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Control of linear systems with input saturation and matched uncertainty and disturbance

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Abstract— **In this paper, we consider the problems of semi-global and global internal stabilization along with disturbance rejection for the case of so-called matched uncertainties and disturbances for linear systems subject to actuator saturation. We develop here low-and-high gain and scheduled low-and-high gain state as well as measurement feedback design methodologies to solve the posed stabilization and disturbance rejection problems.**

I. INTRODUCTION

Almost all physical systems operate under a variety of inherent constraints, limitations, as well as uncertainties and external disturbances. One of the ubiquitous physical limitations is input saturation. Over the last two decades, stabilization of linear systems subject to input saturation have received intense renewed interest from control community, see for instance [1], [2], [3], [12], [13], [15] and references therein. Based on these works, what are known as low-gain and scheduled low-gain desgin methodologies were developed in [5], [6] for semi-global stabilization, and in [9] for global stabilization of linear systems subject to input saturation.

Uncertainties and disturbances are also inevitable in many control engineering applications, where we have to face a situation of both input saturation and various uncertainties and disturbances. There have been several studies on the problems of semi-global and global robust stabilization and disturbance rejection for linear systems with input saturation. For the case when the uncaertainties and disturbances are input additive, these problems have been resolved in [4], [7], [8], [10]. Note that in the presence of disturbances, the low-gain feedback cannot solve the disturbance rejection problems (see for instance [4]). A so-called low-and-high gain design methodology was first proposed in [7] for systems with a special structure of having only a chain of intergrators, and then it was generalized later on in [8] and [10] for general linear systems subject to actuator saturation. The low-and-high gain feedback when appropriately designed can achieve semi-global stabilization and disturbance rejection. In [4], a scheduled low-and-high gain design methodology was developed where both low-gain and high-gain parameters are scheduled simultaneously to solve the global stabilization problems studied in [10]. Here the scheduling of low-gain parameter is based on [9], however the high gain parameter is scheduled in a different fashion.

In this paper, following the innovations of [10] and [4], we investigate the semi-global and global robust stabilization and disturbance rejection in the presence of input saturation and matched uncertainty and disturbance. That is, we consider a system of the form,

$$\dot{x} = Ax + B\sigma(u(t)) + Bf(x,t).$$

Clearly, the magnitude of $f(x,t)$ has to be restricted since the input is bounded. To illustrate this, simply consider a double integrator with a constant disturbance $|f(x,t)| = 1$. Because of input saturation, the state will be driven to infinity regardless of the controller we use. Therefore, we assume that $|f(x,t)| < 1-\delta$, for any a priori given $\delta \in (0,1)$. Under such an assumption, we expand and generalize the low-and-high gain design and scheduled low-and-high gain design methodologies from the input-additive case to the matched case.

This paper is organized as follows: In Section II, we formulate formally the problems to be studied in the paper. A low-and-high gain design is introduced in Section III which solves the semi-global robust stabilization and disturbance rejection problem by state feedback. In Section IV, a scheduled low-and-high gain controller is constructed to solve the global counterpart by state feedback. Section V considers semi-global observer based measurement feedback designs.

II. PROBLEM FORMULATION

Consider a linear system:

$$\Sigma : \begin{cases} \dot{x} = Ax + B\sigma(u(t)) + Bf(x,t), \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are the state, control input, and measured output respectively, and $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a standard saturation defined as $\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_m(u_m)]$ where $\sigma_i(s) = \operatorname{sgn}(s)\min\{|s|,1\}$. Moreover, the term $f(x,t)$ represents an unknown uncertainty or disturbance. Without loss of generality, we assume here that B and C have full rank.

We make the following assumptions:

Assumption 1: The given system (1) is *asymptotically null controllable with bounded control* (ANCBC), or equivalently the given system (1) in the absence of saturation is stabilizable and has all its open-loop poles in the closed left-half plane.

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Assumption 2: The given system (1) in the absence of saturation and uncertain element $f(x,t)$, which is then characterized by the triple (A,B,C), is left invertible and minimum phase. Moreover, we assume that the matrix pair (A,C) is detectable.

Assumption 3: The uncertainty and disturbance $f(x,t)$ is piecewise continuous in t and locally Lipschitz in x, and satisfies

$$\|f(x,t)\| \leq 1-\delta \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

for some $\delta \in (0,1)$.

Assumption 4: The norm of $f(x,t)$ is bounded by a known function

$$\|f(x,t)\| \leq f_0(\|x\|) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

where $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz and satisfies $f_0(0) = 0$.

We present two formal problem statements, first one for the semi-global case and the second one for the global case, utilize state feedback.

Problem 1: Consider the given system (1), and let Assumptions 1 and 3 be satisfied. The semi-global stabilization problem is to find, if possible, for any arbitrary large bounded subset $\mathscr{W} \subset \mathbb{R}^n$ and arbitrary small bounded subset $\mathscr{W}_0 \subset \mathbb{R}^n$ containing the origin, a state feedback law $u = Fx$, such that the closed-loop system satisfies the following conditons:

1) Any trajectory starting in $\mathscr{W}$ will enter $\mathscr{W}_0$ and remain in $\mathscr{W}_0$ thereafter.
2) If $f(x,t)$ satisfies Assumption 4 for a certain f_0, then the equalibrium point $x = 0$ is locally asympotically stable with $\mathscr{W}$ contained in its domain of attraction.

