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Integrated Estimator and $\mathcal{L}_1$ Adaptive Controller for Well Drilling Systems

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***Abstract*— This paper presents an integrated adaptive estimator-controller scheme for a class of systems with only partially measured states. To estimate the non-measured states, a fast adaptive estimator is applied. The estimation is used in the $\mathcal{L}_1$ adaptive controller, which adapts to time-varying unknown parameters and time-varying bounded disturbances in the system without restricting their rate of variation. The results demonstrate that the $\mathcal{L}_1$ adaptive controller has guaranteed performance bounds for system's both input and output, while using the estimation of the regulated outputs. The approach is used to control the bottom hole pressure of a well drilling system, in which the measurement of the pressure is updated at a low rate. Simulations verify the theoretical findings.**

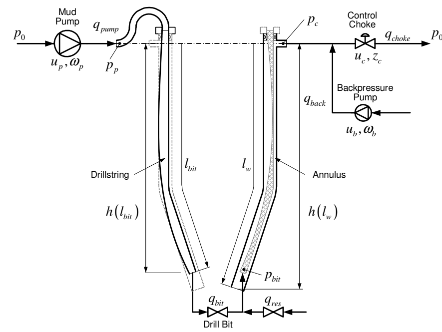

Fig. 1: Well drilling system scheme

I. Introduction

This paper extends the results of [1], [2], [3] to the case, when a part of the system states cannot be measured. When the non-measured states satisfy certain mild assumptions, they can be estimated by the fast adaptive estimator developed in [4]. The estimation is then directly used in the standard $\mathcal{L}_1$ adaptive controller. This paper proves that by replacing the true value of states with their estimates in the $\mathcal{L}_1$ adaptive controller, the steady state and transient performance of the closed-loop system can be systematically improved by increasing the rate of adaptation, similar to the full-state feedback case.

The integrated adaptive estimator-controller structure of this paper can be efficiently used to control the managed pressure drilling (MPD) system. During well drilling, a fluid circulation system is used to maintain the pressure profile along the well with specified lower and upper bounds and carry out the cuttings. The drill fluid (mud) is pumped into the drill string, which is a structure of a series of connected pipes. The fluid then flows down to the drill bit, sprays out through the bit, circulates back up the annulus, and finally exits through a choke valve. The scheme of an oil well drilling system is shown in Fig. 1.

The pressure balance between the well section and the reservoir is critical to the drilling system [5]. The main objective of MPD is to precisely control the well pressure profile throughout the well, i.e. to maintain the pressure above the pore or to collapse pressure below the fracture or sticking pressure. This amounts to stabilizing the downhole pressure within its margins. Since nowadays many wells are depleted with narrow pressure margin, to extract oil from these wells efficiently requires more precise control of the bottom hole pressure.

One of the main challenges of MPD control is the measurement of the bottom hole pressure, which is updated at low rate, and can be viewed as unmeasured state. Another drawback is the uncertainty in the model for the bottom-hole, due to uncertainties in the friction and mud compressibility parameters. Moreover, the model parameters are subject to significant changes among different stages of the drilling process, i.e. from normal drilling to the pipe connection process. These challenges motivate the design of an integrated adaptive estimator and controller scheme. The guaranteed performance bounds of $\mathcal{L}_1$ adaptive controller make it an ideal candidate for addressing the high-precision control of the bottom hole pressure.

The paper is organized as follows. Section II states some preliminary definitions. Section III gives the problem formulation. Section IV presents the fast estimator. Section V presents the $\mathcal{L}_1$ adaptive controller and the uniform performance bounds. Section VI applies the integrated estimator-controller scheme to a well drilling system under different operation conditions. Section VII concludes the paper.

II. Mathematical Preliminaries

In this Section, we recall some basic definitions and facts from linear systems theory [6], [7].

Definition 1: For a signal $\xi(t) \in \mathbb{R}^n$, $t \geq 0$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are defined as $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{0\leq\tau\leq t} |\xi_i(\tau)|)$, and $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{\tau\geq 0} |\xi_i(\tau)|)$, where ξ_i is the i^{th} component of ξ.

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Definition 2: The $\mathcal{L}_1$ norm of a stable proper single–input single–output system $H(s)$ is defined to be $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$.

Definition 3: For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$ norm is defined as $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,\cdots,n} \sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1}$, where $H_{ij}(s)$ is the i^{th} row j^{th} column element of $H(s)$.

Lemma 1: For a stable proper MIMO system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}, \quad \forall\ t > 0$.

Lemma 2: For a cascaded system $H(s) = H_2(s)H_1(s)$, where $H_1(s)$ is a stable proper system with m inputs and l outputs and $H_2(s)$ is a stable proper system with l inputs and n outputs, we have $\|H(s)\|_{\mathcal{L}_1} \leq \|H_1(s)\|_{\mathcal{L}_1} \|H_2(s)\|_{\mathcal{L}_1}$.

Lemma 3: If (A, b) is controllable and $(s\mathbb{I} - A)^{-1}b$ is strictly proper and stable, there exists $c \in \mathbb{R}^n$ such that $c^\top (s\mathbb{I} - A)^{-1} b$ is minimum phase with relative degree one.

III. Problem Formulation and Definitions

A. Problem Formulation

Consider the following system dynamics with only partly measured states:

$$\dot{x}(t) = A_m x(t) + b(\mu(t) + \theta^\top(t)x(t) + \sigma_0(t)), \quad (1)$$
$$\mu(s) = F(s)u(s), \quad (2)$$
$$\dot{z}(t) = A_1 z(t) + x(t), \quad (3)$$
$$y(t) = c^\top x(t), \quad x(0) = x_0, \quad z(0) = z_0, \quad (4)$$

where $x \in \mathbb{R}^n$ is the system state whose measurement is updated at a significantly low rate, and thus can be treated as a non-measured state, $z \in \mathbb{R}^n$ is the system state that is continuously measured, $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m and A_1 are known $n \times n$ Hurwitz matrices, (A_m, b) is controllable, $\theta(t) \in \mathbb{R}^n$ is a vector of time-varying unknown parameters, $\sigma(t) \in \mathbb{R}$ is a time-varying disturbance, and $F(s)$ is an unknown stable transfer function that presents the uncertainties due to the unmodeled actuator dynamics.

