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Transformations of Markov Processes in Fault Tolerant Interconnected Systems

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***Abstract*— Safety-critical control systems use fault tolerant interconnections of components to minimize the effect of randomly triggered faults. The system availability process indicates whether or not the interconnection is operating correctly at each time instant. It is a 2-state process that results from the transformation of the stochastic processes characterizing the availability processes of the interconnected components. To analyze closed-loop systems controlled by these fault tolerant interconnected components, it is important to determine the characteristics of the system availability process. When the availability processes of the interconnected components are independent homogeneous Markov chains, the statistical nature of the system availability process is characterized. In particular, it is shown that the system availability process is not necessarily Markov, but has a well-defined one-step transition probability matrix that approaches a constant stochastic matrix at steady-state. Since it is simpler to analyze switched closed-loop systems when the switching process is Markov, conditions for the system availability process to be a Markov chain for all initial distributions are determined. A sufficient stability condition is given when the system availability process is a non-homogeneous Markov chain for a class of initial distributions.**

I. INTRODUCTION

In fault tolerant systems such as those found in flight control systems, the system availability process indicates whether or not an interconnection of digital logic devices can perform its intended operation at a given time [1]–[3]. When these devices are operating in a harsh environment, the system availability process is induced by the stochastic upsets affecting each device. The devices can be redundant flight control computers (FCC's) or a lower level interconnection of components. The paper focuses on fault tolerant interconnections of FCC's, but the results apply to an arbitrary interconnection of devices. In general, the system availability process depends on the correct operation of a sufficient number of interconnected devices. Consider a particular operation performed by the fault tolerant interconnection of $N \geq 2$ devices and assume that the devices are affected by N independent and identically distributed (i.i.d.) upset processes. Let the state of operation at time $k \in \mathbb{Z}^+ \triangleq \{0, 1, \dots\}$ of the m-th device be denoted by $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N \triangleq \{1, \dots, N\}$ such that $\boldsymbol{z}_m(k) = 0$ ($\boldsymbol{z}_m(k) = 1$) denotes that the m-th device is working (not working). Boldfaced characters will denote a random variable or process. The process $\boldsymbol{z}_m(k)$ represents the state of the device with state space $\mathcal{I} \triangleq \{0, 1\}$ and input given by an i.i.d. upset process. When the transition to the next state, $\boldsymbol{z}_m(k+1)$, from the current state and input is representable by a state transition table, it is known that the state processes of these devices are independent, homogeneous, first-order Markov chains [4]. The ambient probability space over which these processes are defined is given by $(\Omega, \mathcal{F}, \mathrm{Pr})$. In this paper, all Markov chains (MC) satisfy the first-order Markov property, that is, if $\boldsymbol{z}(k)$ is a MC then

$$\mathrm{Pr}\big(\boldsymbol{z}(k+1) = \zeta(k+1) \mid \boldsymbol{z}(k) = \zeta(k), \dots, \boldsymbol{z}(0) = \zeta(0)\big) = \mathrm{Pr}\big(\boldsymbol{z}(k{+}1) = \zeta(k{+}1) \mid \boldsymbol{z}(k) = \zeta(k)\big)$$

where $\mathrm{Pr}(\boldsymbol{z}(k) = \zeta(k), \dots, \boldsymbol{z}(0) = \zeta(0)) > 0$. The system availability process, $\boldsymbol{y}(k)$, is given by a memoryless transformation of these N Markov chains $\boldsymbol{z}_m(k)$. This paper analyzes some of the characteristics of $\boldsymbol{y}(k)$ which need to be known in order to analyze a closed-loop system controlled by such interconnected devices. Such a system appears, for example,

