_{ref}(t)$, which was chosen to adequately track its commanded value $z_{cmd}(t)$, with bounded errors. Hence, the system regulated output $z(t)$ will track its commanded signal $z_{cmd}(t)$, with bounded errors. This completes the design and stability analysis of the adaptive tracking controller (4.1). We summarize our formally proven results in the statement below.

Theorem 4.1

Consider the MIMO system dynamics (2.1),

$$\dot{x} = A_{ref}\, x + B\, \Lambda\left(u + \Theta^T\, \Phi(x)\right) + B_{ref}\, z_{cmd}$$

with the regulated output $z = C_z\, x$, and the reference model (3.1),

$$\dot{x}_{ref} = A_{ref}\, x_{ref} + B_{ref}\, z_{cmd} + L_v\, C\left(x - x_{ref}\right)$$

whose feedback gain matrix $L_v = P_v\, C^T\, R_v^{-1}$ satisfies the ARE (4.6),

$$\begin{aligned} &P_v\left(A_{ref} + \eta\, I_{n \times n}\right)^T + \left(A_{ref} + \eta\, I_{n \times n}\right)P_v \\ &- P_v\, C^T\, R_v^{-1}\, C\, P_v + Q_v = 0 \end{aligned}$$

with a constant $\eta > 0$, symmetric positive definite matrices $\left(Q_v,\, R_v\right)$ from (4.7),

$$Q_v = Q_0 + \left(\frac{v+1}{v}\right) B\, B^T, \quad R_v = \frac{v}{v+1} R_0$$

and a constant parameter $v > 0$. Let $\tilde{P}_v = P_v^{-1}$. Then for any $v > 0$, the adaptive controller (4.1),

$$u = -\hat{\Theta}^T\, \Phi(x)$$

with the projection-based adaptive laws (4.20),

$$\dot{\hat{\Theta}} = \Gamma_\Theta\, \operatorname{Proj}\left(\hat{\Theta},\; \Phi(x)\, e^T\, \tilde{P}_v\, B\right)$$

enforces GAS of the closed-loop error dynamics (4.11). Moreover, this controller makes the system regulated output z track any bounded time-varying command z_{cmd} with bounded errors, while keeping all other signals in the closed-loop system uniformly bounded in time.

As seen from (4.20), the adaptive law dynamics depend on the so-called *training error*,

$$\bar{e}^T = e^T \tilde{P}_v B \tag{4.24}$$

and the columns of

$$C_v^T = \tilde{P}_v B \tag{4.25}$$

define "directions of adaptation". In other words, the adaptive law dynamics (4.20) integrate linear combinations of the tracking error signal e, which is scaled by the regressor vector $\Phi(x)$. The corresponding linear scaling coefficients are defined by the columns of C_v^T. Effectively, MRAC design tuning process entails a proper selection of C_v. In the conventional MRAC[2,3,4], this selection can be achieved by solving the Algebraic Lyapunov Equation,

$$A_v^T \tilde{P} + \tilde{P} A_v = -Q_{ref} \tag{4.26}$$

while choosing a positive definite symmetric matrix Q_{ref} such that the resulting $\tilde{P} B$ has the desired structure. As a consequence, the matrix triplet (A_v, B, C_v) in the error dynamics (4.11) becomes Strictly Positive Real (SPR). The latter is the key ingredient in proving closed-loop stability of the error dynamics.

Our proposed design is different. First of all, instead of solving the Lyapunov equation (4.26), we use the Riccati equation (4.6). Secondly, our choice of v-dependent weights (4.7) forces C_v in (4.25) approach the pre-selected output matrix in (4.14), as v tends to zero. In what follows, we are going to utilize this asymptotic relation to modify the adaptive laws (4.20) such that the adaptation process proceeds along the directions defined by the pre-selected output matrix C.