Assumption 1: The unknown time-varying parameters and the disturbance are uniformly bounded: $\theta(t) \in \Theta$, $|\sigma_0(t)| \leq \Delta$, $\forall t \geq 0$, where Θ is a known compact set, and Θ and Δ are known conservative bounds. Let $L \triangleq \max_{\theta(t)\in\Theta} \sum_{i=1}^n |\theta_i(t)|$, with θ_i being the i^{th} element of θ.

Assumption 2: $\theta(t)$ and $\sigma_0(t)$ are continuously differentiable and their derivatives are uniformly bounded: $\|\dot{\theta}(t)\|_2 \leq d_\theta < \infty$, $|\dot{\sigma}_0(t)| \leq d_\sigma < \infty$, $\forall t \geq 0$.

Assumption 3: There exists L_f such that $\|F(s)\|_{\mathcal{L}_1} \leq L_f$.

The control objective is to design an adaptive controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ both in transient and steady state, while all other error signals remain bounded.

B. Definitions

The design of the $\mathcal{L}_1$ adaptive controller involves a gain $k \in \mathbb{R}^+$ and a strictly proper transfer function $D(s) = \frac{1}{s}\bar{D}(s)$, where $\bar{D}(s)$ is proper and stable, which leads to a strictly proper stable low-pass filter:

$$C(s) = kF(s)D(s)/(1 + kF(s)D(s)) \quad (5)$$

with DC gain $C(0) = 1$. The simplest choice is $D(s) = 1/s$, which yields $C(s) = kF(s)/(s + kF(s))$. Similarly, the design of the fast estimator involves a low-pass filter $C_1(s)$ with $C_1(0) = 1$, e.g., $C_1(s) = \frac{c}{s+c}$, where $c > 0$.

For system in (1)-(4), define $H(s) = (s\mathbb{I} - A_m)^{-1}b$, $G(s) = H(s)(1 - C(s))$. It follows from Lemma 3 that there exists $c_o \in \mathbb{R}^n$ such that

$$c_o^\top H(s) = N_n(s)/N_d(s), \quad (6)$$

where $\deg(N_d(s)) - \deg(N_n(s)) = 1$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials. For the proof of stability and performance bounds, the choice of $D(s)$ and k needs to ensure that:

$$\|G(s)\|_{\mathcal{L}_1} L < 1. \quad (7)$$

For arbitrary $\gamma_0 > 0$, define

$$\gamma_1 \triangleq C_m/(1 - G_m L)\gamma_0 + \beta_1, \quad \gamma_2 \triangleq C_a\gamma_0 + LC_n\gamma_1, \quad (8)$$

where $C_m = \max_{F(s)} \|C(s)\|_{\mathcal{L}_1}$, $G_m = \max_{F(s)} \|G(s)\|_{\mathcal{L}_1}$, $C_a = \max_{F(s)} \left\| \frac{C(s)}{F(s)} \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1}$, $C_n = \max_{F(s)} \left\| \frac{C(s)}{F(s)} \right\|_{\mathcal{L}_1}$, and $\beta_1 > 0$ is an arbitrary constant which satisfies $0 < \beta_1 < \gamma_1$. We will prove that by increasing the adaptive gain, γ_0 can serve as an upper bound for the prediction error.

Let $r_0(t)$ be the signal with its Laplace transformation $r_0(s) = (s\mathbb{I} - A_m)^{-1}x_0$. Since A_m is Hurwitz, $\|r_0\|_{\mathcal{L}_\infty}$ is finite. For arbitrary $\gamma_0 > 0$, and bounded reference signal $r(t) \in \mathbb{R}$, define $\rho = \rho_r + \gamma_1$, $\rho_u = \rho_{u_r} + \gamma_2$, where

$$\rho_r \triangleq (\|G(s)\|_{\mathcal{L}_1}\Delta + k_g\|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty})/(1 - \|G(s)\|_{\mathcal{L}_1}L), \quad (9)$$
$$\rho_{u_r} \triangleq \|C(s)/F(s)\|_{\mathcal{L}_1}(L\rho_r + \Delta + k_g\|r\|_{\mathcal{L}_\infty}), \quad (10)$$

and k_g is defined as $k_g = -1/(c^\top A_m^{-1} b)$.

IV. Adaptive Estimator

In equation (3), we treat $x(t)$ as a time-varying parameter and apply the fast estimator in [4] to get the estimation of $x(t)$. The fast estimator consists of state predictor, adaptive law and low-pass filter. The state predictor is given by:

$$\dot{\hat{z}}(t) = A_1\hat{z}(t) + \hat{x}_e(t), \quad \hat{z}(0) = z_0, \quad (11)$$

which has the same structure as the system in (3), except that the unknown parameter $x(t)$ is replaced by its estimation $\hat{x}_e(t)$, which is governed by the following adaptation law:

$$\dot{\hat{x}}_e(t) = \Gamma_1 \text{Proj}(\hat{x}_e(t), -P_1\tilde{z}(t)), \quad \hat{x}_e(0) = \hat{x}_0, \quad (12)$$

where $\tilde{z}(t) \triangleq \hat{z}(t) - z(t)$, $\Gamma_1 > 0$ is the adaptation gain, $P_1 = P_1^\top$ is the solution of the algebraic equation $A_1^\top P_1 + P_1 A_1 = -Q_1$, $Q_1 > 0$, Proj$(\cdot,\cdot)$ is the projection operator which keeps $\hat{x}_e(t)$ within the pre-specified bound $\|\hat{x}_e(t)\|_\infty \leq \rho$ [8]. The final estimation $x_e(t)$ for $x(t)$ is given by:

$$x_e(s) = C_1(s)\hat{x}_e(s). \quad (13)$$

To streamline the subsequent analysis, we need to introduce several notations. Define $B_1 = 4n\rho^2 + 4n\rho d\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(Q_1)}$, $H_1(s) = (s\mathbb{I}-A_1)^{-1}$, $d = \|A_m\|_{\mathcal{L}_1}\rho+|b|(L_f\rho_u+L\rho+\mu_\sigma)$.