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in [5]. In particular, for a general system availability transformation, the state probabilities (i.e., $\Pr(\boldsymbol{y}(k) = y), y \in \mathcal{I}$) and the one-step transition probabilities (i.e., $\Pr(\boldsymbol{y}(k+1) = j | \boldsymbol{y}(k) = i), i, j \in \mathcal{I}$) are derived. Steady-state values of these probabilities are also given. Clearly, if $\boldsymbol{y}(k)$ is not a MC, these transition probabilities and initial distribution do not characterize the state probabilities at each time instant. But $\boldsymbol{y}(k)$ has a well-defined time-dependent transition probability matrix $\Pi_y(k)$ satisfying $\lim_{k\to\infty} \Pi_y(k) = \Pi$. This steady-state characterization is used to analyze the long term characteristics of a closed-loop system switched by this process under an additional assumption. To simplify the analysis of switched closed-loop systems, it is useful to determine conditions for the system availability process to be a homogeneous Markov chain (HMC). Based on the extensive literature for lumped Markov processes, two results are presented. First, necessary and sufficient conditions for the system availability process to be an HMC are provided [6], [7]. Second, a result from [8] is given which shows that the system availability process can be a non-homogeneous Markov chain (NHMC) only for a class of initial distributions. For this special set of system availability processes, a sufficient condition for exponential second moment stability exists. The switched system can also be analyzed with the joint process $(\boldsymbol{z}(k), \boldsymbol{y}(k))$, which is known to be a HMC [9]. The analysis presented here is motivated by the desirability of lower dimensional computations, resulting from directly working with a 2-state process instead of tests based on a higher dimensional joint process.

The rest of the paper is organized as follows. Section II characterizes the statistical nature of the system availability process. In Section III, two lumpability results and an example are given. The stability analysis of a switched closed-loop system for a special class of system availability processes is done in Section IV, and the conclusions are summarized in Section V.

II. A General System Availability Transformation

For each k, the system availability process $\boldsymbol{y}(k)$ is a transformation, ϕ, of the random vector $\boldsymbol{z}(k) \triangleq (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_N(k))$ into $\mathcal{I}$, where $\phi : \mathcal{I}^N \to \mathcal{I}$, $\mathcal{I}^N \triangleq \underbrace{\mathcal{I} \times \cdots \times \mathcal{I}}_{N \text{ times}}$. It is assumed that ϕ is an onto map. It is known that the joint process $\boldsymbol{z}(k)$ is an HMC, since the component Markov processes $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$, are independent and homogeneous [10]. The transition probability matrix of $\boldsymbol{z}(k)$ is also known to be $\Pi_z = \Pi_{z_1} \otimes \cdots \otimes \Pi_{z_N}$, where Π_{z_m} is the transition probability matrix for each $\boldsymbol{z}_m(k)$ and $\otimes$ denotes the Kronecker product. The system availability transformation ϕ partitions the state space of $\boldsymbol{z}(k)$ as follows: $\mathcal{I}^N = I_\phi \cup \overline{I}_\phi$, where $I_\phi = \phi^{-1}(0) = \{\zeta \in \mathcal{I}^N : \phi(\zeta) = 0\}$.

The statistical nature characterization of $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ is given in this section. It is known that, in general, $\boldsymbol{y}(k)$ will not be a MC for all initial distribution of $\boldsymbol{z}(k)$ [8]. First, the state probability vector and steady-state probability vector are characterized in Lemma 1 and Theorem 1, respectively. Second, the transition probabilities and their steady-state values are characterized in Lemma 2 and Theorem 2, respectively.

Lemma 1: Let each $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$, be an independent HMC with initial distribution $\pi_{z_m}(0)$ such that the joint process $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_N(k))$ is an HMC on $\mathcal{I}^N$ with transition probability matrix $\Pi_z = \Pi_{z_1} \otimes \cdots \otimes \Pi_{z_N}$ and initial distribution vector $\pi_z(0)$. Let $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ be the system availability process. Then the state probability vector of $\boldsymbol{y}(k)$, $\pi_y(k) \triangleq \begin{bmatrix}\Pr(\boldsymbol{y}(k) = 0) & \Pr(\boldsymbol{y}(k) = 1)\end{bmatrix}$, is characterized by

$$\Pr(\boldsymbol{y}(k) = 0) = \sum_{\zeta \in I_\phi} \prod_{m=1}^{N} \pi_{z_m}(0) \Pi_{z_m}^k \begin{bmatrix} 1_{\{\zeta_m = 0\}} \\ 1_{\{\zeta_m = 1\}} \end{bmatrix} \quad (1)$$

and $\Pr(\boldsymbol{y}(k) = 1) = 1 - \Pr(\boldsymbol{y}(k) = 0)$, where $1_{\{\cdot\}}$ is the indicator function of $\{\cdot\}$, and ζ_m is the m-th component of ζ.

