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Computationally Implementable Sufficient Conditions for the Synchronisation of Coupled Dynamical Systems with Time Delays in the Coupling

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***Abstract*— Understanding the mechanisms behind synchronisation of coupled systems is important in biology, chemistry, physics and engineering. For example, research on the synchronisation of coupled circadian clocks is of much importance to better understand many different disease states. Or, different applications in engineering require certificates for synchronised behaviour between the different coupled subsystems. Such systems could correspond to tethered space vehicles or to vehicles in formation flight. In this paper, we provide sufficient conditions for the synchronisation of coupled identical nonlinear dynamical systems with time delays in the coupling, which makes the modelling more realistic. Importantly, the conditions can be efficiently checked using semi-definite programming techniques, which we illustrate through applications.**

NOTATION

$\mathbb{R}$, $\mathbb{R}^{m\times n}$	real numbers, $m\times n$ real matrices
e	$[1,\cdots,1]^{\mathrm{T}}$
A_{ij}	(i,j)th entry of matrix $A\in\mathbb{R}^{m\times n}$
I	the identity matrix
$\mathrm{diag}(A)$	a vector of length n if $A\in\mathbb{R}^{n\times n}$, where $\mathrm{diag}(A)_i = A_{ii}$
$\mathrm{diag}(x)$	a diagonal matrix $\in\mathbb{R}^{n\times n}$ if $x\in\mathbb{R}^n$, where $\mathrm{diag}(x)_{ii} = x_i$
A^{T}	transpose of matrix $A\in\mathbb{R}^{m\times n}$
$\dot{x}$	derivative of x w. r. t. the time variable t
$\otimes$	Kronecker product
$A>0$	positive definite matrix: $x^{\mathrm{T}}Ax>0\ \forall x$ if $x\neq 0$, $x\in\mathbb{R}^n$, $A\in\mathbb{R}^{n\times n}$
$\mathrm{trace}(A)$	trace of matrix A: $\sum_{i=1}^{n} A_{ii}$, $A\in\mathbb{R}^{n\times n}$

I. INTRODUCTION

Synchronisation of behaviour plays an important role in many fields, for example, in decentralised control of automatic space or underwater vehicles, where cohort members communicate only with close neighbours [1]. Another example is the spread of epidemics [2], [3], where the dynamics are governed by major cities with a constant outflow and inflow of infected, who travel between them. Generally speaking, in biology, on the one hand, identifying the conditions that lead to synchronisation is often important for the understanding of many systems [4], [5], [6]. In engineering, on the other hand, for many connected systems synchronisation has to be guaranteed [7], [8].

A numerical approach to check whether coupled identical dynamical systems synchronise locally was introduced by Pecora and Carroll in [9]. Sufficient conditions that guarantee global complete synchronisation (or, in other word, asymptotic stability of the synchronised state) of a system of coupled identical oscillators and that can be checked analytically, albeit not easily, were developed independently of each other by Wu [10], [11], Belykh and colleagues [12], [13] and Slotine and colleagues [14], [15]. The approaches are different but related and based on graph theory and Lyapunov stability theory (or contraction theory). Significantly, these sufficent conditions can be efficiently checked computationally for systems involving trigonometric, polynomial or rational functions using semi-definite programming techniques, which we demonstrated in [16].

Clearly, including delays in the coupling makes the models much more realistic, as often responses to signals, that are sent from each subsystem, arrive with delay. In this paper, we provide novel sufficient conditions for synchronisation of coupled identical nonlinear systems with time delays in the coupling. These conditions are independent of the time delays and – if fulfilled – hold also for strong coupling albeit the delay in the coupling between subsystems. Importantly, they can also be efficiently checked computationally for systems involving trigonometric, polynomial or rational functions. Others have investigated synchronisation in systems with time lags in the coupling. However, their work differs notably from ours. Wu's approach in [17] requires that the time-delayed coupling is relatively weak as opposed to the results presented here. In [18], the systems considered consist only of a delayed linear coupling protocol, while in [19] only two coupled identical systems are considered. Our approach is more general and considers N coupled nonlinear systems, where N can be arbitrarily large.

Section II provides novel sufficient conditions for synchronisation of coupled dynamical systems with time delays in the coupling. In Section II-A, we first present a theorem on boundedness of solutions, since this is required by the main results of this paper, the conditions for synchronisation of

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Section II-B and Section II-C. In Section II-B, we consider an arbitrary coupling topology, while in Section II-C, we investigate all-to-all coupling with equal strength between subsystems. The latter allows us to solve problems of larger size when using the computational approach presented in Section III to check whether our sufficient conditions for synchronisation are fulfilled. In Section IV, we apply our method to different systems. Section V concludes the paper.