Let $\mu_0 > 0$ be an arbitrarily small positive constant. Next we show that if the choice of Γ_1 and $C_1(s)$ verifies:

$$\|C_1(s)H_1^{-1}(s)\|_{\mathcal{L}_1}\sqrt{\frac{B_1}{\Gamma_1\lambda_{\min}(P_1)}} + \|1-C_1(s)\|_{\mathcal{L}_1}\rho \leq \mu_0\,, \tag{14}$$

then the norm of the estimation error is bounded by μ_0.

Lemma 4: For the system in (1)-(3), and the adaptive estimator given in (11)-(13), for any $\mu_0 > 0$ if

$$\|x_t\|_{\mathcal{L}_\infty} \leq \rho\,,\quad \|u_t\|_{\mathcal{L}_\infty} \leq \rho_u\,, \tag{15}$$

and Γ_1 and $C_1(s)$ satisfy the design constraint in (14), then:

$$\|(x_e - x)_t\|_{\mathcal{L}_\infty} \leq \mu_0\,. \tag{16}$$

Proof. The proof is similar to Theorem 3.4 in [4]. □

V. $\mathcal{L}_1$ Adaptive Controller Design and Analysis

A. Controller Design

Using the estimation $x_e(t)$ for feedback, we design the $\mathcal{L}_1$ adaptive controller, which consists of state predictor, adaptive law and control law. The state predictor is given by:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m\hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}^\top(t)x_e(t) + \hat{\sigma}(t))\,,\\ \hat{y}(t) &= c^\top\hat{x}(t)\,,\ \hat{x}(0) = x_0\,,\end{aligned} \tag{17}$$

where $t = 0$ is the time instant when the measurement of $x(t)$ is available.

The parameter estimations $\hat{\omega}(t)$, $\hat{\theta}(t)$ and $\hat{\sigma}(t)$ are governed by the following adaptive laws:

$$\dot{\hat{\omega}}(t) = \Gamma_c\text{Proj}(\hat{\omega}(t), -(\hat{x}(t)-x_e(t))^\top Pbu(t))\,, \tag{18}$$

$$\dot{\hat{\theta}}(t) = \Gamma_c\text{Proj}(\hat{\theta}(t), -(\hat{x}(t)-x_e(t))^\top Pbx_e(t))\,, \tag{19}$$

$$\dot{\hat{\sigma}}(t) = \Gamma_c\text{Proj}(\hat{\sigma}(t), -(\hat{x}(t)-x_e(t))^\top Pb)\,, \tag{20}$$

$$\hat{\omega}(0) = \hat{\omega}_0\,,\ \hat{\theta}(0) = \hat{\theta}_0\,,\ \hat{\sigma}(0) = \hat{\sigma}_0\,,$$

where $P = P^\top > 0$ is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$, $Q > 0$, $\Gamma_c > 0$ is the adaptation gain. The projection operators keep $\hat{\omega}(t)$, $\hat{\theta}(t)$ and $\hat{\sigma}(t)$ in the pre-specified compact sets $[\omega_l,\ \omega_u]$, Θ and $[-\sigma_b,\ \sigma_b]$, respectively, where ω_l and ω_u are chosen to be nonzero constants with the same sign, and σ_b is given by

$$\sigma_b = \Delta + \|F(s) - (\omega_l+\omega_h)/2\|_{\mathcal{L}_1}\,\rho_u + L\mu_0\,, \tag{21}$$

where μ_0 is the solution of the quadratic function

$$\begin{aligned}&4\|Pb\|L\mu_0^2 + 2\mu_0\|Pb\|(2\|F(s)-(\omega_l+\omega_h)/2\|_{\mathcal{L}_1}\rho_u + 2\Delta\\ &+ (\omega_u-\omega_l)\rho_u + 2L\rho) = (\lambda_{\max}(P)\gamma_0^2 - \beta_2)/\Lambda\,,\end{aligned} \tag{22}$$

while $\Lambda = \lambda_{\max}(P)/\lambda_{\min}(Q)$, and $0 < \beta_2 < \lambda_{\max}(P)\gamma_0^2$.

The control signal $u(t)$ is generated through the feedback of the following system:

$$\chi(s) = D(s)\bar{r}(s)\,,\ u(s) = -k\chi(s)\,, \tag{23}$$

where $\bar{r}(t) = \hat{\omega}(t)u(t) + \hat{\theta}^\top(t)\hat{x}(t) + \hat{\sigma}(t) - k_g r(t)$.

The $\mathcal{L}_1$ adaptive controller consists of (17)-(20) and (23), subject to (7).

B. Closed-loop Reference System

First we consider the closed-loop reference system with its control signal and system response defined as:

$$\begin{aligned}\dot{x}_{ref}(t) =& A_m x_{ref}(t) + b\left(\mu_{ref}(t) + \theta^\top(t)x_{ref}(t) + \sigma_0(t)\right),\\ \mu_{ref}(s) =& F(s)u_{ref}(s)\,,\ \ x_{ref}(0) = x_0\,,\\ u_{ref}(s) =& -kD(s)\bar{r}_{ref}(s)\,,\ \ y_{ref}(t) = c^\top x_{ref}(t)\,,\end{aligned} \tag{24}$$

where $\bar{r}_{ref}(s)$ is the Laplace transformation of $\bar{r}_{ref}(t) = \mu_{ref}(t) + \theta^\top(t)x_{ref}(t) + \sigma_0(t) - k_g r(t)$. The next Lemma establishes stability of the closed-loop reference system (24).