Problem 2: Consider the system (1) satisfying Assumption 1 and 3. The global stabilization problem is to find, if possible, for any arbitrary small bounded subset $\mathscr{W}_0 \subset \mathbb{R}^n$ containing the origin, a state feedback law $u = s(x,t)$, such that the closed-loop system satisfies the following conditons:

1) For all initial conditions in $\mathbb{R}^n$, the trajectories will enter $\mathscr{W}_0$ and remain in $\mathscr{W}_0$ thereafter.
2) If $f(x,t)$ satisfies Assumption 4 for a certain f_0, then the equalibrium point $x = 0$ is globally asympotically stable.

Next, we present a formal problem statement for the semi-global case, utilizes measurement feedback.

Problem 3: Consider the given system (1), and let Assumptions 1, 2, and 3 be satisfied. The semi-global stabilization problem is to find, if possible, for any arbitrary large bounded subset $\mathscr{W} \subset \mathbb{R}^{2n}$ and arbitrary small bounded subset $\mathscr{W}_0 \subset \mathbb{R}^{2n}$ containing the origin, a measurement feedback law,

$$\begin{cases} \dot{\hat{x}} = g(\hat{x},y,t), \hat{x} \in \mathbb{R}^n \\ u = h(\hat{x},t), \end{cases}$$

such that the closed-loop system satisfies the following conditons:

1) Any trajectory starting in $\mathscr{W}$ will enter $\mathscr{W}_0$ and remain in $\mathscr{W}_0$ thereafter.
2) if $f(x,t)$ satisfies Assumption 4 along with a given f_0, the equalibrium point $x = 0$ is locally asympotically stable with $\mathscr{W}$ contained in its domain of attraction.

III. Semi-global state feedback designs

In this section, we construct a low-high-gain feedback control law which can solve Problem 1.

Let $P_\varepsilon > 0$ be the solution of the continuous-time algebraic Riccati equation,

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon I = 0. \tag{2}$$

Since the system is stabilizable, such a P_ε always exists. Moreover, since all eigenvalues of A are in the closed left half plane, $P_\varepsilon \to 0$ as $\varepsilon \to 0$. The low-gain controller is then given by

$$u_L = -B'P_\varepsilon x$$

We choose a high gain state feedback law of the form, $u_H = -\rho B'P_\varepsilon x$ and P_ε is the same as in the low-gain feedback design while $\rho \geq 0$ is to be determined.

The low-high-gain state feedback control law is formed by adding together a low and high gain feedback control. We have

$$u = F_{LH}(\varepsilon,\rho)x = u_L + u_H = -(1+\rho)B'P_\varepsilon x. \tag{3}$$

We claim that the controller (3) solves Problem 1 for appropriately chosen ε and ρ, as stated formally in the following theorem:

Theorem 1: Consider the given system (1) that satisfies Assumption 1. For any bounded subsets $\mathscr{W} \subset \mathbb{R}^n$ and $\mathscr{W}_0 \subset \mathbb{R}^n$ containing the origin, there exists an ε^* such that for each $\varepsilon \in (0,\varepsilon^*]$ there exists a ρ^* with the property that for $\rho > \rho^*$, the low-high-gain feedback $u = F_{LH}x$ solves Problem 1.

Proof: Let c be such that

$$c = \sup\{x'P_\varepsilon x \mid \varepsilon \in (0,1], x \in \mathscr{W}\}.$$

Define $V(x) = x'P_\varepsilon x$ and $\mathscr{L}_v(c) = \{x | V(x) < c\}$. There exists an $\varepsilon^* \in (0,1]$ such that for all $x \in \mathscr{L}_v(c)$ we have $\|B'P_\varepsilon x\| \leq \delta$.

Consider the derivative of $V = x'P_\varepsilon x$ along the trajectory of the closed-loop system. We have

$$\dot{V}(x) \leq -x'Q_\varepsilon x - 2x'P_\varepsilon B[\sigma((1+\rho)B'P_\varepsilon x) - B'P_\varepsilon x - f(x,t)].$$

Denote $B'P_\varepsilon x$ by v and denote the ith component of v and $f(x,t)$ by v_i and f_i respectively. We have

$$\dot{V} \leq -\frac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)}V(x) - 2v'[\sigma(v+\rho v) - v - f(x,t)].$$

We know that

$$|v_i| < \delta, \quad \text{and} \quad |f_i| < 1-\delta. \tag{4}$$

This implies that $|v_i + f_i| < 1$.

If $|\rho v_i| > |f_i|$, then

$$|v_i + \rho v_i| = |v_i| + |\rho v_i| \geq |v_i| + |f_i| \geq |v_i + f_i|.$$

Together with (4), we get

$$-v_i[\sigma(v_i + \rho v_i) - (v_i + f_i)] < 0.$$

If $|\rho v_i| < |f_i|$, we have

$$|v_i + \rho v_1| = |v_i| + |\rho v_i| < |v_i| + |f_i|.$$

Then (4) implies that $|v_i + \rho v_1| < 1$. Therefore, we get

$$-2v_i[\sigma(v_i + \rho v_i) - (v_i + f_i)] \le -2v_i[\rho v_i - f_i] \le \tfrac{f_i^2}{2\rho}.$$

Hence,

$$\dot{V}(x) \le -\tfrac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)} V(x) + \sum_{i=1}^{m} \tfrac{f_i^2}{2\rho}. \tag{5}$$

Since $|f_i| < 1$, we get

$$\dot{V}(x) \le -\tfrac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)} V(x) + \tfrac{m}{2\rho}.$$

Choose ν such that $\mathscr{L}_v(\nu) \subset \mathscr{W}_0$. Define

$$\rho_1^* = \tfrac{m\lambda_{\max}(P_\varepsilon)}{2\nu\lambda_{\min}(Q_\varepsilon)}.$$

If $\rho > \rho_1^*$, we have $\dot{V} < 0$ for all $x \in \mathscr{L}_v(c)$ for which $x \notin \mathscr{L}_v(\nu)$. This implies that any trajectory starting from $\mathscr{W}$ will enter and remain in $\mathscr{W}_0$ within finite time.