II. Sufficient Conditions for Synchronisation of Coupled Dynamical Systems with Time Delays in the Coupling

Consider N coupled identical n-dimensional oscillators given by $x_i \in \mathbb{R}^n$, $i = 1, \ldots, N$, such that the behaviour of the coupled system is described by

$$\dot{x} = \tilde{f}(x) + \kappa(C \otimes D)x \tag{1}$$

where $x^{\mathrm{T}} = (x_1^{\mathrm{T}}, \ \ldots, \ x_N^{\mathrm{T}})$ and $\tilde{f}(x)^{\mathrm{T}} = (f(x_1)^{\mathrm{T}}, \ \ldots, \ f(x_N)^{\mathrm{T}})$. The dynamics of each individual system is determined by function $f(\cdot)$. The second summand in the right hand side of Eq. (1) is the coupling term. The positive constant κ corresponds to the coupling strength. Matrix $D \in \mathbb{R}^{n\times n}$ is the nonnegative output matrix (that is, $D_{ij} \geq 0 \ \forall i,j$) for each oscillator; in other words, it denotes the variables that are used in the coupling. Matrix $C \in \mathbb{R}^{N\times N}$ is the Laplacian matrix of the coupling topology. Its off-diagonal entries are nonnegative and diagonal ones are nonpositive, moreover, $e^{\mathrm{T}}C = 0$ and $Ce = 0$. It follows from Geršgorin's theorem that the real part of the eigenvalues of C is nonpositive.

Consider the following system that has delays in the coupling

$$\dot{x}(t) = \tilde{f}(x(t)) - \kappa(C_d \otimes D)x(t) + \kappa(A \otimes D)x(t-\tau) \tag{2}$$

where $C_d = \mathrm{diag}(\mathrm{diag}(C))$ and $A = C - C_d$. In the following sections, we provide conditions for solution trajectories of each individual system in Eq. (2) to approach with time those of any other subsystem and, thus, for asymptotic stability of the synchronisation manifold. These conditions require that the solution trajectories are bounded (otherwise, they might diverge while approaching each other) or, formally, that $x(t) \in \mathcal{D} \subset \mathbb{R}^{nN} \ \forall t$. For this reason, in the next section, we first provide a methodology to check for boundedness of solutions, before proceeding to the main results. For brevity, we denote $x(t)$ by x and $x(t-\tau)$ by x^τ.

A. Boundedness of solutions of coupled dynamical systems with delay

The following two theorems provide a methodology to check for boundedness of solutions. They make use of the following Lyapunov function and its time derivative

$$\begin{aligned} V(x) &= \frac{1}{2}x^{\mathrm{T}}\tilde{P}_b x + \int_{t-\tau}^{t} x(s)^{\mathrm{T}}\tilde{P}_b S x(s)\mathrm{d}s \\ & \quad S \geq 0, \ \tilde{P}_b > 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{V}(x) &= x^{\mathrm{T}}\tilde{P}_b\dot{x} + x^{\mathrm{T}}\tilde{P}_b S x - x^{\tau\,\mathrm{T}}\tilde{P}_b S x^\tau \\ &= x^{\mathrm{T}}\tilde{P}_b\tilde{f}(x) + \begin{bmatrix} x \\ x^\tau \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x \\ x^\tau \end{bmatrix} \\ Q &= \begin{bmatrix} \tilde{P}_b S - \kappa\tilde{P}_b(C_d\otimes D) & \kappa\tilde{P}_b(A\otimes D) \\ 0 & -\tilde{P}_b S \end{bmatrix} \end{aligned} \tag{4}$$

Theorem 1: Consider the system given by Eq. (2). Let function $V(x)$ be given by Eq. (3) and $S = \alpha I$, where constant $\alpha > 0$. If there exists a constant $\gamma > 0$ such that

$$\dot{V}(x) < 0 \text{ if } g(x) = \frac{1}{2}x^{\mathrm{T}}\tilde{P}_b x > \gamma \tag{5}$$

and the initial conditions are such that $V(x) < \gamma$, for $0 \leq t \leq \tau$, then $g(x) \leq \gamma$ for all time.

Proof: We will prove the theorem by showing that for initial conditions such that $V(x^*) < \gamma$, $0 \leq t^* \leq \tau$, which implies that $g(x^*) < \gamma$, $g(x) > \gamma$ at a later time point only if $\dot{V} > 0$, which contradicts Eq. (5) and, thus, is not possible. For instance, let us assume that $g(x(t_0)) = \gamma$, $t_0 > \tau$, and $g(x(t_1)) = \gamma + \epsilon$, where $0 < t_1 - t_0 \ll 1$ and $0 < \epsilon \ll 1$. Let $\Delta V = V(x(t_1)) - V(x(t_0))$. Then,

$$\begin{aligned} \Delta V &= \epsilon + 2\alpha\int_{t_1-\tau}^{t_1} g(x(s))\mathrm{d}s - 2\alpha\int_{t_0-\tau}^{t_0} g(x(s))\mathrm{d}s \\ &= \epsilon + 2\alpha\int_{t_0}^{t_1} g(x(s))\mathrm{d}s - 2\alpha\int_{t_0-\tau}^{t_1-\tau} g(x(s))\mathrm{d}s \end{aligned}$$

