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Stability Analysis for 2-Dimensional Switched Linear Systems

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Abstract

A method of determining stability of 2-dimensional switched linear systems is developed. Specifically, given available system matrices and switching rules, we show that the mode sequence eventually become periodic. To determine stability, we show that it suffices to examine the behavior of the system after the mode sequence has become periodic. Finally, we show a couple of illustrative examples to demonstrate efficacy of the proposed approach.

1. Introduction

In a recent paper [1], we developed a novel framework of determining stability for piecewise linear planar systems. The approach is based on the uniformity of the radial growth rate of linear systems and provide necessary and sufficient conditions for stability of the zero solution of such systems. Stabilization of piecewise linear and affine systems have been attracting much attention in the literature (see, for example, [2–5]). In particular, even 2-dimensional piecewise linear systems have rich characteristics.

Based on the results given in [1], in this paper we develop a method of determining stability of 2-dimensional switched linear systems. Specifically, first we derive the discrete dynamics representing the mode transition of the system and describe the mode transition as a graph. Next, we consider the graph Laplacian matrix associated with the graph and show that the eigenvectors of the Laplacian associated with the zero eigenvalue play an important role in our framework. To determine stability, we show that it suffices to examine the behavior of the system after the mode sequence has become periodic. Finally, we show a couple of illustrative examples to demonstrate efficacy of the proposed approach.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{N}$ denotes the set of natural numbers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose and I for identity matrix.

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2. Switched Systems

In this section, we introduce general multi-dimensional switched nonlinear systems for the description of switched linear systems that we deal with in this paper and derive a difference equation in terms of mode switching times. Specifically, consider the state-driven switched nonlinear system $\mathcal{G}_{\mathrm{NL}}$ given by the continuous-state dynamics

$$\dot{x}(t) = f(\sigma(t), x(t)), \quad x(0) = x_0, \quad t \geq 0, \qquad (1)$$

and discrete state dynamics which govern the mode sequence given by

$$\sigma(t) = h(\sigma(t^-), x(t^-)), \quad \sigma(0) = \sigma_0, \qquad (2)$$

where $x(t) \in \mathbb{R}^n$ is the continuous-state vector, $\sigma(t) \in \mathcal{Q}$ is the discrete state which represents the operation mode of the system, $\mathcal{Q} = \{1, 2, \ldots, m\}$, $f_\sigma : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}$ is a function that shows transition of the modes. We assume that $\sigma(t)$ is right continuous and there is no Zeno type phenomenon so that infinitely many switching times do not occur in finite time.

In this paper, we assume that the mode of the system $\mathcal{G}_{\mathrm{NL}}$ jumps to another mode when the state (σ, x) satisfies a switching condition. In particular, for $i \neq j$, we call the set of x the *switching set* from mode i to j if x satisfies $j = h(i, x)$.

Now, for the state-driven switched system $\mathcal{G}_{\mathrm{NL}}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$ be the number of mode switches that occur for a particular initial condition (σ_0, x_0) and let t_i, $i = 1, \ldots, \kappa$, be the corresponding switching instants. If κ is finite, there is no mode switch after t_κ so that the stability of the state-driven switched system $\mathcal{G}_{\mathrm{NL}}$ can be determined by examining the stability of the continuous-state dynamics (1) for the mode $\sigma(t_\kappa)$ fixed. On the contrary, if $\kappa = \infty$, then the stability analysis is likely to become much more complex since the mode sequence should be properly taken into account.

Suppose that for any initial condition (σ_0, x_0) of $\mathcal{G}_{\mathrm{NL}}$ the number of switches κ is infinite and consider the