V. Adaptive Control Along Selected Directions

Consider again the time derivative (4.19) of the Lyapunov function candidate (4.18), computed along the trajectories of the error dynamics (4.11),

$$\begin{aligned} \dot{V}(e, \Delta\Theta) &= -e^T \left(C^T R_v^{-1} C + \tilde{P}_v Q_v \tilde{P}_v + 2\eta \tilde{P}_v \right) e \\ &\quad -2 e^T P_v B \Lambda \Delta\Theta^T \Phi(x) + 2\,\mathrm{trace}\left(\Lambda \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} \right) \end{aligned} \tag{5.1}$$

and substitute the asymptotics (4.14) into the second term. Inserting the weights from (4.7), gives,

$$\begin{aligned} &\dot{V}(e, \Delta\Theta) = \\ &-e^T \left(C^T \left(1 + \frac{1}{v}\right) R_0^{-1} C + \tilde{P}_v \left(Q_0 + \left(\frac{v+1}{v}\right) B B^T \right) \tilde{P}_v \right) e \\ &-2\eta e^T \tilde{P}_v e - 2 e^T \left(C^T R_0^{-\frac{1}{2}} W + \mathrm{O}(v) \right) \Lambda \Delta\Theta^T \Phi(x) \\ &+2\,\mathrm{trace}\left(\Lambda \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} \right) \end{aligned} \tag{5.2}$$

or, equivalently

$$\begin{aligned} &\dot{V}(e, \Delta\Theta) = \\ &-e^T C^T \left(1 + \frac{1}{v}\right) R_0^{-1} C e - 2\eta e^T \tilde{P}_v e - e^T \tilde{P}_v Q_0 \tilde{P}_v e \\ &-\left(1 + \frac{1}{v}\right) \left\| e^T \tilde{P}_v B \right\|^2 - 2 e^T \mathrm{O}(v) \Lambda \Delta\Theta^T \Phi(x) \\ &+2\,\mathrm{trace}\left(\Lambda \Delta\Theta^T \left\{ \Gamma_\Theta^{-1} \dot{\hat{\Theta}} - \Phi(x) e^T C^T R_0^{-\frac{1}{2}} W \right\} \right) \end{aligned} \tag{5.3}$$

Projection-based adaptive laws can now be selected to force the trace term become non-positive.

$$\dot{\hat{\Theta}} = \Gamma_\Theta \,\mathrm{Proj}\left(\hat{\Theta},\ \Phi(x) e^T C^T R_0^{-\frac{1}{2}} W \right) \tag{5.4}$$

Then,

$$\begin{aligned} &\dot{V}(e, \Delta\Theta) \le \\ &-e^T C^T \left(1 + \frac{1}{v}\right) R_0^{-1} C e - 2\eta e^T \tilde{P}_v e - e^T \tilde{P}_v Q_0 \tilde{P}_v e \\ &-\left(1 + \frac{1}{v}\right) \left\| e^T \tilde{P}_v B \right\|^2 - 2 e^T \mathrm{O}(v) \Lambda \Delta\Theta^T \Phi(x) \end{aligned} \tag{5.5}$$

where $\mathrm{O}(v) \in R^{n \times m}$ is a constant matrix, whose induced norm is upper bounded, $\|\mathrm{O}(v)\| \le k\,v$, with $k > 0$ being a constant, independent of v. Suppose that the regressor vector $\Phi(x)$ is globally Lipschitz. Then because of (3.1), one can show that the following inequality takes place[1],

$$\|\Phi(x)\| \le b_1 + b_2 \|e\| \tag{5.6}$$

where $b_1 \ge 0$ and $b_2 > 0$ are known (computable) constants. Using (5.6), an upper bound for $\dot{V}$ can be computed,

$$\begin{aligned} &\dot{V}(e, \Delta\Theta) \le -e^T C^T \left(1 + \frac{1}{v}\right) R_0^{-1} C e - 2\eta e^T \tilde{P}_v e \\ &-e^T \tilde{P}_v Q_0 \tilde{P}_v e - \left(1 + \frac{1}{v}\right) \left\| e^T \tilde{P}_v B \right\|^2 \\ &+2\,k\,v\,\Lambda_{\max}\,\Delta\Theta_{\max} \|e\| \left(b_1 + b_2 \|e\| \right) \end{aligned} \tag{5.7}$$

where $(\Lambda_{\max}, \Delta\Theta_{\max})$ are constant norm upper bounds for the control uncertainty Λ and the parameter estimation error $\Delta\Theta$, respectively.