Furthermore, with $\Delta t = t_1 - t_0$,

$$\begin{aligned} \int_{t_0-\tau}^{t_1-\tau} g(x(s))\mathrm{d}s &\leq \max(g(x(t_0-\tau)), g(x(t_1-\tau)))\Delta t \\ &< \gamma\Delta t \\ \int_{t_0}^{t_1} g(x(s))\mathrm{d}s &\geq \min(g(x(t_0)), g(x(t_1)))\Delta t \\ &= \min(\gamma, g(x(t_1)))\Delta t \end{aligned}$$

It follows that $\Delta V > 0$ and, thus, $\dot{V}(x) > 0$, which contradicts Eq. (5). Hence, $g(x) \leq \gamma$ for all time. ■

Note that Theorem 1 implies that if Eq. (5) holds then solutions of the system given by Eq. (2) are bounded. Eq. (5) is a test for boundedness that can also be computationally implemented efficiently (Section III). However, while $\tilde{P}_b$ and γ can be searched for, we must define α *a priori*. Usually, a small α suffices. For $D = I$ and symmetric C, we can circumnavigate this problem as the following theorem shows.

Theorem 2: Let $D = I$ and $C = C^{\mathrm{T}}$. If there exists a matrix $P_b > 0$, $P_b \in \mathbb{R}^n$, such that, for all i and all x_i, if $\frac{1}{2}x_i^{\mathrm{T}}P_b x_i \leq \gamma$ then $x_i^{\mathrm{T}}P_b f(x_i) \leq \beta$ and if $\frac{1}{2}x_i^{\mathrm{T}}P_b x_i > \gamma$ then $x_i^{\mathrm{T}}P_b f(x_i) < -M\beta < 0$, where $M = N - 1$, then solutions of the system given by Eq. (2) are bounded.

Proof: Let $\tilde{P}_b = I \otimes P_b$. Then, the requirements on matrix P_b imply that

$$\dot{V}(x) = \sum_{i=1}^{n} x_i^{\mathrm{T}}P_b f(x_i) + \begin{bmatrix} x \\ x^\tau \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x \\ x^\tau \end{bmatrix} < 0$$

if $g(x) > \gamma$. Thus, solutions are bounded if additionally $Q \leq 0$, which we prove next. Let $S = \frac{\kappa}{2}(C_d \otimes I)$. Then,

$$Q = \frac{\kappa}{2}\begin{bmatrix} C_d\otimes P_b & A\otimes P_b \\ A\otimes P_b & C_d\otimes P_b \end{bmatrix} = \frac{\kappa}{2}\begin{bmatrix} C_d & A \\ A & C_d \end{bmatrix} \otimes P_b$$

Since $e^{\mathrm{T}}C = 0$, $Ce = 0$ and $C = C^{\mathrm{T}}$, it follows from Geršgorin's theorem that $\tilde{C} = \begin{bmatrix} C_d & A \\ A & C_d \end{bmatrix} \leq 0$. Moreover, the eigenvalues of $\tilde{C} \otimes P_b$ are given by all possible products of the eigenvalues of $\tilde{C}$ and P_b, which implies that $Q \leq 0$ and completes the proof. ■

B. Synchronisation of coupled dynamical systems with delay

Let the Laplacian matrix of the completely connected graph be given by $U \in \mathbb{R}^{N \times N}$. That is,

$$U = E - NI, \ E = ee^{\mathrm{T}}$$

Theorem 3: Consider the Lyapunov function given by

$$V(x) = -\frac{1}{2}x^{\mathrm{T}}(U \otimes P)x - \int_{t-\tau}^{t} x(s)^{\mathrm{T}}(U \otimes S)x(s)\mathrm{d}s$$

where $S \geq 0$ and $P > 0$. Then, $V(x) > 0$ if $x_i \neq x_j$ for any i, j, $i \neq j$, and $V(x) = 0$ if $x_i = x_j$ for all i, j. Let $x_i - x_j \in \overline{\mathcal{D}}$ for all i, j, $\overline{\mathcal{D}} \subset \mathbb{R}^n$, and $\overline{\mathcal{D}}$ is convex. Let $J(y) = \frac{\partial f(y)}{\partial y}$, $y \in \overline{\mathcal{D}}$. If

$$PW + W^{\mathrm{T}}P < 0 \ \forall y \tag{6}$$

where $W = J(y) + \kappa_1 \max(C_d)D$, $\kappa = \kappa_1 + \kappa_2$, $\kappa_1, \kappa_2 > 0$, then the system given by Eq. (2) synchronises in the sense that $\|x_i - x_j\| \to 0$, $\forall i, j$, as $t \to \infty$.