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state at the switching times, i.e., $(\sigma(t_i), x(t_i))$, $i \in \mathbb{N}$. Now, letting $t_0 \triangleq 0$ and defining $(\sigma_{\rm d}[k], x_{\rm d}[k]) \triangleq (x(t_k), \sigma(t_k))$, $k \in \mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$, the difference equation in terms of the switching instants t_k, $k \in \mathbb{N}_0$, is given by

$$x_{\rm d}[k+1] = f_{\rm d}(\sigma_{\rm d}[k], x_{\rm d}[k]), \quad x_{\rm d}[0] = x_0, \quad k \in \mathbb{N}_0, \tag{3}$$

$$\sigma_{\rm d}[k+1] = h_{\rm d}(\sigma_{\rm d}[k], x_{\rm d}[k]), \quad \sigma_{\rm d}[0] = \sigma_0, \tag{4}$$

where $h_{\rm d} : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}$ describes the mode sequence and $f_{\rm d} : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ represents the impact map [6] that maps the continuous state $x(t_k)$ on a switching set to $x(t_{k+1})$ at the next switching set. For convenience, in the case of $k = 0$, the functions $f_{\rm d}$ and $h_{\rm d}$ map the initial state (σ_0, x_0) to the state $(\sigma_{\rm d}[1], x_{\rm d}[1])$ which locates at the first switching set from (σ_0, x_0).

As previously mentioned, it is important to analyze the mode sequence $\sigma_{\rm d}[k]$ in determining stability of the state-driven switched system $\mathcal{G}_{\rm NL}$. However, it is extremely difficult to obtain the explicit mode sequence $\sigma_{\rm d}[k]$ for general high-dimensional nonlinear switched systems. In the case of switched linear system with dimension 2, it is possible to obtain the sequence $\sigma_{\rm d}[k]$ as shown in the following sections.

3. Two-Dimensional Switched Linear System

Based on the characterization presented in Section 2, in this section we specialize $\mathcal{G}_{\rm NL}$ to the case where the continuous-state dynamics are given as linear functions of x for each fixed $\sigma \in \mathcal{Q}$ and $n = 2$. Specifically, consider the switched linear planar system $\mathcal{G}$ given by

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad x(0) = x_0, \quad t \geq 0, \tag{5}$$

$$\sigma(t) = \begin{cases} \sigma(t^-), & \text{if } c_{\sigma(t^-),j} x(t^-) \neq 0, \quad j \in \mathcal{Q}_{\sigma(t^-)}, \\ j, & \text{if } c_{\sigma(t^-),j} x(t^-) = 0, \end{cases} \quad \sigma(0) = \sigma_0, \tag{6}$$

where $x(t) \in \mathbb{R}^2$ is the continuous-state vector, $\sigma(t) \in \mathcal{Q}$ is the discrete state, $A_i \in \mathbb{R}^{2\times 2}$ is the system matrix for mode $i \in \mathcal{Q}$, $c_{i,j} \in \mathbb{R}^{1\times 2}$ is the unit row vector that characterizes the switching surface from mode $i \in \mathcal{Q}$ to mode $j \in \mathcal{Q}$ (if the current mode is i, then the mode changes to j ($\neq i$) as soon as the continuous state $x(t)$ satisfies $c_{i,j} x(t) = 0$ (see Figure 3.1)), $\mathcal{Q} = \{1, 2, \ldots, m\}$, m is the possible number of the modes, and $\mathcal{Q}_i \subset \mathcal{Q}$ is the set of modes to which the mode can jump from mode i.

As mentioned in Section 2, stability analysis for the case where $\kappa < \infty$ is straightforward. Henceforth, we assume that there are infinitely many switching instants. This case can be assured if, for example, A_i has complex conjugate eigenvalues and $\mathcal{Q}_i \neq \emptyset$ for all $i \in \mathcal{Q}$.

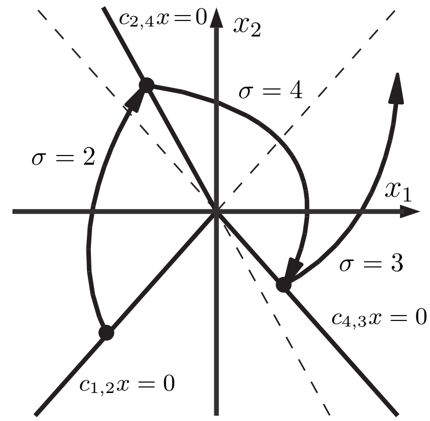

Figure 3.1: Example of a continuous-state trajectory and switching surfaces

3.1. Automata of Mode Transition

In this section, we derive an automata that shows the mode transition of the switched linear planar system $\mathcal{G}$. Specifically, we consider the difference equations (3), (4) for the switched linear planar system $\mathcal{G}$. Note that in this case $x_{\rm d}[k]$ at the switching instants t_k satisfies