If $f(x,t)$ satisfies Assumption 4, we can define

$$M = \sup\{\tfrac{f_0(s)}{s} \mid s \in (0, c/\sqrt{\lambda_{\min}(P_\varepsilon)}]\}.$$

Such a M exists because f_0 is locally Lipschitz.

Therefore, from (5), we can conclude that for $x \in L_v(c)$,

$$\dot{V}(x) \le [-\lambda_{\min}(Q_\varepsilon) + \tfrac{mM^2}{2\rho}]\|x\|^2.$$

Define

$$\rho_2^* = \tfrac{mM^2}{2\lambda_{\min}(Q_\varepsilon)}.$$

If $\rho > \rho_2^*$, we have $\dot{V} < 0$ for all $x \in \mathscr{L}_v(c)$. Hence the origin is asymptotically stable with $\mathscr{W}$ contained in its domain of attraction. ■

IV. Global low-high-gain state feedback designs

In this paper, we use the same scheduling of low-gain parameter as in [4] which is developed in [9]. Consider

$$\varepsilon_s(x) = \max\{\, r \in (0,1] \mid (x'P_r x)\,\mathrm{trace}\left[B'P_rB\right] \le \delta^2 \,\}. \tag{6}$$

Choose the scheduled high-gain parameter ρ_s as

$$\rho_s(x) = \tfrac{\rho_0[g^2(\|x\|)+1]\lambda_{\max}(P_{\varepsilon_s(x)})}{2\lambda_{\min}(Q_{\varepsilon_s(x)})} \tag{7}$$

where ρ_0 is to be determined and $g(x)$ is defined as follows: if Assumption 4 is not satified, $g(x) \equiv 0$; if Assumption 4 is satisfied, $g(x)$ is any locally Lipschitz function such that $g(x) \ge \frac{f_0(x)}{x}$. Such a $g(x)$ exists since $f_0(x)$ is locally Lipschitz and $f_0(0) = 0$.

We claim that the controller constructed in the preceding section together with the scheduling low-gain and high-gain parameters solves Problem 2.

Theorem 2: Consider the given system (1) that satisfies Assumptions 1 and 3. For any bounded subset $\mathscr{W}_0$ there exists ρ_0^* such that the low-high-gain feedback controller (3) when ε and ρ are replaced with the scheduling parameters $\varepsilon_s(x)$ as in (6) and $\rho_s(x)$ as in (7) with $\rho_0 > \rho_0^*$ solves Problem 2.

Proof: Consider the derivative of $V = x'P(\varepsilon_s)x$ along any trajectory,

$$\dot{V}(x) \le -\lambda_{\min}(Q)\|x\|^2 - 2v'[\sigma(v+\rho v) - v - f(x,t)] + x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x.$$

As shown in the previous section,

$$\begin{aligned}\dot{V}(x) &\le -\tfrac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})}V(x) + \sum_{i=1}^{m}\tfrac{f_i^2}{2\rho_s} + x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x \\ &\le -\tfrac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})}[V(x) - \tfrac{m}{\rho_0}] + x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x.\end{aligned}$$

Let $\nu \le 1$ be such that $\mathscr{L}_v(\nu) \subset \mathscr{W}_0$. Define $\rho_0^* = \frac{m}{\nu}$. We have

$$\dot{V} < x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x, \quad \forall \rho_0 \ge \rho_0^* \text{ and } x \notin \mathscr{L}_V(\nu). \tag{8}$$

Assume that $\dot{V} \ge 0$ for some $x(t) \notin \mathscr{L}_V(\nu)$. We have two possible cases:

1) *Case I:* $\varepsilon_s(x) = 1$. We have $\frac{dP_{\varepsilon_s}}{dt} = 0$. But then (8) implies that $\dot{V} < 0$. This yields a contradiction.
2) *Case II:* $\varepsilon_s(x) \ne 1$. Note that $V\,\mathrm{trace}\left[B'P_{\varepsilon_s(x)}B\right] = \delta^2$ whenever $\varepsilon_s \ne 1$. Hence $\dot{V} \ge 0$ implies that $\frac{dP_{\varepsilon_s}}{dt} \le 0$. But (8) gives $\dot{V} < 0$. This yield a contradiction.

Therefore, we conclude that $\dot{V} < 0$ for all $x \notin \mathscr{L}_V(\nu)$. Any trajectory will enter and remain in $\mathscr{W}_0$ after finite time.