Lemma 5: For the closed-loop reference system in (24), subject to the $\mathcal{L}_1$-norm condition in (7), we have

$$\|x_{ref}\|_{\mathcal{L}_\infty} \leq \rho_r\,,\quad \|u_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{u_r}\,, \tag{25}$$

where ρ_r and ρ_{u_r} are defined in (9)-(10)

Proof. The proof is similar to Lemma 1 in [9]. □

C. Equivalent Linear Time-Varying System

In this section, we demonstrate that the system with unmodeled actuator dynamics in (1) can be transformed into an equivalent linear system with unknown time-varying parameters. Define $\rho_\omega = \|ksD(s)\|_{\mathcal{L}_1}(\rho_u\omega_m+L\rho+\sigma_b+k_g\|r\|_{\mathcal{L}_1})$, where $\omega_m = \max\{|\omega_l|, |\omega_h|\}$. Since $sD(s) = \bar{D}(s)$ is stable and proper, $\|ksD(s)\|_{\mathcal{L}_1}$ is finite.

Lemma 6: Let $\mu(s) = F(s)u(s)$, where $F(s)$ is a stable unknown transfer function.

(i) If $\|u_t\|_{\mathcal{L}_\infty} \leq \rho_u$, there exist ω and $\bar{\sigma}(\tau)$ over $[0,\ t]$ such that $\mu(\tau) = \omega u(\tau) + \bar{\sigma}(\tau)$, where $\omega_l < \omega < \omega_h$ and $|\bar{\sigma}(\tau)| < \|F(s) - (\omega_l+\omega_u)/2\|_{\mathcal{L}_1}\rho_u$.

(ii) If in addition to (i) , $\|\dot{u}_t\|_{\mathcal{L}_\infty} \leq \rho_\omega$, then $\bar{\sigma}(\tau)$ is differentiable and for any $0 \leq \tau \leq t$, $|\dot{\bar{\sigma}}(\tau)| \leq \|F(s) - (\omega_l+\omega_u)/2\|_{\mathcal{L}_1}\rho_\omega$.

Proof. The proof is similar to Lemma 2 in [9]. □

Remark 1: For the $\mathcal{L}_1$ adaptive controller in (17)-(20), (23), suppose $\|x_t\|_{\mathcal{L}_\infty} \leq \rho$ and $\|u_t\|_{\mathcal{L}_1} \leq \rho_u$. Since the projection operators ensure that for any $0 \leq \tau \leq t$, $\theta \in \Theta$, $\omega(\tau) \leq |\omega_m|$, and $|\hat{\sigma}(\tau)| \leq \sigma_b$, we have $\|\bar{r}_t\|_{\mathcal{L}_\infty} \leq \omega_m\rho_u+L\rho+\sigma_b+k_g\|r\|_{\mathcal{L}_\infty}$. The control law in (23) implies $u(s) = -kD(s)\bar{r}(s)$, and hence, $su(s) = -ksD(s)\bar{r}(s)$. It follows from the definition of ρ_ω that $\|\dot{u}_t\|_{\mathcal{L}_\infty} \leq \rho_\omega$.

Remark 2: If $\|u_t\|_{\mathcal{L}_1} \leq \rho_u$, it follows from Lemma 6 (i) that the system in (1) can be rewritten over $[0, t]$ as

$$\dot{x}(\tau) = A_m x(\tau) + b(\omega u(\tau) + \theta^\top(\tau)x(\tau) + \sigma(\tau))\,, \tag{26}$$

where $\sigma(\tau) = \sigma_0(\tau) + \bar{\sigma}(\tau)$ satisfies $|\sigma_t| \leq \Delta + \|F(s) - (\omega_h+\omega_l)/2\|_{\mathcal{L}_1}\rho_u$. Also, if $\|\dot{u}_t\|_{\mathcal{L}_1} \leq \rho_\omega$, the condition in Lemma 6 (ii) ensures $\|\dot{\sigma}_t\|_{\mathcal{L}_\infty} \leq d_\sigma + \|F(s) - (\omega_h+\omega_l)/2\|_{\mathcal{L}_1}\rho_\omega$.

D. Prediction Error Signal

To prove the uniform transient tracking between the closed-loop system with the estimator and $\mathcal{L}_1$ adaptive controller and the reference system in (24), we first need to quantify the prediction error performance. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$ be the prediction error, γ_0 be the desired performance bound for $\|\tilde{x}\|_{\mathcal{L}_\infty}$, and μ_0 be the desired bound for the estimation error introduced in (22).

In preparation for the development that follows, we introduce the following notations: $\Lambda = \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}$, $\kappa_1 = 4\sigma_b^2 + (\omega_u - \omega_l)^2 + 4L\max_i|\theta_i|$, $\kappa_2 = 4Ld_\theta + 4\sigma_b d_{\sigma_1}$, $\kappa_3 = 2\mu_0\|Pb\|((\omega_u-\omega_l)\rho_u + 2L\rho + 2\sigma_b)$, where σ_b is defined in (21), and $d_{\sigma_1} = \|F(s) - (\omega_h+\omega_l)/2\|_{\mathcal{L}_1}\rho_\omega + d_\sigma + d_\theta\mu_0 + 2L\rho$ is the upper bound for $\|(\dot{\sigma}_1)_t\|_{\mathcal{L}_\infty}$.

Lemma 7: For the system in (1)-(4), the adaptive estimator (11)-(13) satisfying the design condition (14), and the $\mathcal{L}_1$ adaptive controller in (17)-(20) and (23), subject to (7), if $\|x_t\|_{\mathcal{L}_\infty} \le \rho$, $\|u_t\|_{\mathcal{L}_\infty} \le \rho_u$, and the adaptive gain Γ_c verifies

$$\Gamma_c \quad > \quad (\Lambda\kappa_1 + \kappa_2)/(\lambda_{\min}(P)\gamma_0^2 - \Lambda\kappa_3)\,, \qquad (27)$$

then the prediction error is bounded $\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \gamma_0$.

Proof. The proof is given in Appendix.

E. Transient and Steady State Performance

We notice that the reference system is not implementable, since it uses the unknown parameters. This closed-loop system is only used for analysis purposes. Next we prove stability and transient performance of the integrated estimator-controller closed-loop system with respect to this reference system.

Theorem 1: Given the system in (1)-(4), the adaptive estimator in (11)-(13) and the $\mathcal{L}_1$ adaptive controller in (17)-20) and (23), subject to (7), if $\|x_0\|_\infty \le \rho_r$, and the design constraints in (14) and (27) hold, then

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \quad \le \quad \gamma_0\,, \qquad (28)$$

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \quad < \quad \gamma_1\,, \qquad (29)$$

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \quad < \quad \gamma_2\,. \qquad (30)$$

Proof. The proof is given in Appendix.