$$\frac{x_{\rm d}[k]}{\|x_{\rm d}[k]\|} \in \left\{ G c^{\rm T}_{\sigma_{\rm d}[k-1],\sigma_{\rm d}[k]}, -G c^{\rm T}_{\sigma_{\rm d}[k-1],\sigma_{\rm d}[k]} \right\}, \quad k \in \mathbb{N}, \tag{7}$$

where $G \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation matrix with the rotation angle 90 degrees in counterclockwise direction. Using this fact, we obtain the following result.

Lemma 3.1. Consider the switched linear planar system $\mathcal{G}$ given by (5), (6) and the associated difference equations (3), (4) for $\mathcal{G}$. Then, for $k \geq 1$, $\sigma_{\rm d}[k+1]$ is uniquely determined from $\sigma_{\rm d}[k]$ and $\sigma_{\rm d}[k-1]$.

Proof. The proof is immediate from (7) and the fact that $h_{\rm d}(\sigma, ax) = h_{\rm d}(\sigma, x)$, $a \in \mathbb{R}$. □

From Lemma 3.1 it follows that the existence of a function $s : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ such that

$$\sigma_{\rm d}[k+1] = s(\sigma_{\rm d}[k], \sigma_{\rm d}[k-1]), \tag{8}$$

is guaranteed. This function returns the next mode from the current mode and the one before, without using the variable $x_{\rm d}[\cdot]$. In particular, in the case where A_i, $i \in \mathcal{Q}$, have complex conjugate eigenvalues, the function $s(\cdot,\cdot)$ is given by

$$s(i,j) \triangleq \begin{cases} \displaystyle \arg\min_{k \in \mathcal{Q}_i} \frac{c_{i,k} c^{\rm T}_{j,i}}{c_{i,k} G c^{\rm T}_{j,i}} & \text{if } \operatorname{rot}(A_i, G c^{\rm T}_{j,i}) = -1, \\ \displaystyle \arg\max_{k \in \mathcal{Q}_i} \frac{c_{i,k} c^{\rm T}_{j,i}}{c_{i,k} G c^{\rm T}_{j,i}} & \text{if } \operatorname{rot}(A_i, G c^{\rm T}_{j,i}) = 1, \end{cases} \tag{9}$$

where $\mathrm{rot}(A, x)$ denotes the rotational direction of Ax at the point of $x \in \mathbb{R}^2$ such that

$$\mathrm{rot}(A, x) \triangleq \begin{cases} 1, & \text{if } \det\big([x, Ax]\big) > 0, \\ -1, & \text{if } \det\big([x, Ax]\big) < 0, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Even though $s(i, i)$ is not well defined since $c_{i,i}$ is not defined, we set $s(i, i) = d$, where $d \neq i$ is an arbitrary scalar, for simplicity of exposition.

For developing a special class of automata expressing the mode transition described by (8), we define the binary variables $\delta[k] \in \mathbb{B}^m$ and $\tilde{\delta}[k] \in \mathbb{B}^{m^2}$ at the time instant t_k, where $\mathbb{B}^e = \{z \in \{0, 1\}^e : z^{\mathrm{T}} z = 1\}$. Specifically, let $\delta[k]$ be defined as the binary variable describing the current mode $\sigma_{\mathrm{d}}[k]$ such that

$$\delta[k] \triangleq \zeta(\sigma_{\mathrm{d}}[k]), \quad (11)$$

where $\zeta : \mathcal{Q} \to \mathbb{B}^m$ is the function given by

$$\zeta(\phi) \triangleq \begin{bmatrix} \zeta_1(\phi) \\ \vdots \\ \zeta_m(\phi) \end{bmatrix}, \quad \zeta_i(\phi) \triangleq \begin{cases} 1, & \text{if } \phi = i, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