If $f(x,t)$ satisfies Assumption 4, we have

$$\begin{aligned}\dot{V}(x) &\le -\tfrac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})}V(x) + \sum_{i=1}^{m}\tfrac{f_i^2}{2\rho_s} + x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x \\ &\le -\tfrac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})}V(x)(1 - \tfrac{m}{\rho_0}) + x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x.\end{aligned}$$

we get

$$\dot{V} < x'\tfrac{dP_{\varepsilon_s(x)}}{dt}x, \quad \forall x \ne 0 \text{ and } \rho_0 > \rho_0^*.$$

We have $\dot{V} < 0$ for all $x \ne 0$. Therefore the origin is globally asymptotically stable. ■

V. Semi-global observer based measurement feedback designs

Before we proceed with our design, it is necessary to introduce a Special Coordinate Basis (SCB) of the given system (1) in the absence of saturation and uncertain element $f(x,t)$. Consider

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Without loss of generality, we assume that B and C have full rank. Then, there exist nonsingular transformation matrices Γ_x, Γ_y and Γ_u such that

$$\bar{x} = \Gamma_x^{-1}x \quad \bar{y} = \Gamma_y^{-1}y \quad \bar{u} = \Gamma_u^{-1}u,$$

where

$$\begin{gathered}\bar{x} = \begin{pmatrix} x_a' & x_b' & x_c' & x_d' \end{pmatrix}', \bar{y} = \begin{pmatrix} y_b' & y_d' \end{pmatrix}', \bar{u} = \begin{pmatrix} u_c' & u_d' \end{pmatrix}', \\ x_b = \begin{pmatrix} x_{b,1}' & \dots & x_{b,m_b}' \end{pmatrix}', \quad x_{b,i} = \begin{pmatrix} x_{b,i_1} & \dots & x_{b,i_{r_i}} \end{pmatrix}', \\ x_d = \begin{pmatrix} x_{d,1}' & \dots & x_{d,m_d}' \end{pmatrix} \quad x_{d,i} = \begin{pmatrix} x_{d,i_1} & \dots & x_{d,i_{q_i}} \end{pmatrix}', \\ y_b = \begin{pmatrix} y_{b,1} & \dots & y_{b,m_b} \end{pmatrix}', \quad y_d = \begin{pmatrix} y_{d,1} & \dots & y_{d,m_d} \end{pmatrix}', \\ u_d = \begin{pmatrix} u_{d,1} & \dots & u_{d,m_d} \end{pmatrix}',\end{gathered}$$

and where x_a, x_b, x_c and x_d are of dimension n_a, n_b, n_c and n_d respectively, y_b and y_d are of dimension m_b and m_d respectively, u_c and u_d are of dimension $m-m_d$ and m_d respectively,

$$n = n_a+n_b+n_c+n_d, \quad n_b = \textstyle\sum_{i=1}^{m_b} r_i, \quad n_d = \textstyle\sum_{i=1}^{m_d} q_i, \\ m_b+m_d = p.$$

In the new coordinate basis, we have

$$\begin{aligned} \dot{x}_a &= A_{aa}x_a + L_{ab}y_b + L_{ad}y_d; \\ \dot{x}_c &= A_{cc}x_c + L_{cb}y_b + L_{cd}x_d + B_c[E_{ca}x_a + u_c]. \end{aligned}$$

For $i = 1,\dots,m_b$,

$$\begin{aligned} \dot{x}_{b,i} &= A_{r_i}x_{b,i} + L_{b_ib}y_b + L_{b_id}y_d \\ y_{b,i} &= C_{r_i}x_{b,i} = x_{b,i_1}. \end{aligned}$$

For $i = 1,\dots,m_d$,

$$\begin{aligned} \dot{x}_{d,i} &= A_{q_i}x_{d,i} + L_iy_d + B_{qi}[u_{d,i} + E_{ia}x_a + E_{ib}x_b + E_{ic}x_c \ + E_{id}x_d] \\ y_{d,i} &= C_{q_i}x_{d,i} = x_{d,i_1}, \end{aligned}$$

where

$$A_r = \begin{pmatrix} 0 & I_{r-1} \\ 0 & 0 \end{pmatrix}, \qquad \begin{aligned} B_r &= \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}', \\ C_r &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

The SCB given above displays explicitly both finite and infinite zero structure of the system given in (9), and has a number of important properties (see [11], [14]). We need to stress that in view of these SCB properties, Assumption 2 implies that A_{aa} is Hurwitz, and x_c and u_c do not exist. Hence we have $m_d = m$ and $m_b = p-m$. Moreover, the input transformation $\Gamma_u = I$, in another word, we don't need to transform the input.

We now proceed to implement the low-and-high gain controller designed in Section III using a high-gain observer under Assumption 2.

The measurement feedback is of the form:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(\ell)(y - C\hat{x}) \\ u = F_{LH}(\varepsilon,\rho)\hat{x} \end{cases} \tag{10}$$

where $F_{LH}(\varepsilon,\rho)$ is given by (3) and where parameterized observer gain $L(\ell)$ is designed shortly as given in (11).