VI. Application to Well Drilling System

A. Plant Model

We use a newly developed third order nonlinear model in [10] to describe the dynamics of the well drilling system. The model has been shown by experiments to be simple and have acceptable fidelity level for calculating the non-measured states and for parameter estimation [11]. Let p_{pump} denote the pressure on the pump side, p_{choke} denote the pressure on the choke side, q_{bit} denote the flow rate through the bit, and p_{bit} denote the bottom hole bit pressure, which is the pressure to be controlled. The system dynamics are given by:

$$\frac{V_d}{\beta_d}\dot{p}_{pump}(t) = \; q_{pump} - q_{bit}(t)\,, \qquad (31)$$

$$\frac{V_a}{\beta_a}\dot{p}_{choke}(t) = \; -K_c z_c(t)\sqrt{\frac{2}{\rho_{a0}}(p_{choke}(t) - p_0)} - \dot{V}_a + q_{bit}(t) + q_{res} + q_{back}(t)\,, \qquad (32)$$

$$M\dot{q}_{bit}(t) = \; p_{pump}(t) - p_{choke}(t) - F_a(q_{bit}(t) + q_{res})^2 - F_d q_{bit}^2(t) + (\rho_{d0} - \rho_{a0})gh_{bit}\,, \qquad (33)$$

$$p_{bit}(t) = \; p_{choke}(t) + M_a\dot{q}_{bit}(t) + F_a q_{bit}^2(t) + \rho_{a0}gh_{bit}\,. \qquad (34)$$

with $p_{pump}(0) = p_{p0}$, $p_{chock}(0) = p_{c0}$, and $q_{bit}(0) = q_{b0}$.

The input signal $z_c(t)$ has the following dynamics

$$\dot{z}_c(s) = F(s)u_c(s)\,, \qquad (35)$$

where $F(s)$ presents the unmodeled dynamics for choke valve, and u_c is the choke opening signal.

Due to the measurement constraints, $q_{bit}(t)$ and $p_{bit}(t)$ are updated at a low rate, and thus are viewed as non-measured signals for controller design. All the coefficients except β_d and V_d are unknown and time-varying, with known conservative bounds. Plugging (33) into (34), and taking derivatives on both sides, we write the dynamics of p_{bit} as:

$$\begin{aligned}\dot{p}_{bit}(t) =& \frac{1}{M}[M_d\dot{p}_c(t) + 2(MF_a - M_aF_d)q_{bit}(t)\dot{q}_{bit}(t) \\ &+ M_a\dot{p}_p(t) - 2M_aF_a(q_{bit}(t) + q_{res}(t))\dot{q}_{bit}(t)] \\ =& \frac{M_a}{M}\frac{\beta_d}{V_d}(q_{pump}(t) - q_{bit}(t)) + \frac{M_d}{M}\frac{\beta_a}{V_a}[q_{bit}(t) - \dot{V}_a \\ &+ q_{res}(t) + q_{back}(t) - K_c z_c\sqrt{\frac{2}{\rho_{a0}}(p_{choke}(t) - p_0)}] \\ &+ \frac{2}{M}[q_{bit}(t)(M_dF_a - M_aF_d) - q_{res}(t)M_aF_a] \\ &[p_{pump}(t) - p_{choke}(t) - F_d q_{bit}^2(t) + (\rho_{d0} - \rho_{a0})gh_{bit} \\ &- F_a(q_{bit}(t) + q_{res}(t))^2]\,.\end{aligned} \qquad (36)$$

B. Estimator Design

Since q_{bit} can be viewed as a time-varying parameter in the p_{pump} dynamics in (31) , and p_{bit} is the linear combination of q_{bit} and q_{bit}^2 in (34) , we can estimate $p_{bit}(t)$ indirectly by two steps. First we apply the fast estimator (11)-(13) to (31). The estimator for q_{bit} is given by

$$\frac{V_d}{\beta_d}\dot{\hat{p}}_{pump}(t) = a_2\tilde{p}_{pump}(t) + q_{pump}(t) - \hat{q}_{bit}(t)\,, \qquad (37)$$

$$\dot{\hat{q}}_{bit}(t) = \Gamma_2\text{Proj}(\hat{q}_{bit}(t)\,, \tilde{p}_{pump}(t))\,, \qquad (38)$$

$$\bar{q}_{bit}(s) = C_2(s)\hat{q}_{bit}(s)\,, \qquad (39)$$

where $\tilde{p}_{pump} = \hat{p}_{pump} - p_{pump}$, $a_2 < 0$, $\Gamma_2 > 0$ is the adaptation gain, and $C_2(s)$ is a low-pass filter. Notice that we modify the state predictor so that subtracting (31) from (37) yields the expected prediction error dynamics $\frac{V_d}{\beta_d}\dot{\tilde{p}}_{pump} = a_2\tilde{p}_{pump} - (\hat{q}_{bit} - q_{bit})$. From (16) we can render the estimation error arbitrarily small by increasing Γ_2 and the bandwidth of $C_2(s)$.