$\phi \in \mathcal{Q}$. Note that the $\sigma_{\mathrm{d}}[k]$th element of $\delta[k] \in \mathbb{B}^m$ is 1 and the other elements are 0. Furthermore, ζ has the inverse function $\zeta^{-1} : \mathbb{B}^m \to \mathcal{Q}$ since ζ has one-to-one correspondence with ϕ. In addition, let $\tilde{\delta}[k]$ be defined by the tensor product of the current mode $\delta[k]$ and the previous mode $\delta[k-1]$ such that

$$\tilde{\delta}[k] \triangleq \xi(\delta[k], \delta[k-1]), \quad (13)$$

where $\xi : \mathbb{B}^m \times \mathbb{B}^m \to \mathbb{B}^{m^2}$ is given by

$$\xi(\psi_1, \psi_2) \triangleq \psi_1 \otimes \psi_2, \quad (14)$$

$\psi_1, \psi_2 \in \mathbb{B}^m$. Note that $\tilde{\delta}[k] \in \mathbb{B}^{m^2}$ also has only one element that is 1. The binary variable $\tilde{\delta}[k]$ holds the information of both the current and the previous modes. Furthermore, ξ also has the inverse function $\xi^{-1} : \mathbb{B}^{m^2} \to \mathbb{B}^m \times \mathbb{B}^m$. Hence, we can obtain $\sigma_{\mathrm{d}}[k]$ and $\sigma_{\mathrm{d}}[k-1]$ from $\tilde{\delta}[k]$ using the inverse functions ζ^{-1} and ξ^{-1}.

Now, viewing $\tilde{\delta}[k]$ as a node of automata, the mode transition (8) is described by the one-step difference equation

$$\tilde{\delta}[k+1] = E\tilde{\delta}[k], \quad \tilde{\delta}[1] = \tilde{\delta}_0, \quad k \in \mathbb{N}, \quad (15)$$

where $E \in \mathbb{R}^{m^2 \times m^2}$ has elements $E_{i,j}$ given by

$$E_{(a_1-1)m+a_2,(b_1-1)m+b_2} = \begin{cases} 1, & \text{if } s(b_1, b_2) = a_1 \text{ and } b_1 = a_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_1, a_2, b_1, b_2 = 1, \ldots, m, \quad (16)$$

and the initial state $\tilde{\delta}_0$ is uniquely determined from σ_0 and $\sigma_{\mathrm{d}}[1]$.

With respect to the automata (15), in the sense of digraph, there is only one arrow emanating from each node. From this fact, it follows that there exists at least one closed path in (15) and that no node is shared by two or more different closed paths. Hence, each connected graph possesses only one closed path so that the state $\tilde{\delta}[k]$ of the automata (15) reaches one of the closed paths in finite switch depending on the initial state $\tilde{\delta}_0$.

It is important to note that given the initial state $\tilde{\delta}_0$ corresponding to σ_0 and $\sigma_{\mathrm{d}}[1]$, the discrete dynamics (15) is implied by (4) of the switched linear planar system $\mathcal{G}$. Hence, it is possible to obtain the mode sequence $\sigma_{\mathrm{d}}[k]$, $k \in \mathbb{N}$, of the original system $\mathcal{G}$ by analyzing the trajectory $\tilde{\delta}[k]$ given by (15) and then analyzing the trajectory $x(\cdot)$ governed by (5). Furthermore, since each connected graph yielded by (15) has only one closed path, the closed path reached in finite switch is uniquely determined by the initial state $\tilde{\delta}_0$. This fact is elucidated in the following section.

4. Detection of Closed Path

As mentioned in the previous section, trajectory of the automata (15) reaches one of the closed paths in finite switch and remains in the closed path for all time onwards. Furthermore, since the switched linear planar system $\mathcal{G}$ is piecewise linear in time, the trajectory of the planar continuous state does not converge or diverge in finite time. For these reasons, if we can show that the trajectory of continuous state $x(t)$ converges to zero for each initial condition (σ_0, x_0) (and hence for each closed path of the directed graph), it follows that the zero solution of (5) is globally asymptotically stable. Hence, the number of closed paths and the information of nodes constituting the closed paths are extremely important to analyze stability of the zero solution $x(t) \equiv 0$ of the continuous-state dynamics of $\mathcal{G}$. In this section, we present a way of obtaining the closed paths.