Proof: Note that

$$-x^{\mathrm{T}}(U \otimes P)x = \sum_{i=1}^{N}\sum_{i<j}^{N}(x_i - x_j)^{\mathrm{T}}P(x_i - x_j)$$

Moreover,

$$\begin{aligned} \dot{V}(x) \ &= \ \sum_{i=1}^{N}\sum_{i<j}^{N}(x_i - x_j)^{\mathrm{T}}P(f(x_i) - f(x_j)) + \\ & \quad \sum_{i=1}^{N} x_i^{\mathrm{T}}\kappa_1 C_{ii}PDx_i + \begin{bmatrix} x \\ x^{\tau} \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x \\ x^{\tau} \end{bmatrix} \end{aligned} \tag{7}$$

where

$$Q = -\begin{bmatrix} UC_d \otimes \kappa_2 PD + U \otimes S & UA \otimes \kappa PD \\ 0 & -U \otimes S \end{bmatrix}$$

It follows from the mean value theorem (for a description of the theorem see [20]) that, for any x_i and x_j, there exists a $y \in \overline{\mathcal{D}}$ such that

$$(x_i - x_j)^{\mathrm{T}}P(f(x_i) - f(x_j)) = (x_i - x_j)^{\mathrm{T}}PJ(y)(x_i - x_j)$$

Thus, if $x_i \neq x_j$ for any i, j, $i \neq j$, and Eq. (6) holds, and

$$Q + Q^{\mathrm{T}} \leq 0 \tag{8}$$

then $\dot{V}(x) < 0$ and $\dot{V}(x) = 0$ only if $x_i = x_j$ for all i, j. This implies that Eq. (2) synchronises in the sense that $\|x_i - x_j\| \to 0$, $\forall i, j$, as $t \to \infty$, and completes the proof. ■

Note that, given κ_1 and κ_2, checking directly, whether

(i) there exists matrix P s. t. Eq. (7) is negative if $x_i \neq x_j$, is harder than checking, whether

(ii) there exists matrix P s. t. Eq. (6) and Eq. (8) hold.

However, the latter (simpler) check, given by (ii), comes at a cost. It is more conservative in the sense that there might exist a positive definite matrix P such that $\dot{V}(x) < 0$ if $x_i \neq x_j$, while such a matrix satisfying Eq. (6) and Eq. (8) might not exist. Nevertheless, we recommend check (ii). Next, for all-to-all coupling, only a problem of even smaller size that lacks the conservativeness mentioned, needs to be solved to check for synchronisation, as we show in the following section.

C. Synchronisation of all-to-all coupled dynamical systems with delay

Consider the system given by Eq. (2) with an all-to-all coupling topology; that is, $C = U$. Let $X_i = x_i - x_1$, $X = X(t)$, and $X^{\tau} = X(t - \tau)$. Then, it follows from the mean value theorem in its integral form as presented in [13] that

$$\dot{X} = J(y)X - \kappa MDX - \kappa DX^{\tau}, \ y \in \overline{\mathcal{D}}$$

where $M = N - 1$, $x_i \in \mathcal{D} \subseteq \overline{\mathcal{D}} \subset \mathbb{R}^n$ for all i, and $\overline{\mathcal{D}}$ is convex. We say that the system given by Eq. (2) synchronises if $X \to 0$ as $t \to \infty$. Note that $X = 0$ and $X^{\tau} = 0$ imply $\dot{X} = 0$. This corresponds to the synchronised state if solution trajectories of (2) are bounded. In the following, we provide conditions for asymptotic stability of this state.

Theorem 4: Consider the Lyapunov function given by

$$V(X) = \frac{1}{2}X^{\mathrm{T}}P_s X + \int_{t-\tau}^{t} X(s)^{\mathrm{T}}SX(s)\mathrm{d}s$$

where $S > 0$, $P_s > 0$. Then, $V(X) > 0$ if $X \neq 0$ and $V(0) = 0$. If

$$Q + Q^{\mathrm{T}} < 0 \ \forall y \in \overline{\mathcal{D}} \tag{9}$$

where

$$Q = \begin{bmatrix} S + P_s(J(y) - \kappa MD) & -\kappa P_s D \\ 0 & -S \end{bmatrix} \tag{10}$$

then $\dot{V}(X) < 0$ if $X \neq 0$ and $\dot{V}(0) = 0$.

Proof: The following proves the theorem: $\dot{V}(X) = \begin{bmatrix} X \\ X^{\tau} \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} X \\ X^{\tau} \end{bmatrix}$. ■

Interestingly, above implies that when compared to the case of systems without delay in the coupling, for which $Q = P_s(J(y) - \kappa ND)$ [16], the number of coupled subsystems N plays a much more important role in guaranteeing synchronisation for systems with delay. Indeed, we can observe this in the examples presented in Section IV. Moreover, although the assumption of all-to-all coupling might seem restrictive, at times, it can be quite realistic. For example, for centralised control, the control might be such that signals from each subsystem are transmitted to a central receiver, where they are collected before they are forwarded to all other individual members. This is the case for automatic vehicle cohorts, when each vehicle transmits signals to a satellite first, before they are forwarded to the others at the next passing of the satellite [21], [8]. For the spread of epidemics, it is conceivable that, for many cases, travel time (corresponding to the delay) can be assumed independent of distance because of different means of transportation. Finally, note that the result presented in this section is a general stability result for systems with delay, which we present next, for completeness.