Notice that (34) can be written as

$$p_{bit} = \theta_1 p_{pump}(t) + \theta_2 p_{chock}(t) + \theta_3 q_{bit}(t) + \theta_4 q_{bit}^2(t) + \theta_5\,,$$

where $\theta_i\,, i = 1, \ldots, 5$, are unknown constants. Estimation of p_{bit} can be achieved by the RLS algorithm, which is given by

$$\begin{aligned}&\bar{p}_{bit}(t) = p_{bit}(t_i)\,, \quad t = t_i\,, \\ &\bar{p}_{bit}(t) = \hat{\theta}_1(i)p_{pump}(t) + \hat{\theta}_2(i)p_{choke}(t) + \hat{\theta}_3(i)\bar{q}_{bit}(t) \\ &\qquad + \hat{\theta}_4(i)\bar{q}_{bit}^2(t) + \hat{\theta}_5(i)\,, \quad t_i < t < t_{i+1}\,.\end{aligned} \qquad (40)$$

The parameters $\hat{\theta}_i$ are updated by the RLS algorithm

$$\begin{aligned} L(i) &= \frac{P(i-1)\phi(i)}{\lambda+\phi^\top(i)P(i-1)\phi(i)} \\ \hat{\theta}(i) &= \hat{\theta}(i-1)+L(i)(\omega_{e_i}(i)-\phi^\top(i)\hat{\theta}(i-1)) \\ P(i) &= \frac{1}{\lambda}\left(P(i)-\frac{P(i-1)\phi(i)\phi^\top(i)P(i-1)}{\lambda+\phi^\top(i)P(i-1)\phi(i)}\right), \end{aligned}$$

where $\phi(i) = [p_{pump}(t_i)\ p_{choke}(t_i)\ q_{bit}(t_i)\ q^2_{bit}(t_i)\ 1]^\top$. $P(0) = pI_{5\times5}$ and $\lambda \in (0,1)$. The parameter p is chosen large (10^6), while λ is chosen between 0.95 and 0.99.

C. $\mathcal{L}_1$ Adaptive Controller

The dynamics of p_{bit} in (36) can be written as:

$$\dot{p}_{bit}(t) = a_m p_{bit}(t) + \omega z_c(t) + \theta(t)p_{bit}(t) + \sigma(t), \quad (41)$$

where $a_m < 0$ and $\theta(t)$, ω and $\sigma(t)$ are bounded unknown parameters. The $\mathcal{L}_1$ adaptive controller has the following structure.

The state predictor is given by:

$$\dot{\hat{x}}(t) = a_m\hat{x}(t) + b(\hat{\omega}(t)u(t)+\hat{\theta}^\top(t)\bar{x}(t)+\hat{\sigma}(t)),\hat{x}(0) = x_0 \quad (42)$$

The adaptive laws in equations (18)-(20) take the form:

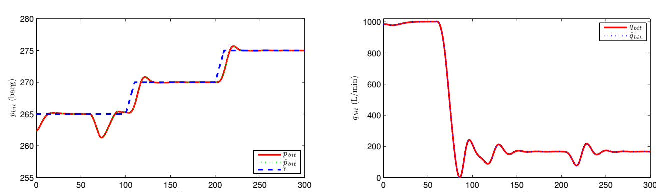
$$\begin{aligned} \dot{\hat{\omega}}(t) &= \Gamma_c\text{Proj}(\hat{\omega}(t), -(\hat{x}(t)-\bar{x}(t))Pbu(t)), \quad (43) \\ \dot{\hat{\theta}}(t) &= \Gamma_c\text{Proj}(\hat{\theta}(t), -(\hat{x}(t)-\bar{x}(t))Pb\bar{x}(t)), \quad (44) \\ \dot{\hat{\sigma}}(t) &= \Gamma_c\text{Proj}(\hat{\sigma}(t), -(\hat{x}(t)-\bar{x}(t))Pb), \quad (45) \end{aligned}$$

The control low, following (23), takes the form:

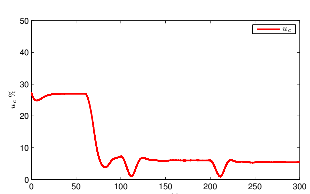
$$z_c(s) = k\chi(s), \ \chi(s) = D(s)r_u(s),$$

where $k > 0$, $r_u(s)$ is the Laplace transformation of $r_u(t) = \hat{\omega}(t)z_c(t) + \bar{r}(t)$, $\bar{r}(t) = \hat{\theta}(t)p_{bit}(t) + \hat{\sigma}(t) - k_g r(t)$, $k_g = -a_m$, while $D(s) = 1/s$.

Remark 3: If $D(s) = 1/s$, the control law becomes $\dot{z}_c(t) = -k(\hat{\omega}(t)z_c(t) + \hat{\theta}(t)p_{bit}(t) + \hat{\sigma}(t) - k_g r(t))$.

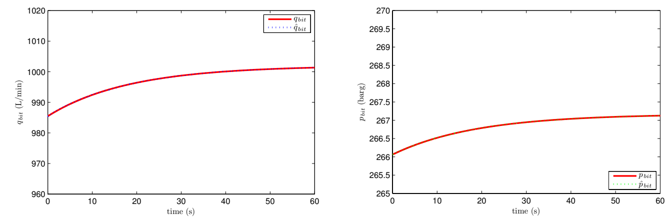


(a) r (dashed), p_{bit} (solid), $\hat{p}_{bit}$ (dotted) (b) q_{bit} (solid), $\bar{q}_{bit}$ (dashed)



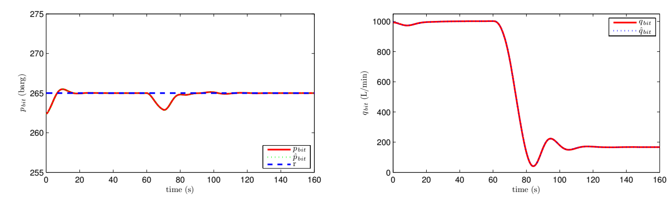
(c) Control signal: $z_c(t)(\%)$

Fig. 2: Drilling under normal condition, $r = 275$ barg.

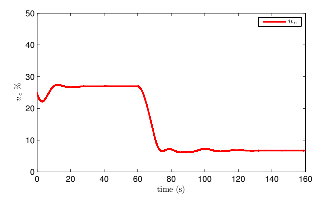


(a) r (dashed), p_{bit} (solid), $\hat{p}_{bit}$ (dotted) (b) q_{bit} (solid), $\bar{q}_{bit}$ (dashed)

Fig. 3: Estimator performance, $z_c = 25\%$.



(a) r (dashed), p_{bit} (solid), $\hat{p}_{bit}$ (dotted) (b) q_{bit} (solid), $\bar{q}_{bit}$ (dashed)



(c) Control signal: $z_c(t)(\%)$

Fig. 4: Control during pipe-connection (q_{pump} drops).