To this end, consider the discrete dynamics (15) as a digraph and the associated graph Laplacian matrix given by

$$L \triangleq D - E, \quad (17)$$

where $D \in \mathbb{R}^{m^2 \times m^2}$ is the degree matrix which is identity in the case of (15) where there is only one arrow from one node and $E \in \mathbb{R}^{m^2 \times m^2}$ is the adjacency matrix defined by (16). Now, using this graph Laplacian matrix L, the following lemma provides a way of characterizing the closed paths contained in (15).

Lemma 4.1. The number of the closed paths with respect to (15) is equal to the geometric multiplicity μ of the zero eigenvalue of L given by (17). Furthermore, there exist eigenvectors $v_p \in \{0,1\}^{m^2}$, $p = 1, \ldots, \mu$, satisfying $Lv_p = 0$, and $l_p \in \mathbb{N}$ such that each v_p is given by

$$v_p = \tilde{\delta}_p^1 + \tilde{\delta}_p^2 + \cdots + \tilde{\delta}_p^{l_p}, \tag{18}$$

and satisfies $\sum_{p=1}^{\mu} v_p \in \{0,1\}^{m^2}$, where $\tilde{\delta}_p^q$, $q = 1, \ldots, l_p$, are the elements of the standard basis of $\mathbb{R}^{m^2}$. In addition,

$$\mathcal{N}_p \triangleq \left\{\tilde{\delta}_p^1, \tilde{\delta}_p^2, \ldots, \tilde{\delta}_p^{l_p}\right\}, \tag{19}$$

is the set of the nodes that constitute the pth closed path.

Proof. The proof is immediate. □

From this lemma, we can obtain the information of the number of the closed paths and the nodes that constitute each closed path from the graph Laplacian L. In fact, the order of the nodes forming each closed path can also be identified by the information of the adjacency matrix E. Specifically, for each $p = 1, \ldots, \mu$, the circular permutation $\mathcal{D}_p$ arranged in order of passing the nodes from $\tilde{\delta}_p^1 \in \mathcal{N}_p$ is given by

$$\mathcal{D}_p \triangleq \left\{E^0\tilde{\delta}_p^1, \ldots, E^{l_p-1}\tilde{\delta}_p^1\right\}. \tag{20}$$

Note that the nodes contained in $\mathcal{N}_p$ are identical to the ones in $\mathcal{D}_p$. Furthermore, since $\mathcal{D}_p$ is a circular permutation, it follows that $\mathcal{D}_p$ satisfies

$$\mathcal{D}_p = \left\{E^0\tilde{\delta}_p^q, \ldots, E^{l_p-1}\tilde{\delta}_p^q\right\}, \quad q = 1, \ldots, l_p. \tag{21}$$

5. Stability Analysis

In this section, we characterize a way of analyzing stability of the switched linear planar system $\mathcal{G}$ with respect to the origin. We begin by introducing key results concerning 2-dimensional linear dynamical systems that are necessary for developing the method of stability analysis.

5.1. Radial Growth Rate of Trajectories of Linear Planar Systems [1]

Consider the linear planar dynamical system given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{22}$$

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}} \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Furthermore, consider the polar form (r, θ) as shown in Figure 5.1. Then, the radial growth rate of the trajectories of (22) at x is characterized by

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{\mathrm{d}\theta}{\mathrm{d}t}} = \frac{r\eta^{\mathrm{T}}(\theta)A\eta(\theta)}{\det[\eta(\theta), A\eta(\theta)]}. \tag{23}$$

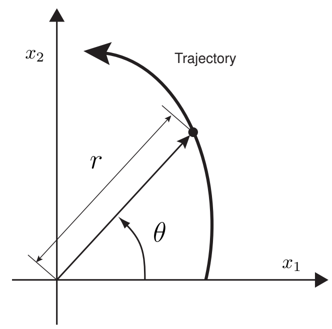

Figure 5.1: Polar form

Since the rate of radial growth with respect to θ is proportional to the distance r from the origin, it follows that the 'normalized' radial growth rate with respect to θ defined by

$$\rho(\theta) \triangleq \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\eta^{\mathrm{T}}(\theta)A\eta(\theta)}{\det[\eta(\theta), A\eta(\theta)]}, \tag{24}$$

depends solely on θ but r. Note that the function $\rho(\theta)$ is periodic of period π, that is, $\rho(\theta + \pi) = \rho(\theta)$.