1) Stability of coupled dynamical systems with delay: Consider the following dynamical system with time delays in some of the linear components

$$\dot{x} = f(x) + A_d x^{\tau}, \; x \in \mathbb{R}^n \tag{11}$$

and the Lyapunov function given by

$$V(x) = \frac{1}{2} x^{\mathrm{T}} P_a x + \int_{t-\tau}^{t} x(s)^{\mathrm{T}} S x(s) \mathrm{d}s, \; S > 0, \; P_a > 0$$

It follows that

$$\dot{V}(x) = x^{\mathrm{T}} P_a f(x) + \begin{bmatrix} x \\ x^{\tau} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S & P_a A_d \\ 0 & -S \end{bmatrix} \begin{bmatrix} x \\ x^{\tau} \end{bmatrix}$$

Let the origin be the unique equilibrium point of the system given by Eq. (11). Then,

$$\dot{V}(x) < 0 \; \forall x, \; x \neq 0, \; \dot{V}(0) = 0 \tag{12}$$

implies asymptotic stability of the system given by Eq. (11) with respect to the origin [20]. Finally, note that this stability result can be easily extended to systems with different time delays, which are of the form given by:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n} a_i x_i(t - \tau_i)$$

III. Computational Implementation

In this section, we provide some mathematical background on the computational tools we use to efficiently check the different requirements given in this paper. That is, the following can be efficiently checked using the MATLAB toolboxes YALMIP [22] and SOSTOOLS [23] for systems involving bounded (for example, trigonometric) functions, polynomial or rational functions:

- given α, whether there exist a constant $\gamma > 0$ and a matrix $\tilde{P}_b > 0$ such that Eq. (5) holds,
- given κ_1 and κ_2, whether there exist a matrix $P > 0$ and a matrix $S \geq 0$ such that Eq. (6) and Eq. (8) hold,
- whether there exist a matrices $S, P_s > 0$ such that Eq. (9) holds (note that, often, $J(y)$ will not be a constant matrix),
- or whether there exist a matrices $S, P_a > 0$ such that Eq. (12) holds.

A. Semidefinite programming and the sum of squares decomposition

The main computational tool used in this paper is semidefinite programming. Programmes of this type can be solved efficiently using interior-point methods. In semidefinite programming, we replace the nonnegative orthant constraint of linear programming by the cone of positive semidefinite matrices and pose the following minimisation problem:

$$\begin{aligned} \text{minimise} \quad & \mathrm{c}^T x \\ \text{subject to} \quad & F(x) \geq 0, \; \text{where} \\ & F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \end{aligned} \tag{13}$$

Here, $x \in \mathbb{R}^n$ is the free variable. The so called problem data, which are given, are the vector $c \in \mathbb{R}^n$ and the matrices $F_j \in \mathbb{R}^{m \times m}$, $j = 0, \ldots, n$. Note that convexity of the set of symmetric positive semidefinite matrices in (13) implies that the minimisation problem has a global minimum.

B. Sum of squares decomposition

For problem data that consists of polynomials of any degree the requirement of positivity can be relaxed to the condition that the polynomial function is a sum of squares. On one hand, this is only a sufficient condition for positivity and it can, at times, be quite conservative; in other words, a function can be positive without being a sum of squares. On the other hand, testing positivity of a polynomial is NP-hard [24].

Consider the real-valued polynomial function $F(x)$ of degree $2d$, $x \in \mathbb{R}^n$. A sufficient condition for $F(x)$ to be nonnegative is that it can be decomposed into a sum of squares [25]: $F(x) = \sum_i f_i^2(x) \geq 0$, where f_i are polynomial functions. Now, $F(x)$ is a sum of squares if and only if there exists a positive semidefinite matrix R and

$$F(x) = \chi^{\mathrm{T}} R \chi, \; \chi = [1, \; x_1, \; x_2, \ldots, \; x_n, x_1 x_2, \ldots, \; x_n^d]$$

The length of vector χ is $\ell = \binom{n+d}{d}$. Note that R is not necessarily unique. However, $\sum_i f_i^2(x) = \chi^{\mathrm{T}} R \chi$ poses certain constraints on R of the form $\mathrm{trace}(A_j R) = c_j$, where A_j and c_j are appropriate matrices and constants, respectively (for an illustration, see Example 3.5 in [25]).

In general, in order to find R, we solve the optimisation problem associated with the following semidefinite program:

$$\begin{aligned} \text{minimise} \quad & \mathrm{trace}(A_0 R) \\ \text{subject to} \quad & \mathrm{trace}(A_j R) = c_j, \; j = 1, \ldots, m \\ & R \geq 0 \end{aligned} \tag{14}$$

Here are some additional remarks on how to check positivity of rational functions and when functions are restricted to a specific region of the state or/and parameter space.