We construct the high gain observer in following steps:

Step 1: Transform the system into the Special coordinate basis. Given Assumption 2 satisfied, we have

$$\begin{aligned} \bar{x} &= \begin{pmatrix} x_a' & x_b' & x_d' \end{pmatrix}', \ \bar{y} = \begin{pmatrix} y_b' & y_d' \end{pmatrix}', \ \bar{u} = \begin{pmatrix} u_{d,1} & \dots & u_{d,m} \end{pmatrix}' \\ x_b &= \begin{pmatrix} x_{b,1}' & \dots & x_{b,p-m}' \end{pmatrix}', \quad x_{b_i} = \begin{pmatrix} x_{b,i_1} & \dots & x_{b,i_{r_i}} \end{pmatrix}' \\ x_d &= \begin{pmatrix} x_{d,1}' & \dots & x_{d,m}' \end{pmatrix}', \quad x_{d_i} = \begin{pmatrix} x_{d,i_1} & \dots & x_{d,i_{q_i}} \end{pmatrix}' \\ y_b &= \begin{pmatrix} y_{b,1} & \dots & y_{b,p-m} \end{pmatrix}', \quad y_d = \begin{pmatrix} y_{d,1} & \dots & y_{d,m} \end{pmatrix}' \end{aligned}$$

$$\dot{x}_a = A_{aa}x_a + L_{ab}y_b + L_{ad}y_d,$$

and for $i = 1,\dots,p-m$,

$$\begin{aligned} \dot{x}_{b,i} &= A_{r_i}x_{b,i} + L_{b_ib}y_b + L_{b_id}y_d \\ y_{b,i} &= C_{r_i}x_{b,i} = x_{b,i_1} \end{aligned}$$

For $i = 1,\dots,m$,

$$\begin{aligned} \dot{x}_{d,i} &= A_{q_i}x_{d,i} + L_iy_d + B_{qi}[u_{d,i} + E_{ia}x_a + E_{ib}x_b + E_{id}x_d] \\ y_{d,i} &= C_{q_i}x_{d,i} = x_{d,i_1}, \end{aligned}$$

Step 2, Since (A_{r_i}, C_{r_i}) is observable, for $i = 1$ to $p-m$, choose $L_{b,i} \in \mathbb{R}^{r_i\times 1}$ such that $A^c_{r_i} = A_{r_i} - L_{b,i}C_{r_i}$ is Hurwitz.

Similarly, (A_{q_i}, C_{q_i}) is observable. For $i = 1$ to m, choose $L_{d,i} \in \mathbb{R}^{q_i\times 1}$ such that $A^c_{q_i} = A_{q_i} - L_{d,i}C_{q_i}$ is Hurwitz

Step 3, For any $\ell \in (0,1]$, define a matrix $L(\ell) \in \mathbb{R}^{n\times p}$ as

$$L(\ell) = \Gamma_x \begin{pmatrix} L_{ab} & L_{ad} & \\ L_{bb} + L_b(\ell) & L_{bd} & \\ 0 & L_{dd} & L_d(\ell) \end{pmatrix} \Gamma_y^{-1}, \tag{11}$$

where

$$L_{bb} = \begin{pmatrix} L_{b_1b} \\ L_{b_2b} \\ \vdots \\ L_{b_{p-m}b} \end{pmatrix}, \ L_{bd} = \begin{pmatrix} L_{b_1d} \\ L_{b_2d} \\ \vdots \\ L_{b_md} \end{pmatrix}, \ L_{dd} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix},$$

$$\begin{aligned} L_b(\ell) &= \text{blkdiag}\left\{S_{r_i}(\ell)L_{b,i}\right\}_{i=1}^{p-m}, \\ L_d(\ell) &= \text{blkdiag}\left\{S_{q_i}(\ell)L_{d,i}\right\}_{i=1}^{m} \end{aligned}$$

and $S_r(\ell) = \text{blkdiag}\left\{\ell^i\right\}_{i=1}^{r}$ for any integer $r \geq 1$.

We have following theorem

Theorem 3: Consider the system (1). Let Assumptions 1, 2, 3 be satisfied. There exist ε^*, $\rho*$ and ℓ^* such that for any $\varepsilon \in (0,\varepsilon^*]$, $\rho > \rho^*$ and $\ell > \ell^*$, the measurement feedback controller (10) solves the Problem 3.

In order to prove this theorem, we need to establish the following lemmas. Let $\tilde{\Sigma}$ denote the system,

$$\tilde{\Sigma}: \begin{cases} \dot{x} = Ax + B[\sigma(u) + f(x+Te,t) + Ee] \\ \dot{e} = A_oe, \end{cases} \tag{12}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $e \in \mathbb{R}^k$ and A_o is Hurwitz stable. Let P_o be the solution of the Lyapunov equation, i.e.

$$A_o'P_o + P_oA_o = -I.$$

Define $\tau = \sqrt{\lambda(E'E)}$ and $\kappa = \sqrt{\lambda(T'T)}$.

Lemma 1: Given $\delta, \varepsilon \in (0,1)$. Let $c > 0$ be such that

$$\|B'P_\varepsilon x\| < 1 \quad \forall x \in \{x \in \mathbb{R}^n : x'P_\varepsilon x < c^2+1\},$$

where P_ε is as in (2). Define

$$\begin{aligned} &\gamma = \frac{\max\{1,(\tau^2+1)\lambda_{\max}(P_o)\}}{\min\{1,\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(Q_\varepsilon)}\}}, \quad M = \sup_{s\in(0,F)}\left\{\frac{g_0(s)}{s}\right\}, \\ &F = \sqrt{c^2+1}\left(\sqrt{\lambda_{\min}(P_\varepsilon)^{-1}} + \kappa\sqrt{[(\tau^2+1)\lambda_{\min}(P_o)]^{-1}}\right), \\ &\rho_1^* = \frac{2m}{(c^2+1)\gamma}, \quad \rho_2^* = \frac{2mM^2}{\lambda_{\min}(Q_\varepsilon)}, \quad \rho_3^* = 2mM^2\kappa^2. \end{aligned}$$

Assume $\rho > \max\{\rho_1^*, \rho_2^*, \rho_3^*\}$. For the system $\tilde{\Sigma}$ that satisfies Assumption 1 and 3, and with controller (3), there exists a continuous function $\psi : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^+$ such that the function

$$V(x,e) = x'P_\varepsilon x + (\tau^2+1)e'P_oe$$

satisfies $\dot{V} \leq -\psi(x,e)$.