By integrating the radial growth rate given by (24) from θ_0 to θ_{f}, it can be examined how the distance of the trajectory of (22) is changed over $\theta_{\mathrm{f}} - \theta_0$. Specifically, suppose that the matrix A in (22) has complex conjugate eigenvalues and that the rotational direction of the trajectories is in the counterclockwise direction. In this case, assuming the initial distance of the trajectories is given by r_0, it follows from (24) that the distance r_{f} when the trajectory first intersects the semi-infinite straight line with phase θ_{f} satisfies

$$\int_{\theta_0}^{\theta_{\mathrm{f}}} \rho(\theta)\mathrm{d}\theta = \log\frac{r_{\mathrm{f}}}{r_0}. \tag{25}$$

If this value is positive (resp., negative), then it implies $\log r_{\mathrm{f}} > \log r_0$ (resp., $\log r_{\mathrm{f}} < \log r_0$) and hence the distance r_{f} from the origin is larger (resp., smaller) than the original distance r_0.

5.2. Global Stability Analysis for Switched Linear Planar Systems

In this section, we propose a method of analyzing stability properties for the switched linear planar system $\mathcal{G}$. As mentioned in Section 4, if the trajectory of the continuous state $x(t)$ converges to zero for each closed path with respect to (15), then the zero solution of (5) is globally asymptotically stable. Now, we consider the convergence of $x_{\mathrm{d}}[k]$ given by (3) which is the sequence of the continuous state $x(t)$ at the switching instants t_k because if $x_{\mathrm{d}}[k]$ converges to zero then so does $x(t)$. To obtain the change of the norm from $x_{\mathrm{d}}[k]$ to $x_{\mathrm{d}}[k+1]$, we can use (25) with θ_0 as the phase of $x_{\rm d}[k]$, $\theta_{\rm f}$ as the phase of $x_{\rm d}[k+1]$, and A in (24) replaced by $A_{\sigma_{\rm d}[k]}$.

Lemma 5.1. Consider the switched linear planar system $\mathcal{G}$ given by (5), (6). If the initial state (σ_0, x_0) of the system $\mathcal{G}$ satisfies $\tilde{\delta}_0 \in \mathcal{N}_p$, where $\mathcal{N}_p$ is given by (19), then there exists a period T_p such that

$$\|x(t+T_p)\| = e^{\gamma_p}\, \|x(t)\|, \quad t \geq 0. \tag{26}$$

where γ_p is defined by

$$\gamma_p \triangleq \sum_{w=1}^{l_p} \int_{\theta_{\sigma_p^{w-1}, \sigma_p^w}}^{\theta_{\sigma_p^w, \sigma_p^{w+1}}} \rho_{\sigma_p^w}(\theta) \mathrm{d}\theta, \tag{27}$$

$\rho_i(\cdot)$ is the normalized radial growth rate (24) with A replaced by A_i, $\{\sigma_p^1, \ldots, \sigma_p^{l_p}\}$ is the periodic mode sequence associated with $\mathcal{D}_p$ given by (20) and $\theta_{i,j}$ is the phase of switching surface from mode i to mode j.

Proof. The proof is omitted due to space limitations. □

From this lemma, we can determine the convergence (divergence) rate of the trajectory $x(\cdot)$ when the initial state (σ_0, x_0) is given such that $\tilde{\delta}_0$ is on the closed path $\mathcal{N}_p$. Thus, if γ_p given by (27) is negative (resp., positive), then the trajectory of the continuous state $x(\cdot)$ associated with the initial condition $\tilde{\delta}_0 \in \mathcal{N}_p$ is convergent (resp., divergent). Due to the fact that for any initial state (σ_0, x_0), the trajectory $\tilde{\delta}[\cdot]$ reaches a closed path in finite switch and remains in it, we can determine stability of the switched linear planar system $\mathcal{G}$ from the convergence properties of the trajectory $x(\cdot)$ for all closed paths. Furthermore, (26) shows that if the initial state (σ_0, x_0) satisfies $\tilde{\delta}_0 \in \mathcal{N}_p$ and $\gamma_p < 0$, then the norm of the trajectory $x(\cdot)$ exponentially converges to zero with the exponent γ_p/T_p. Now, we are ready to state the main result of this paper.