Proof: Since ϕ is a measurable mapping, it follows that

$$\Pr(\boldsymbol{y}(k) = 0) = \sum_{\zeta \in I_\phi} \Pr(\boldsymbol{z}(k) = \zeta).$$

From the assumption that the processes $\boldsymbol{z}_m(k)$ are independent HMC's, the following equalities hold

$$\begin{aligned}\Pr(\boldsymbol{y}(k) = 0) &= \sum_{\zeta \in I_\phi} \prod_{m=1}^{N} \Pr(\boldsymbol{z}_m(k) = \zeta_m) \\ &= \sum_{\zeta \in I_\phi} \prod_{m=1}^{N} \pi_{z_m}(k) \begin{bmatrix} 1_{\{\zeta_m = 0\}} \\ 1_{\{\zeta_m = 1\}} \end{bmatrix},\end{aligned}$$

where $\pi_{z_m}(k) \triangleq \left[\Pr(\boldsymbol{z}_m(k) = 0) \quad \Pr(\boldsymbol{z}_m(k) = 1)\right]$ is the state probability vector of $\boldsymbol{z}_m(k)$. Since $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$ is an HMC, it follows that

$$\Pr(\boldsymbol{y}(k) = 0) = \sum_{\zeta \in I_\phi} \prod_{m=1}^{N} \pi_{z_m}(0)\Pi_{z_m}^k \begin{bmatrix} 1_{\{\zeta_m=0\}} \\ 1_{\{\zeta_m=1\}} \end{bmatrix}.$$

The probability $\Pr(\boldsymbol{y}(k) = 1)$ can be similarly shown and it satisfies $\Pr(\boldsymbol{y}(k) = 1) = 1 - \Pr(\boldsymbol{y}(k) = 0)$. ■

The probability $\lim_{k\to\infty} \Pr(\boldsymbol{y}(k) = 0)$ is called the *availability* of the system. It is computed in Theorem 1 and shown to be constant under the additional assumptions that the independent HMC's $\boldsymbol{z}_m(k)$ are aperiodic and irreducible. Notice that under these conditions the joint process $\boldsymbol{z}(k)$ is also aperiodic and irreducible [10].

Theorem 1: Let $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$ be aperiodic and irreducible HMC's with stationary probability vectors π_{z_m}. Then the availability of the system is

$$\lim_{k\to\infty} \Pr(\boldsymbol{y}(k) = 0) = \sum_{\zeta \in I_\phi} \prod_{m=1}^{N} \pi_{z_m} \begin{bmatrix} 1_{\{\zeta_m=0\}} \\ 1_{\{\zeta_m=1\}} \end{bmatrix}. \quad (2)$$

Proof: Under the given assumptions, the limit exists and (2) follows directly from (1). ■

Since ϕ reduces the 2^N states of the HMC $\boldsymbol{z}(k)$ down to two, it is a type of general lumping Markov transformation that has been extensively studied since the 1950's (cf. [6]–[8], [11], [12]). Thus, necessary and sufficient conditions are well known for: $\boldsymbol{y}(k)$ to be a MC, an HMC, and a NHMC. To simplify the presentation, the 2^N possible states of $\boldsymbol{z}(k)$, labeled in their natural last-lexical order [12], are assigned values in $L = \{1, 2, \ldots, 2^N\}$. Let $\xi : \mathcal{I}^N \to L$ denote the bijective function that maps a state to an integer label in L, such as, $\xi((0,0,\ldots,0)) = 1$ and $\xi((1,1,\ldots,1)) = 2^N$. Thus, ϕ induces through ξ the partition $L = L_\phi \cup \overline{L}_\phi$, where $L_\phi = \{l \in L : l = \xi(\zeta), \forall\ \zeta \in I_\phi\}$. The following $2^N \times 2$ lumping matrix [7] characterizes this partition and is useful in the analysis of the lumping operation: $M_\phi = [m_{ij}]$, where $m_{ij} = 1$ whenever $\phi(\xi^{-1}(i)) = 1$; otherwise, $m_{ij} = 0$, for $i \in L$ and $j \in \{1, 2\}$. The columns of M_ϕ will be denoted by M_j, $j \in \{1, 2\}$.