If Assumption 4 is not satisfied, then

$$(x',e')' \in L_V(c^2+1) \Rightarrow \psi(x,e) \geq \gamma\left(V - \tfrac{\rho_1^*}{\rho}\tfrac{c^2+1}{2}\right).$$

If Assumption 4 is satisfied, then

$$(x',e')' \in L_V(c^2+1) \Rightarrow \psi(x,e) \geq 0.5\gamma V.$$

Proof: Note that $u = -(1+\rho)B'P_\varepsilon x$. We denote $B'P_\varepsilon x$ by v and denote the ith component of v and $f(x+Te,t)$ by v_i

and f_i respectively. Consider the derivative of V along the trajectory in the set $L_V(c^2+1)$,

$$\dot{V} = -x'Q_\varepsilon x - 2v'[\sigma((1+\rho)v) - f(x+Te,t) - v] - v'v + 2v'Ee - (\tau^2+1)e'e. \quad (13)$$

Similar with the proof in Section III, we have

$$2v'[\sigma((1+\rho)v) - f(x+Te,t) - v] \le \sum_{i=1}^{m} \frac{f_i^2}{2\rho} \le \frac{m}{2\rho}.$$

Hence

$$\dot{V} \le -x'Q_\varepsilon x + \sum_{i=1}^{m} \frac{f_i^2}{2\rho} - e'e \le -\gamma[V(x) - \frac{\rho_1^*}{\rho}\frac{c^2+1}{2}].$$

Moreover, if Assumption 4 is satisfied and $\rho > \max\{\rho_1^*, \rho_2^*\}$, we have

$$\begin{aligned}\dot{V}(x) =& -x'Q_\varepsilon x + \tfrac{mM^2\|x+Te\|^2}{2\rho} - e'e \\ \le& -x'Q_\varepsilon x + \tfrac{mM^2(\|x\|^2+\kappa^2\|e\|^2)}{\rho} - e'e \\ \le& -[0.5\lambda_{\min}(Q_\varepsilon) - \tfrac{mM^2}{\rho}]\|x\|^2 - [0.5 - \tfrac{mM^2\kappa^2}{\rho}]\|e\|^2 \\ & - 0.5\gamma V(x) \\ \le& -0.5\gamma V(x).\end{aligned}$$

■

The following Lemma is the same as Lemma 4 in [10], which is adapted from [16].

Lemma 2: Consider the nonlinear system

$$\begin{cases} \dot{z} = f(z,e,t), \quad z \in \mathbb{R}^n; \\ \dot{e} = \ell Ae + g(z,e,t), \quad e \in \mathbb{R}^m \end{cases}$$

where $\ell > 0$ and A is Hurwitz matrix. Assume that for the system $\dot{z} = f(z,0,t)$, there exists a neighborhood $\mathscr{W}_1$ of the origin in $\mathbb{R}^n$ and a $\mathscr{C}_1$ function $V_1 : \mathscr{W}_1 \to R^+$ which is positive definte on $\mathscr{W}_1 \setminus \{0\}$ and proper on $\mathscr{W}_1$ and satisfies

$$\tfrac{\partial V_1}{\partial z} f(z,0,t) \le -\psi_1(z),$$

where $\psi_1(z)$ is continuous on $\mathscr{W}_1$ and positive definite on $\{z : \nu_1 < V_1(z) \le c_1+1\}$ for some nonnegative real number $\nu_1 < 1$ and some real number $c_1 \ge 1$. Also assume that there exist positive real numbers α and β and a bounded function γ with $\gamma(0) = 0$ satisfying

$$\left.\begin{aligned} \|f(z,e,t) - f(z,0,t)\| \le \gamma(\|e\|) \\ \|g(z,e,t)\| \le \alpha\|e\| + \beta \end{aligned}\right\}$$
$$\forall (z,e,t) \in \{z \in \mathbb{R}^n : V_1(z) \le c_1+1\} \times \mathbb{R}^m \times \mathbb{R}^+$$

Let $c_2(\ell)$ be a class $\mathscr{K}_\infty$ function satisfying $\lim_{\ell\to\infty} \frac{\ell}{c_2^4(\ell)} = \infty$ and P solves the Lyapunov equation $A'P + PA = -I$. Define the function

$$V(z,e) = c_1 \frac{V_1(z)}{c_1 + 1 - V_1(z)} + c_2(\ell)\frac{\ln(1+e'Pe)}{c_2(\ell)+1-\ln(1+e'Pe)},$$

and the set

$$\mathscr{W} := \{z : V_1(z) < c_1+1\} \times \{e : \ln(1+e'Pe) < c_2(\ell)+1\}.$$

Then, for $\ell > 0$, $V : \mathscr{W} \to R^+$ is positive definite on $\mathscr{W}$ $\{0\}$ and proper on $\mathscr{W}$. Furthermore, for any $\nu_2 \in (0,1)$, there exists an $\ell^*(\nu_2) > 0$ such that, for all $\ell \in [\ell^*(\nu_2), \infty)$, the derivative of V along the trajectories of systems satisfies $\dot{V} \le -\psi_2(z,e)$ where $\psi_2(z,e)$ is positive definite on $\{(z,e) : \nu_1 + \nu_2 \le V(z,e) \le c_1^2 + c_2^2(\ell) + 1\}$.