Theorem 5.1. Consider the switched linear planar system $\mathcal{G}$ given by (5), (6). Then the zero solution $x(t) \equiv 0$ of (5) is globally exponentially stable if and only if γ_p, $p = 1, \ldots, \mu$, given by (27) are negative.

Proof. The proof is omitted due to space limitations. □

6. Illustrative Numerical Examples

In this section we present several numerical examples to demonstrate the utility of the proposed framework.

Example 6.1 Simple Switched System. Assume that we are given the following system matrices

$$A_1 = \begin{bmatrix} 0 & 8 \\ -2 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & -1.5 \\ 3 & 0.2 \end{bmatrix}, \tag{28}$$

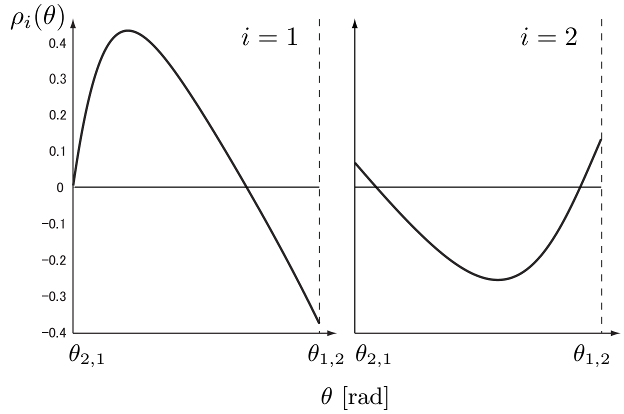

Figure 6.1: Radial growth rate versus phase. The trajectory $x(\cdot)$ rotates counterclockwise for mode $i = 2$ and clockwise for mode $i = 1$.

and switching surfaces

$$c_{1,2} = \begin{bmatrix} 1, 0 \end{bmatrix}, \quad c_{2,1} = \begin{bmatrix} 0, 1 \end{bmatrix}, \tag{29}$$

where A_i, $i = 1, 2$, have unstable eigenvalues.

The graph Laplacian matrix of the system is

$$L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{30}$$

and the geometric multiplicity μ of the zero eigenvalue of L is 1 so that there is only one closed path. The nodes of the closed path are obtained by the binary eigenvector of the zero eigenvalue as

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{31}$$

that is constructed by the vector $[0, 1, 0, 0]^{\rm T}$ representing the transition from 2 to 1 and the vector $[0, 0, 1, 0]^{\rm T}$ representing the transition from 1 to 2. From this fact, the mode sequence of the closed path is $\{1, 2, 1, 2, \ldots\}$.

Now, we apply Lemma 5.1 to the closed path. The initial phase and the terminal phase on the each mode of the closed path are uniquely determined by $c_{1,2}$ and $c_{2,1}$ so that $\rho_i(\cdot)$ on the closed path can be plotted in Figure 6.1 (where $\theta_{i,j}$ means the phase of switching surface from mode i to mode j) . Hence, since

$$\gamma_1 = \int_{\theta_{2,1}}^{\theta_{1,2}} \rho_1(\theta) \mathrm{d}\theta + \int_{\theta_{1,2}}^{\theta_{2,1}} \rho_2(\theta) \mathrm{d}\theta = -0.4115, \tag{32}$$

it follows from Theorem 5.1 that the zero solution of the switched linear planar system shown is globally exponentially stable (Figure 6.2).