In (5.7), the first four terms are non-positive. The last term is of order $\mathrm{O}(v)\mathrm{O}\left(\|e\|^2\right)$. Repeating the derivations from [1, Section 5], it is possible to formulate verifiable sufficient conditions that would guarantee Uniform Ultimate Boundedness[8] (UUB) of the tracking error e, with respect to a neighborhood of the origin whose radius is directly proportional to the design parameter v. In this case, UUB tracking "tends to" global asymptotic tracking at the rate of $\mathrm{O}(v)$, as $v \to 0$. A summary of our design and analysis is given in the statement below.

Theorem 5.1

Consider the system dynamics (2.1), the reference model (3.1), and the control law (4.1) – all as defined in Theorem 4.1. In addition, suppose that an output matrix $C \in R^{m\times n}$ is selected to satisfy the "squaring-up conditions" (2.3),

$$\det\left[C\,B\right] \neq 0, \quad \det\left[C\left(s\,I_{n\times n} - A_{ref}\right)^{-1} B\right] \neq 0$$

for all $\operatorname{Re} s > 0$. Then there exists a sufficiently small parameter $v > 0$, such that the adaptive law (5.4),

$$\dot{\hat{\Theta}} = \Gamma_{\Theta}\, \operatorname{Proj}\left(\hat{\Theta},\ \Phi\left(x\right) e^T\, C^T\, R_0^{-\frac{1}{2}}\, W\right)$$

enforces global UUB property of the state tracking error $e = x - x_{ref}$, with the ultimate bound of order $\mathrm{O}\left(v\right)$.

Remark 5.1

The adaptive law dynamics (5.4) do not explicitly depend on the small parameter v. However, the observer gain L_v from (4.5) is inversely proportional to v.

$$L_v = P_v\, C^T\, R_v^{-1} = \left(1 + \frac{1}{v}\right) P_v\, C^T\, R_0^{-1}$$

In [1, Section III, Eq. 3.41], we have shown that as $v \to 0$,

$$P_v\, C^T = B\, W^T \sqrt{R_0} + \mathrm{O}\left(v\right)$$

In this case,

$$L_v = \left(1 + \frac{1}{v}\right)\left(B\, W^T \sqrt{R_0} + \mathrm{O}\left(v\right)\right) R_0^{-1} = \mathrm{O}\left(\frac{1}{v}\right)$$

and consequently, too-small values of v may lead to undesirable effects associated with a high gain observer, such as increased sensitivity to process noise and loss of robustness[9,10]. These considerations should be taken into account during the control design process by placing a lower bound on the tuning parameter v.

Remark 5.2

From a practical perspective, the adaptation process (5.4) has the advantage of being invariant to ARE solutions. However, it provides bounded tracking performance. Using the original adaptive laws (4.20), gives asymptotic tracking and, because of (4.14), the desired directions of adaptation can be recovered asymptotically with respect to the design tuning parameter v.

VI. CONCLUSIONS

In this paper, we presented practical methods to design state feedback MRAC systems for MIMO dynamics with linear-in-parameters matched uncertainties. We have employed asymptotic properties of ARE solutions to construct state feedback adaptive laws such that their dynamics evolve near or along pre-specified directions in the system state space. Future research directions will focus on formal analysis and quantification of transient performance of the developed MRAC architectures.

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