The following lemma gives the transition probabilities of the process $\boldsymbol{y}(k)$.

Lemma 2: Let $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$ be aperiodic and irreducible HMC's, and let the transition probability matrix of $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_N(k))$ be $\Pi_z = \left[p_{ij}^z\right]$, $i, j \in L$. Then the diagonal entries of $\Pi_y(k)$, the transition probability matrix of $\boldsymbol{y}(k)$, are

$$p_{11}^y(k) = \frac{1}{\pi_z(0)\Pi_z^k M_1} \sum_{i,j \in L_\phi} p_{ij}^z \pi_z(0)\Pi_z^k e_i \quad (3)$$

and

$$p_{22}^y(k) = \frac{1}{\pi_z(0)\Pi_z^k M_2} \sum_{i,j \in \overline{L}_\phi} p_{ij}^z \pi_z(0)\Pi_z^k e_i \ , \quad (4)$$

where $\overline{p}_{11}^y = \Pr(\boldsymbol{y}(k+1) = 0 | \boldsymbol{y}(k) = 0)$, $\overline{p}_{22}^y = \Pr(\boldsymbol{y}(k+1) = 1 | \boldsymbol{y}(k) = 1)$ and $e_i \in \mathbb{R}^{2^N}$ is the vector of zeros with a single 1 in the i-th position.

Proof: The proof is given for $p_{11}^{\boldsymbol{y}}(k)$ since the other case is similar.

$$\begin{aligned}
&\Pr(\boldsymbol{y}(k+1) = 0 \mid \boldsymbol{y}(k) = 0) \\
&= \Pr\big(\boldsymbol{z}(k+1) \in \cup_{j\in L_\phi}\{\xi^{-1}(j)\} \mid \\
&\qquad \boldsymbol{z}(k) \in \cup_{i\in L_\phi}\{\xi^{-1}(i)\}\big) \\
&= \Bigg(\sum_{i,j\in L_\phi} \Pr\big(\boldsymbol{z}(k+1) = \xi^{-1}(j) \mid \\
&\qquad \boldsymbol{z}(k) = \xi^{-1}(i)\big) \Pr\big(\boldsymbol{z}(k) = \xi^{-1}(i)\big)\Bigg) \Bigg/ \\
&\qquad \sum_{i\in L_\phi} \Pr\big(\boldsymbol{z}(k) = \xi^{-1}(i)\big) \\
&= \frac{\sum_{i,j\in L_\phi} p_{ij}^z \Pr(\boldsymbol{z}(k) = \xi^{-1}(i))}{\sum_{i\in L_\phi} \Pr(\boldsymbol{z}(k) = \xi^{-1}(i))} \\
&= \frac{\sum_{i,j\in L_\phi} p_{ij}^z \pi_z(k) e_i}{\pi_z(k) M_1} \\
&= \frac{\sum_{i,j\in L_\phi} p_{ij}^z \pi_z(0)\Pi_z^k e_i}{\pi_z(0)\Pi_z^k M_1}.
\end{aligned}$$

Observe that the scalars $\pi_z(0)\Pi_z^k M_j, j = 1, 2$ are positive because each column of M_ϕ has at least one 0 and one 1 from the onto assumption concerning ϕ, and Π_z has positive entries since each 2-state HMC $\boldsymbol{z}_m(k)$ is aperiodic and irreducible. ■

As a consequence of Lemma 2 the following steady-state result is obtained.