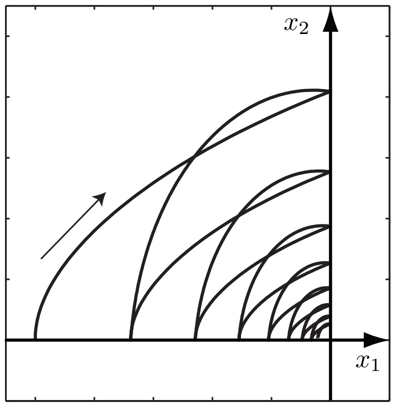

Figure 6.2: Phase portrait

Example 6.2 Large Switched System. Consider the switched linear planar system which has more than 2 modes. Specifically, assume that we are given the system matrices $A_i, i = 1, \ldots, 10$, described by

$$\begin{bmatrix} -0.09 & -1.10 \\ 1.10 & -0.68 \end{bmatrix}, \begin{bmatrix} 0.69 & 3.36 \\ -3.36 & 0.33 \end{bmatrix}, \begin{bmatrix} -0.02 & -0.22 \\ 0.22 & -0.04 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.29 \\ -0.29 & 0.00 \end{bmatrix}, \begin{bmatrix} -0.49 & -3.01 \\ 3.01 & -0.61 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.81 \\ -0.81 & 0.06 \end{bmatrix}, \begin{bmatrix} -0.55 & -2.71 \\ 2.71 & -0.09 \end{bmatrix}, \begin{bmatrix} -0.42 & 3.46 \\ -3.46 & -0.50 \end{bmatrix}, \begin{bmatrix} -0.73 & -1.16 \\ 1.12 & -0.38 \end{bmatrix}, \begin{bmatrix} 0.58 & 0.65 \\ -0.65 & 0.45 \end{bmatrix}, \quad (33)$$

and 50 switching surfaces (actual values are omitted in this paper). In this case, the size of the graph Laplacian matrix L is 100×100 and the geometric multiplicity μ of the zero eigenvalue of L is 2. Taking the similar approach to the previous section, the mode sequences of the closed paths are $\{2, 3, 4, 5, 2, 3, 4, 5, \ldots\}$ and $\{4, 9, 4, 9, \ldots\}$. For instance, the $\rho_i(\cdot)$ on the closed path of the former can be plotted in Figure 6.3. Now, since γ_1 is negative, and γ_2 is also negative, it follows from Theorem 5.1 that the zero solution is globally exponentially stable (Figure 6.4).

7. Conclusion

We developed a way of determining global exponential stability for the switched linear planar systems. It is important to note that the piecewise linear planar system considered in [1] is a special class of the switched linear system considered in this paper. Specifically, Theorem 5.1 reduces to Theorem 4.1 of [1] in the case where $m = 1$ in $\mathcal{G}$ given by (5), (6). Future research of this approach includes the extension to the case of switched homogeneous planar systems.

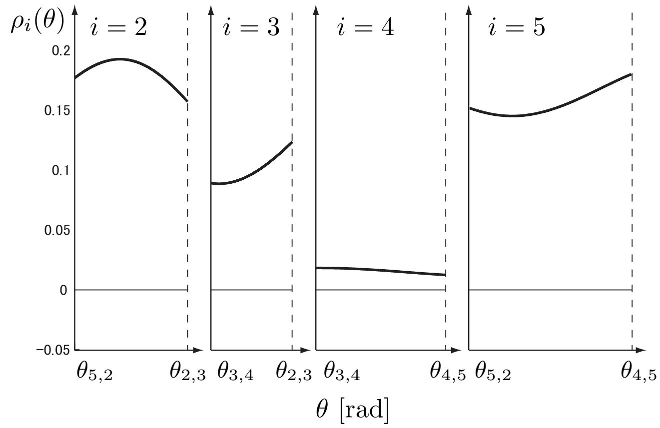

Figure 6.3: Radial growth rate versus phase. The trajectory $x(\cdot)$ rotates counterclockwise for mode $i = 2, 4$ and clockwise for mode $i = 3, 5$.

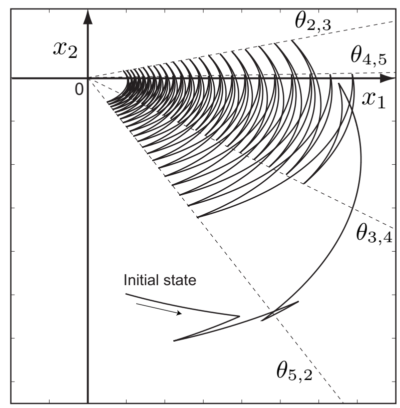

Figure 6.4: Phase portrait

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