D. Simulation Results

In this section we give the simulation results of the integrated estimator-controller scheme for the system introduced in (VI-A). The parameters are given in Section IV of [11], and the unmodeled actuator dynamics is given by $F(s) = 1/(s^2 + 1.4s + 1)$. In the implementation of the estimator and the $\mathcal{L}_1$ adaptive controller, we set $A_1 = -1$, $A_m = -0.2$, $b = 1$, $C_1(s) = \frac{1000}{s+1000}$, $D(s) = \frac{1}{s}$, $k = 2$, $\Gamma_1 = 5000$, $\Gamma_c = 10000$, and the bounds for the parameters are chosen to be: $\omega_l = 0.1$, $\omega_u = 1$, $d_\theta = 10$, $\mu_\sigma = 1.2$.

The simulations are done under two scenarios. First we consider the drilling under normal conditions, when $q_{res} = 0$. The initial steady state value of p_{bit} is 263 barg, and the final reference pressure is 275 barg. The pressure set is done in 3 steps. The results are shown in Fig. 2. The performance of open-loop estimators for q_{bit} and p_{bit} is shown in Fig 3.

The second scenario is the pipe connection, during which the pumping of fluid is first stopped, then a new pipe segment is mounted to the drill string, and finally the pump is restarted. To demonstrate the performance of the controller, we simulate the scenario of power loss, an even more severe scenario, as compared to a sudden drop of q_{pump}. We see from Fig. 4 that with the sudden drop of q_{pump} from 1000 l/min to nearly 0 l/min in 10 seconds at time instant $t = 40s$, the $\mathcal{L}_1$ adaptive estimator and controller regulate the bit pressure with desired transient and steady performance.

VII. Conclusion

The paper presents integrated estimator-controller scheme applicable for MPD control system. The $\mathcal{L}_1$ adaptive controller achieves guaranteed performance bounds for system's input and output signals in the presence of time-varying parameters and disturbances, with the state being sampled at a significantly low rate.

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Appendix

Proof of Lemma 7. Let $\sigma_1(\tau) = \sigma(\tau) + \theta^\top(\tau)(x(\tau) - x_e(\tau))$. Then (26) can be written as

$$\dot{x}(\tau) = A_m x(\tau) + b(\omega u(\tau) + \theta^\top(\tau) x_e(\tau) + \sigma_1(\tau)) . \qquad (46)$$

From (46) and (17) we have the prediction error dynamics

$$\dot{\tilde{x}}(\tau) = A_m \tilde{x}(\tau) + b(\tilde{\omega}(\tau)u(\tau) + \tilde{\theta}^\top(\tau) x_e(\tau) + \tilde{\sigma}_1(\tau)) , \quad (47)$$

where $\tilde{\omega}(\tau) = \hat{\omega}(\tau) - \omega(\tau)$, $\tilde{\theta}(\tau) = \hat{\theta}(\tau) - \theta(\tau)$, and $\tilde{\sigma}_1(\tau) = \hat{\sigma}(\tau) - \sigma_1(\tau)$. From Remark 1, $\|\dot{u}_t\|_{\mathcal{L}_\infty} \le \rho_\omega$.

Now consider the Lyapunov function $V(\tau) = \tilde{x}^\top(\tau) P \tilde{x}(\tau) + \frac{1}{\Gamma}(\tilde{\theta}^\top(\tau)\tilde{\theta}(\tau) + \tilde{\omega}^2(\tau) + \tilde{\sigma}_1^2(\tau))$. The projection based adaptive law ensures the following upper-bound $\dot{V}(\tau) \le -\tilde{x}^\top(\tau) Q \tilde{x}(\tau) - \frac{2}{\Gamma}(\tilde{\theta}^\top(\tau)\dot{\theta}(\tau) + \tilde{\sigma}_1(\tau)\dot{\sigma}_1(\tau)) + 2(x_e(\tau) - x(\tau))^\top P b(\tilde{\omega}(\tau)u(\tau) + \tilde{\theta}^\top(\tau) x_e(\tau) + \tilde{\sigma}_1(\tau))$. From the definitions of Λ, κ_1, κ_2 and κ_3, we have the bound $\|[-\frac{2}{\Gamma}(\tilde{\theta}^\top\dot{\theta} + \tilde{\sigma}_1\dot{\sigma}_1) + 2(x_e - x)^\top P b(\tilde{\omega}u + \tilde{\theta}^\top x_e + \tilde{\sigma}_1)]_t\|_{\mathcal{L}_\infty} \le \kappa_3 + \frac{\kappa_2}{\Gamma_c}$, $\|(\tilde{\theta}^\top\tilde{\theta} + \tilde{\omega}^2 + \tilde{\sigma}_1^2)_t\|_{\mathcal{L}_\infty} \le \kappa_1$.

If $V(\tau) > \Lambda(\kappa_3 + \kappa_2/\Gamma_c) + \kappa_1/\Gamma_c$, then $\tilde{x}^\top(\tau) P \tilde{x}(\tau) > \Lambda(\kappa_3 + \kappa_2/\Gamma_c)$, so $\tilde{x}^\top(\tau) Q \tilde{x}(\tau) \ge \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \tilde{x}^\top(\tau) P \tilde{x}(\tau) > \kappa_3 + \kappa_2/\Gamma_c$. Consequently $\dot{V}(\tau) < 0$. Since $\hat{x}$ is initialized at the time instance that the measurement of x is available, $\tilde{x}(0) = 0$, so $V(0) < \Lambda(\kappa_3 + \kappa_2/\Gamma_c) + \kappa_1/\Gamma_c$. Thus we have $V(\tau) < \Lambda(\kappa_3 + \kappa_2/\Gamma_c) + \kappa_1/\Gamma_c$ for $0 \le \tau \le t$.

From the choice of μ_0 in (22), $\Lambda\kappa_3 = \lambda_{\max}(P)\gamma_0^2 - \beta_2 < \lambda_{\max}(P)\gamma_0^2$. If the adaptive gain Γ_c satisfies (27), we have $\Lambda(\kappa_3 + \kappa_2/\Gamma_c) + \kappa_1/\Gamma_c < \lambda_{\max}(P)\gamma_0^2$, and thus $\tilde{x}^\top(\tau)\tilde{x}(\tau) \le \frac{V(\tau)}{\lambda_{\max}(P)} < \gamma_0^2$. Consequently we have $\|\tilde{x}_t\|_{\mathcal{L}_\infty} < \gamma_0$.