Theorem 2: Let $\boldsymbol{z}_m(k)$, $m \in \mathscr{I}_N$ be aperiodic and irreducible HMC's, and let the transition probability matrix of $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_N(k))$ be $\Pi_z = \left[p_{ij}^z\right]$,

$i, j \in L$ with stationary probability vector given by π_z. Then $\lim_{k\to\infty} \Pi_y(k) = \Pi$, where Π is a stochastic matrix with diagonal entries

$$\overline{p}_{11} = \lim_{k\to\infty} p^y_{11}(k) = \frac{1}{\pi_z M_1} \sum_{i,j\in L_\phi} p^z_{ij}\pi_z e_i$$

and

$$\overline{p}_{22} = \lim_{k\to\infty} p^y_{22}(k) = \frac{1}{\pi_z M_2} \sum_{i,j\in \overline{L}_\phi} p^z_{ij}\pi_z e_i \ .$$

Proof: Because of the hypothesis, the stationary probability vector π_z has positive components. Then the claim follows directly by taking limits in equations (3) and (4), respectively. ■

Since the analysis of a closed-loop system switched by $\boldsymbol{y}(k)$ is simplified when $\boldsymbol{y}(k)$ is an HMC, the next section gives conditions under which $\boldsymbol{y}(k)$ is an HMC.

III. HMC CONDITIONS

Lemma 3 gives necessary and sufficient conditions under which the process $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ will be an HMC for all initial distributions. It is a special case of theorems in the literature (cf. [6], [7]) where $\boldsymbol{y}(k)$ is a 2-state process.

Lemma 3: Let $\boldsymbol{z}(k)$ be an HMC with transition probability matrix Π_z. Then the process $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ is an HMC for every initial distribution $\pi_z(0)$ if and only if there exists constants μ_1 and μ_2 in $[0, 1]$ satisfying

$$\Pi_i M_1 = \mu_1 \ \ \forall\ i \in L_\phi$$

and

$$\Pi_i M_1 = \mu_2 \ \ \forall\ i \in \overline{L}_\phi,$$

where Π_i is the i-th row of Π_z. Furthermore, the transition probability matrix of $\boldsymbol{y}(k)$ is $\Pi_y \triangleq \left[p^y_{ij}\right] = \left[\begin{smallmatrix}\mu_1 & 1-\mu_1\\ \mu_2 & 1-\mu_2\end{smallmatrix}\right]$, $i, j \in \{1, 2\}$.

Proof: The proof follows from [6]. ■

The next result shows that $\boldsymbol{y}(k)$ can be an NHMC only for some but not all initial distributions [8].

Lemma 4: Let $\boldsymbol{z}(k)$ be an HMC. If the process $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ is a MC for all initial distributions $\pi_z(0)$ then it is an HMC.

Proof: This lemma is stated as a question and then answered in [8]. ■

In the reliability literature, the map ϕ is called a *structure function* [2]. A general class of interconnected devices is referred to as κ-out-of-N when the interconnection is operating correctly if at least κ of the devices are working with $\kappa \in \mathscr{I}_N$. Note that there are N κ-out-of-N structure functions uniquely determined by κ. When $\kappa = 1$, the structure function corresponds to a parallel interconnection as in the following example that illustrates the results described in this paper.

TABLE I

TRANSFORMATION TABLE IN EXAMPLE 1

$\boldsymbol{z}_1(k)$	$\boldsymbol{z}_2(k)$	$\boldsymbol{z}(k)$	$\xi(\boldsymbol{z}(k))$	$\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$
0	0	$(0, 0)$	1	0
0	1	$(0, 1)$	2	0
1	0	$(1, 0)$	3	0
1	1	$(1, 1)$	4	1