- Consider a rational function $F(x)$; that is, $F(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions. Then, $F(x) \geq 0$ if (14) is feasible with $\chi^{\mathrm{T}} R \chi = F(x) g^2(x)$ or with $\chi^{\mathrm{T}} R \chi = F(x) g(x)$ if $g(x) > 0$.
- If
$$F(x) + p(x) h(x) = \sum_i g_i^2(x) \geq 0, \; p(x) \geq 0,$$
$$h(x) = \begin{cases} \leq 0 \text{ if } a_i \leq x_i \leq b_i \; \forall i \\ > 0 \text{ otherwise} \end{cases}$$
then $F(x) \geq 0$ if $a_i \leq x_i \leq b_i$ for all i, where a_i, b_i are constants. This can be used to show that $F(x)$

is nonnegative in a specific region of the state or/and parameter space.

Next, as an example, we provide the sum of squares programme used in Section IV-B to check for synchronisability and solved using SOSTOOLS.

$$\begin{array}{rl} \text{given} & J(\cdot),\ N,\ D,\ \kappa,\ \delta > 0 \\ \text{search for} & P_s \in \mathbb{R}^3,\ S \in \mathbb{R}^3,\ p(v) \\ \text{s. t. } \forall y, \forall v & v^{\mathrm{T}}(P-\delta I)v,\ v^{\mathrm{T}}(S-\delta I)v,\ p(v) \text{ are SOS} \\ & p(v)(y_1^2 + y_2^2 + (38-y_3)^2 - 1540.3N) \\ & -v^{\mathrm{T}}(Q+\delta I)v \text{ is SOS} \end{array}$$

where Q is given by Eq. (10).

IV. Applications

A. A system of coupled simple oscillators

Consider the following simple system, describing N coupled oscillators

$$\begin{aligned} \dot{x}_i &= \begin{bmatrix} -0.1 & 0.1 \\ -1 & -0.2 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \sin(x_{i_2}) \end{bmatrix} \\ &+ \kappa \sum_{j=1, j\neq i}^{N} \begin{bmatrix} 0 \\ x_{j_2}^{\tau} - x_{i_2} \end{bmatrix} \end{aligned} \tag{15}$$

Here, $D = \mathrm{diag}([0\ 1]^{\mathrm{T}})$. Using YALMIP, we can prove boundedness and guarantee synchronisation for all τ if $N = 3$ (or $N = 4, 5, 12$) and $\kappa = 0.8$, which is close to the value of κ below which synchronisation was not always observed in numerical simulations; that is, for $\kappa \leq \kappa^* = 0.5$. On the other hand, while we can easily show boundedness for $N = 2$, we can not obtain a certificate for synchronisation. Indeed, as Figure 1 indicates the system does not necessarily synchronise for $N = 2$. However, using the approach presented in this paper, we can show that increasing N reduces the coupling strength required to guarantee synchronisation (for example, $\kappa = 0.4$ for $N = 4$ and $\kappa = 0.1$ for $N = 10$).

Note that above configuration corresponds to all-to-all coupling; that is, $C = U$. For $N = 3$, a coupling constant given by $\bar{\kappa} = 4$ and

$$C = \begin{bmatrix} -2 & 0.5 & 1.5 \\ 0.5 & -2 & 1.5 \\ 1 & 1 & -2 \end{bmatrix}$$

we prove boundedness of solutions and guarantee synchronisation for all τ by checking Eq. (6) and Eq. (8). However, numerical simulations seem to indicate that synchronisation occurs already if the coupling strength κ is such that $\kappa \geq \kappa^* = 0.8$. The larger gap between $\bar{\kappa}$ and κ^* is due to the conservativeness (discussed above) of Eq. (6) and Eq. (8).

B. Coupled Lorenz systems

We show the applicability of our approach to coupled chaotic systems by considering a network of N coupled identical Lorenz systems. The set of equations for each individual system is given by, where $i = 1, \ldots, N$,

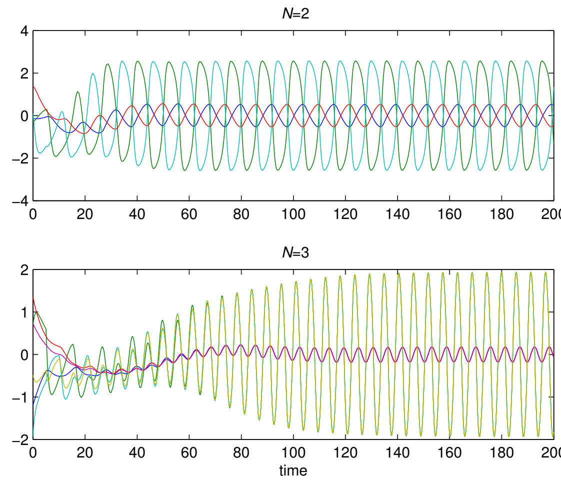

Fig. 1. **The behaviour of the system consisting of N coupled systems of the form given by Eq. (15) for $\kappa = 0.8$ and $\tau = 5$.** The different lines show the behaviour of x_{ij}, $i = 1, 2$ and $j = 1, \ldots, N$. For $N = 3$, we observe synchronisation of the coupled system but not for $N = 2$.