Next, we proceed to prove theorem 3.

Proof of theorem 3: Consider the closed-loop system of (1) and (10),

$$\begin{cases} \dot{x} = Ax + B[\sigma(u) + f(x,t)] \\ \dot{\hat{x}} = A\hat{x} + Bu + L(\ell)(y - C\hat{x}) \\ u = F_{LH}(\varepsilon,\rho)\hat{x}. \end{cases} \quad (14)$$

Using the state and output transformation Γ_x and Γ_y, we transform the system into its SCB form,

$$\bar{x} = \Gamma_x^{-1}x = \begin{pmatrix} x_a' & x_b' & x_d' \end{pmatrix}', \; \bar{\hat{x}} = \Gamma_x^{-1}x = \begin{pmatrix} \hat{x}_a' & \hat{x}_b' & \hat{x}_d' \end{pmatrix}'.$$

We construct a new state as

$$\tilde{x} = \Gamma_x \begin{pmatrix} \hat{x}_a' & x_b' & x_d' \end{pmatrix}, \quad \tilde{e} = \begin{pmatrix} e_a' & e_b' & e_d' \end{pmatrix}',$$

where $e_a = x_a - \hat{x}_a$, $e_b = S_b(\ell)(x_b - \hat{x}_b)$, $e_d = S_d(\ell)(x_d - \hat{x}_d)$,

$$S_b(\ell) = \text{blkdiag}\left\{\ell^{r_i} S_{r_i}^{-1}(\ell)\right\}_{i=1}^{p-m},$$
$$S_d(\ell) = \text{blkdiag}\left\{\ell^{q_i} S_{q_i}^{-1}(\ell)\right\}_{i=1}^{m}.$$

We denote $e_{bd} = (e_b', e_d')'$. Then the closed-loop system in the new basis is

$$\dot{\tilde{x}} = A\tilde{x} + B[\sigma(u) + f(\tilde{x} + \Gamma_{xa}e_a, t) + E_a e_a], \quad (15)$$
$$\dot{e}_a = Ae_a, \quad (16)$$
$$\begin{aligned}\dot{\tilde{e}}_{bd} = \ell A_{bd}e_{bd} + B_{bf}[\sigma(u) + f(\tilde{x} + \Gamma_{xa}e_a, t) \\ - u + E_{bd}S_{bd}^{-1}(\ell)e_{bd}],\end{aligned} \quad (17)$$

$$u = F_{LH}(\varepsilon,\rho)[\tilde{x} - \Gamma_{xbd}S_{bd}(\ell)^{-1}e_{bd}], \quad (18)$$

where

$$E_a = \begin{bmatrix} E_{1a} & \dots & Ema \end{bmatrix}$$
$$A_{bd} = diag\{A_{r_1}^c, A_{r_2}^c, \dots, A_{r_{p-m}}^c, A_{q_1}^c, A_{q_2}^c, \dots, A_{q_m}^c\}$$
$$B_{bd} = [0, diag\{B_{q_1}, B_{q_2}, \dots, B_{q_m}\}']'$$
$$E_{bd} = \begin{pmatrix} E_{1b} & E_{1d} \\ E_{2b} & E_{2d} \\ \vdots & \vdots \\ E_{mb} & E_{md} \end{pmatrix}$$
$$S_{bd}(\ell) = \text{blkdiag}\{S_b(\ell), S_d(\ell)\}$$

and $\Gamma_x = (\Gamma_{xa}, \Gamma_{xbd})$ with $\Gamma_{xa} \in \mathbb{R}^{n\times n_a}$ and $\Gamma_{xbd} \in \mathbb{R}^{n\times(n_b+n_d)}$.

Consider the dynamic of $\tilde{x}$ and e_a. We will apply Lemma 1. Set $e_{bd} = 0$ in the closed-loop equations (15), (17) and (18). Then, we have

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B[\sigma(u) + f(\tilde{x} + \Gamma_{xa}e_a, t) + E_a e_a] \\ \dot{e}_a = A_{aa}e_a \\ u = F_{LH}(\varepsilon,\rho)\tilde{x}. \end{cases}$$

By Assumption 2, A_{aa} is Hurwitz stable. Let $P_a > 0$ be the solution of

$$A_{aa}'P_a + P_a A_{aa} = -I.$$

Following Lemma 1, we define

$$V_1(\tilde{x}, e_a) = \tilde{x}'P_\varepsilon\tilde{x} + (\tau^2+1)e_a'P_a e_a,$$

where $\tau = \sqrt{\lambda_{\max} E_a' E_a}$. Let $c_1 > 1$ be such that

$$c_1^2 > \sup\{V_1(\tilde{x}, e_a) \mid (x, \hat{x}) \in \mathscr{W}, \varepsilon \in (0,1)\}.$$