Proof of Theorem 1. We prove the theorem by contradiction. Assume (29) or (30) do not hold. Then since $\|x(0) - x_{ref}(0)\|_\infty = 0 \le \gamma_1$, $u(0) - u_{ref}(0) = 0$, and x, x_{ref}, u, u_{ref} are continuous, there exists $t \ge 0$ such that

$$\|x(t) - x_{ref}(t)\|_\infty = \gamma_1 , \quad \text{or} \quad \|u(t) - u_{ref}(t)\|_\infty = \gamma_2 \quad (48)$$

while $\|(x - x_{ref})_t\|_{\mathcal{L}_\infty} \le \gamma_1$, $\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \le \gamma_2$. Since Lemma 5 implies that $\|x_{ref}\|_{\mathcal{L}_\infty} \le \rho_r$, $\|u_{ref}\|_{\mathcal{L}_\infty} \le \rho_{u_r}$, we have $\|x_t\|_{\mathcal{L}_\infty} \le \gamma_1 + \rho_r = \rho$, $\|u_t\|_{\mathcal{L}_\infty} \le \gamma_2 + \rho_{ur} = \rho_u$. It follows from Remark 1 that $\|\dot{u}_t\|_{\mathcal{L}_\infty} \le \rho_\omega$. Thus from Lemma 7 we have $\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \gamma_0$.

Let $\tilde{r}(\tau) = \tilde{\omega}(\tau)u(\tau) + \tilde{\theta}^\top(\tau)x_e(\tau) + \tilde{\sigma}_1(\tau)$, $r_1 = \theta^\top(\tau)x_{ref}(\tau) + \sigma_0(\tau)$, $r_2(\tau) = \theta^\top(\tau)x(\tau) + \sigma_0(\tau)$. It follows from (23) and Lemma 6 that $u(s) = -kD(s)(\tilde{r}(s) + r_2(s) - k_g r(s) + \omega u(s) + \bar{\sigma}(s))$. It follows from (2) and (5) that

$$\mu_{ref}(s) = -C(s)(r_1(s) - k_g r(s)) , \quad (49)$$

$$\mu(s) = -C(s)(r_2(s) - k_g r(s) + \tilde{r}(s)) . \quad (50)$$

Then the system in (1) consequently takes the form

$$x(s) = G(s)r_2(s) + H(s)C(s)(k_g r(s) - \tilde{r}(s)) + (s\mathbb{I} - A_m)^{-1}x_0 . \quad (51)$$

Let $e(\tau) = x(\tau) - x_{ref}(\tau)$. Then by (51) we have

$$e(s) = G(s)r_3(s) - H(s)C(s)\tilde{r}(s) , \quad e(0) = 0 , \quad (52)$$

where $r_3(s)$ is the Laplace transformation of $r_3(\tau) = \theta^\top(\tau)e(\tau)$. Lemma 1 gives the following bound:

$$\|e_t\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}\|(r_3)_t\|_{\mathcal{L}_\infty} + \|(r_4)_t\|_{\mathcal{L}_\infty} , \quad (53)$$

where $r_4(t)$ is the signal with its Laplace transformation being $r_4(s) = C(s)H(s)\tilde{r}(s)$. From (47) we have $\tilde{x}(s) = H(s)\tilde{r}(s)$, which leads to $r_4(s) = C(s)\tilde{x}(s)$, and hence $\|(r_4)_t\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. By definition of L we have $\|r_{3t}\|_{\mathcal{L}_\infty} \le L\|e_t\|_{\mathcal{L}_\infty}$. From (53) we have $\|e_t\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1} L \|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. The upper bound from Lemma 7 and the $\mathcal{L}_1$-norm upper bound from (7) lead to the following upper bound

$$\|e_t\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}/(1 - \|G(s)\|_{\mathcal{L}_1}L)\gamma_0 , \quad (54)$$

which along with (8) leads to

$$\|e_t\|_{\mathcal{L}_\infty} \le \gamma_1 - \beta_1 < \gamma_1 . \quad (55)$$

Thus from (49) and (50) we have $u(s) - u_{ref}(s) = (\mu(s) - \mu_{ref}(s))/F(s) = -r_3(s)C(s)/F(s) - r_5(s)$, where $r_5(s) = C(s)/F(s)\tilde{r}(s)$. Hence it follows from Lemma 1 that

$$\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \le L\|\frac{C(s)}{F(S)}\|_{\mathcal{L}_1}\|(x - x_{ref})_t\|_{\mathcal{L}_\infty} + \|(r_5)_t\|_{\mathcal{L}_\infty} . \quad (56)$$

By (6), $r_5(s) = \frac{C(s)}{F(s)}\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{F(s)}\frac{N_d(s)}{N_n(s)}c_o^\top \tilde{x}(s)$. Since $\frac{C(s)}{F(s)}$ is stable and strictly proper, the complete system $\frac{C(s)}{F(s)}\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ norm exists and is finite. Hence, we have

$$\|(r_5)_t\|_{\mathcal{L}_\infty} \le \left\| \frac{C(s)}{F(s)}\frac{1}{c_o^\top H(s)}c_o^\top \right\|_{\mathcal{L}_1} \|\tilde{x}_t\|_{\mathcal{L}_\infty} . \quad (57)$$

Combining (55), (56) , (57) we have

$$\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \le L\left\|\frac{C(s)}{F(s)}\right\|_{\mathcal{L}_1}(\gamma_1 - \beta_1) + \left\| \frac{C(s)}{F(s)}\frac{1}{c_o^\top H(s)}c_o^\top \right\|_{\mathcal{L}_1}\gamma_0 < \gamma_2 \quad (58)$$

We notice that the upper bounds in (55) and (58) contradict the equality in (48), which proves (29)-(30). Since the bounds in (29)-(30) hold uniformly in t, Lemma 7 implies (28).