$$\begin{aligned} \dot{x}_{i_1} &= 10(x_{i_2} - x_{i_1}) + \kappa \sum_{j=1, j\neq i}^{N} (x_{j_1}^{\tau} - x_{i_1}) \\ \dot{x}_{i_2} &= 28x_{i_1} - x_{i_2} - x_{i_1}x_{i_3} + \kappa \sum_{j=1, j\neq i}^{N} (x_{j_2}^{\tau} - x_{i_2}) \\ \dot{x}_{i_3} &= x_{i_1}x_{i_2} - \frac{8}{3}x_{i_3} + \kappa \sum_{j=1, j\neq i}^{N} (x_{j_3}^{\tau} - x_{i_3}) \end{aligned} \tag{16}$$

Here, $D = I$. Using SOSTOOLS, we show boundedness of solutions for $N = 2, 3$ (or $N = 4, 5, 10$) and all τ. However, while we can guarantee synchronisation for $N = 3$ (or $N = 4, 5, 10$), all τ, and, for example, $\kappa = 46.5$, we cannot guarantee it for $N = 2$ (Figure 2). To check for synchronisation if $N = 3$, $\kappa = 46.5$, we use the fact that $\sum_{i=1}^{3} \bar{x}_i^{\mathrm{T}} P_b \bar{x}_i \leq 1540.3N$ [16], where, for all i, $\bar{x}_i = [x_{i_1}\ x_{i_2}\ x_{i_3} - 38]^{\mathrm{T}}$ and $P_b = I$. This example shows that synchronisation of coupled chaotic oscillators is possible even with time delays in the coupling.

V. Conclusions and Discussion

In this paper, we provided sufficient conditions for the synchronisation of coupled identical nonlinear dynamical systems with time delays in the coupling, which makes the modelling of the coupling more realistic. Importantly, we showed how these conditions can be efficiently checked using semi-definite programming techniques. Understanding the mechanisms behind synchronisation in biology and physiology is an important research field. For example, research on the synchronisation of coupled circadian clocks is of importance to better understand many different disease states [26]. Similarly, providing certificates for synchronised behaviour of the different infection states between connected

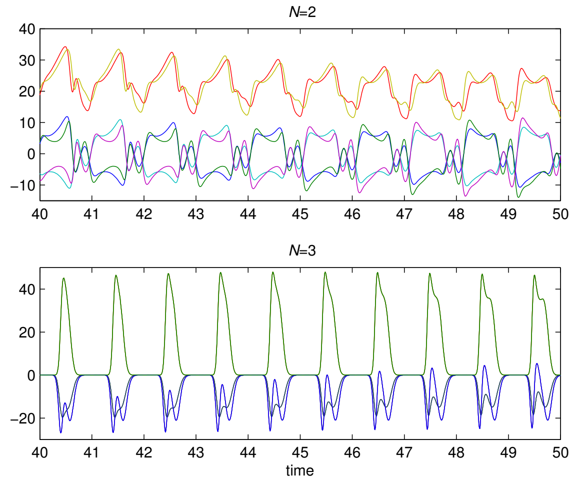

Fig. 2. **The behaviour of the system consisting of N coupled systems of the form given by Eq. (16) for $\kappa = 46.5$ and $\tau = 1$.** The different lines show the behaviour of x_{ij}, $i = 1, 2, 3$ and $j = 1, \ldots, N$. For $N = 3$, we observe synchronisation of the coupled system but not for $N = 2$.

populations during an epidemic can be of high importance. Whether we observe anti-phase synchronisation or in-phase synchronisation (which corresponds to synchronisation in the sense of this paper) in the numbers of infected of the different populations, has an important implication for the progress of an epidemic. In-phase synchronisation tends to lead to global extinction of the disease, while anti-phase synchronisation tends to promote it [3]. Thus, being able to guarantee in-phase synchronisation, for example, through the design of pulse vaccination [3], is of great significance. Moreover, for applications in engineering that require synchronised behaviour, consensus or so called stable flocking behaviour, it is often necessary to be able to provide certificates for synchronised behaviour between the different subsystems [27], [28]. Such systems correspond to tethered space vehicles or, more generally, vehicles in formation flight [1], [21].

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References

[1] S.-J. Chung, U. Ahsun and J.-J. E. Slotine, "Application of Synchronization to Formation Flying Spacecraft: Lagrangian Approach," *Journal of Guidance, Control, and Dynamics*, vol. 32, pp. 512–526, 2009.