There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*)$

$$\|B' P_\varepsilon \tilde{x}\| < 1, \quad \forall (x, \hat{x}) \in L_{V_1}(c_1) = \{V_1(\tilde{x}, e_a) < c_1^2 + 1\}.$$

Fix $\varepsilon \in (0, \varepsilon^*]$. Let P_{bd} satisfy the Lyapunov equation

$$A_{bd}' P_{bd} + P_{bd} A_{bd} = -I,$$

and let $V_3 = e_{bd}' P_{bd} e_{bd}$. Observe that from the definition of $S_r(\ell)$, if we assume $\ell > 1$, there exists a $k > 0$ such that, for any $r \geq 0$, we have

$$\|(\tilde{x}', e_a', e_{bd}')'\| \leq r \Rightarrow \|(x', \hat{x}')'\| \leq kr. \tag{19}$$

Moreover, this k is independent of ℓ provided that $\ell > 1$.

We can choose $\nu \in (0,1)$, a strictly positive real number such that, for all $\ell > 1$, we have

$$L_{V_1}(\nu) \times L_{V_3}(\exp(\nu) - 1) \subset \mathscr{W}_0.$$

Such a ν exsits since $\mathscr{W}_0$ contains zero in its interior, P_ε, P_a and P_{bd} are positive definite and (19) holds for all $\ell > 1$. It follows from Lemma 1 that if

$$\rho > \max\{\rho_1^*, \tfrac{\rho_1^*(c_1^2+1)}{2\nu}\},$$

we get $\dot{V}_1 \leq -\psi_1(\tilde{x}, e_a)$, where

$$(\tilde{x}', e_a')' \in \{\tfrac{\nu}{4} < V_1(\tilde{x}, e_a) \leq c_1^2 + 1\} \Rightarrow -\psi_1(\tilde{x}, e_a) < 0. \tag{20}$$

Let ρ be fixed. Choose

$$c_2(\ell) = \ln(1 + \lambda_{\max}(P_{bf} R^2 \ell^{2(n_b + n_d)})),$$

where R is such that $(x, \hat{x}) \in \mathscr{W}$ implies that $\|x_b - \hat{x}_b\| < R/2$ and $\|x_d - \hat{x}_d\| < R/2$. Obviously $c_2(\ell)$ is of class $\mathscr{K}_\infty$ and satisfies

$$\lim_{\ell \to \infty} \tfrac{\ell}{c_2^4(\ell)} = \infty.$$

We then define the Lyapunov function

$$V_2(\tilde{x}, e_a.e_{bd}) = \frac{c_1^2 V_1}{c_1^2 + 1 - V_1} + \frac{c_2(\ell) \ln(1 + e_{bd}' P_{bd} e_{bd})}{c_2(\ell) + 1 - \ln(1 + e_{bd} P_{bd} e_{bd})},$$

and the set

$$\begin{aligned} \mathscr{W}_2 = \{(\tilde{x} \times e_a) : V_1(\tilde{x}, e_a) < c_1^2 + 1\} \times \\ \{e_{bd} : \ln(1 + e_{bd}' P_{bd} e_{bd}) < c_2(\ell) + 1\}. \end{aligned} \tag{21}$$

It then follows from Lemma 1 that for all $\ell > 0$, V_2 is positive definite on $\mathscr{W}_2$ $\{0\}$ and proper on $\mathscr{W}_2$. Furthermore, there exists an $\ell^*(\varepsilon, \rho, \nu)$ such that, for all $\ell > \ell^*(\varepsilon, \rho, \nu)$, we have

$$\dot{V}_2 \leq -\psi_2(\tilde{x}, e_a, e_{bd}),$$

where $\psi_2(\tilde{x}, e_a.e_{bd})$ is positive definite on

$$\mathscr{W}_3 := \{(\tilde{x}, e_a, e_{bd}), \nu/2 < V_2 < c_1^4 + c_2(\ell)^2 + 1\}.$$

It is clear that $(x, \hat{x}) \in \mathscr{W}$ implies $V_2 < c_1^4 + c_2^2(\ell)$ and $V_2 < \nu/2$ implies $(x, \hat{x}) \in \mathscr{W}_0$. This completes the proof of item 1 in Problem 3.

If Assumption 4 is satisfied, it follows from Lemma 1 that for $\rho \geq \max\{\rho_2^*, \rho_3^*\}$ and for any $\nu \in (0,1)$, we have

$$(\tilde{x}', e_a')' \in \{\nu/4 < V_1(\tilde{x}, e_a) \leq c_1^2 + 1\} \Rightarrow \psi_1(\tilde{x}, e_a) > 0.5\gamma V_1.$$

This implies that the origin of $(\tilde{x}, e_a)$ is locally exponentially stable. Then for any a priori given neighborhood $\mathscr{H}$ of the origin, the local asymptotic stability of the origin of $(\tilde{x}, e_a, e_{bd})$ with a domain of attraction containning $\mathscr{H}$ follows from the standard singular perturbation result. ■

So far we have discussed semi-global stabilization along with disturbance rejection while utilizing measurement feedback. Along the same lines, a similar result for global stabilization can be developed.

VI. Conclusions

Low-and-high gain and scheduled low-and-high gain state and measurement feedback design methodologies are expanded and generalized to solve semi-global and global internal stabilization along with disturbance rejection for the case of matched disturbances and uncertainties for linear systems subject to actuator saturation.

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