[2] B. T. Grenfell, O. N. Bjørnstad and J. Kappey, "Travelling waves and spatial hierarchies in measles epidemics," *Nature*, vol. 414, pp. 716–723, 2001.

[3] D. He and L. Stone, "Spatio-temporal synchronization of recurrent epidemics," *Proceedings of the Royal Society of London, Series B*, vol. 270, pp. 1519–1526, 2003.

[4] S. H. Strogatz, "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D*, vol. 143, pp. 1–20 , 2000.

[5] S. Coombes, "Neuronal networks with gap junctions: A study of piece-wise linear planar neuron models," *SIAM Journal on Applied Dynamical Systems*, vol. 7, pp. 1101–1129, 2008.

[6] E. Steur, I. Tyukin and H. Nijmeijer, "Semi-passivity and synchronization of diffusively coupled neuronal oscillators," *Physica D*, vol. 238, pp. 2119–2128, 2009.

[7] C. W. Wu, "On two approaches to analyzing consensus in complex networks," *IEEE International Symposium on Circuits and Systems*, pp. 2638–2641, 2007.

[8] S.-J. Chung and J.-J. E. Slotine, "Cooperative Robot Control and Concurrent Synchronization of Lagrangian Systems," *IEEE Transactions on Robotics*,vol. 25, pp. 686–700, 2009.

[9] L. M. Pecora and T. L. Carroll, "Master Stability Functions for Synchronized Coupled Systems," *Physical Review Letters*, vol. 80, pp. 2109–2112, 1998.

[10] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, pp. 1057–1064, 2005.

[11] C. W. Wu, "Synchronization in an array of chaotic systems coupled via a directed graph," *IEEE International Symposium on Circuits and Systems*, vol. 6, pp. 6046–6049, 2005.

[12] I. Belykh, V. Belykh, K. Nevidin and M. Hasler, "Persistent clusters in lattices of coupled nonidentical chaotic systems," *Chaos*, vol. 13, pp. 165–178, 2003.

[13] V. N. Belykh, I. V. Belykh and M. Hasler, "Connection graph stability method for synchronized coupled chaotic systems," *Physica D*, vol. 195, pp. 159–187, 2004.

[14] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, pp. 683–696, 1998.

[15] W. Wang and J.-J. E. Slotine, "On partial contraction analysis for coupled nonlinear oscillators," *Biological Cybernetics*, vol. 92, pp. 38–53, 2005.

[16] E. August and M. Barahona, "Obtaining certificates for complete synchronisation of coupled oscillators," *Physica D*, vol. 240, pp. 795–803, 2011.

[17] C. W. Wu, "Synchronization in Arrays of Coupled Nonlinear Systems with Delay and Nonreciprocal Coupling," *IEEE Transactions on Circuits and Systems - II: Express Briefs*, vol. 52, pp. 282–286, 2005.

[18] W. Wang and J.-J. E. Slotine, "Contraction analysis of time-delayed communications and group cooperation," *IEEE Transactions on Automatic Control*, vol. 51, pp. 712–717, 2006.

[19] T. Yamamoto, T. Oguchi and H. Nijmeijer, "Synchronization of coupled nonlinear systems with time delay," *Proceedings of the European Control Conference*, pp. 3056–3061, 2007.

[20] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, 2000.

[21] S.-J. Chung, J.-J. E. Slotine and D. W. Miller, "Nonlinear Model Reduction and Decentralized Control of Tethered Formation Flight," *Journal of Guidance, Control, and Dynamics* vol. 30, pp. 390–400, 2007.

[22] J. Löfberg, "YALMIP : A Toolbox for Modeling and Optimization in MATLAB," *Proceedings of the CACSD Conference*, Available from `control.ee.ethz.ch/~joloef/wiki/pmwiki.php`, 2004.

[23] S. Prajna, A. Papachristodoulou, P. Seiler and P. A. Parrilo, Available from `http://www.cds.caltech.edu/sostools` and `http://www.mit.edu/∼parrilo/sostools`, 2004.

[24] K. G. Murty and S. N. Kabadi, "Some NP-complete problems in quadratic and nonlinear programming," *Mathematical Programming*, vol. 39, pp. 117–129, 1987.

[25] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming, Series B*, vol. 96, pp. 293–320, 2003.

[26] D. Gonze, S. Bernard, C. Waltermann, A. Kramer and H. Herzel, "Spontaneous Synchronization of Coupled Circadian Oscillators," *Biophysical Journal*, vol. 89, pp. 120–129, 2005.

[27] U. Münz, A. Papachristodoulou and F. Allgöwer, "Delay-dependent rendezvous and flocking of large scale multi-agent systems with communication delays," *Proceedings of the* 47^{th} *IEEE Conference on Decision and Control*, pp. 2038–2043, 2008.

[28] A. Papachristodoulou, A. Jadbabaie and U. Münz, "Effects of Delay in Multi-Agent Consensus and Oscillator Synchronization," *IEEE Transactions on Automatic Control*, vol. 55, pp. 1471–1477, 2010.