Example 1: Consider an interconnection of $N = 2$ devices with upset processes given by an HMC with transition probability matrices

$$\Pi_{z_i} \triangleq \begin{bmatrix} p^i_{11} & p^i_{12} \\ p^i_{21} & p^i_{22} \end{bmatrix}, \ i = 1, 2$$

and initial distributions $\pi_{z_i}(0)$, $i = 1, 2$. If the system availability process is given by the transformation ϕ defined in Table I, then the state space is partitioned as $\mathcal{I}^2 = I_\phi \bigcup \overline{I}_\phi$, where $I_\phi = \{(0, 0), (0, 1), (1, 0)\}$ and $\overline{I}_\phi = \{(1, 1)\}$. By Lemma 1, the probability

$$\begin{aligned}\Pr(\boldsymbol{y}(k) = 0) =& \pi_{z_1}(0)\Pi^k_{z_1} \begin{bmatrix}1\\0\end{bmatrix} \pi_{z_2}(0)\Pi^k_{z_2} \begin{bmatrix}1\\1\end{bmatrix} + \\ & \pi_{z_1}(0)\Pi^k_{z_1} \begin{bmatrix}0\\1\end{bmatrix} \pi_{z_2}(0)\Pi^k_{z_2} \begin{bmatrix}1\\0\end{bmatrix}.\end{aligned}$$

The stationary probability of $\boldsymbol{y}(k)$ exists whenever $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ have stationary probabilities π_{z_1} and π_{z_2}, respectively, as given in Theorem 1. In particular

$$\begin{aligned}&\lim_{k\to\infty} \Pr(\boldsymbol{y}(k) = 0)\\ &\quad = \pi_{z_1} \begin{bmatrix}1\\0\end{bmatrix} \pi_{z_2} \begin{bmatrix}1\\1\end{bmatrix} + \pi_{z_1} \begin{bmatrix}0\\1\end{bmatrix} \pi_{z_2} \begin{bmatrix}1\\0\end{bmatrix}.\end{aligned}$$

Lemma 3 is used to determine the conditions for $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ to be an HMC. Observe that ϕ partitions the set of labels as $L = L_\phi \bigcup \overline{L}_\phi$, where $L_\phi = \{1, 2, 3\}$ and $\overline{L}_\phi = \{4\}$. Thus, the matrix

$$M^T_\phi = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the superscript 'T' denotes matrix transposition. So the process $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ will be an HMC if and only if the following equalities are satisfied:

$$\begin{aligned}\Pi_1 M_1 &= \mu_1 = 1 - p_{12}^1 \times p_{12}^2 \\ \Pi_2 M_1 &= \mu_1 = 1 - p_{12}^1 \times p_{22}^2 \\ \Pi_3 M_1 &= \mu_1 = 1 - p_{22}^1 \times p_{12}^2.\end{aligned}$$

Lemma 3 gives a fourth equation, $\Pi_4 M_1 = \mu_2$, that is not needed, since it is dependent on the first three equations. These relations imply that

$$p_{12}^1 \times p_{12}^2 = p_{12}^1 \times p_{22}^2 = p_{22}^1 \times p_{12}^2.$$

If these equalities do not hold, then $\boldsymbol{y}(k)$ will not be an HMC for all initial distributions $\pi_z(0)$. By Lemma 4, however, $\boldsymbol{y}(k)$ could be a NHMC for some but not all initial distributions. Whenever the stationary probability vector for $\boldsymbol{z}(k)$ exists then as $k \to \infty$ $\boldsymbol{y}(k)$ is characterized by a constant transition probability matrix as shown in Theorem 2. If the 2-states HMC's $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ are aperiodic and irreducible then the entries of Π_{z_1} and Π_{z_2} are positive. Thus, the necessary and sufficient conditions for $\boldsymbol{y}(k)$ to be an HMC are

$$p_{12}^1 = p_{22}^1 \text{ and } p_{12}^2 = p_{22}^2.$$

In this case, Π_{z_1} and Π_{z_2} have the form

$$\Pi_{z_1} \triangleq \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}, \quad \Pi_{z_2} \triangleq \begin{bmatrix} b & 1-b \\ b & 1-b \end{bmatrix}, \qquad (5)$$

where $a = 1 - p_{12}^1$ and $b = 1 - p_{12}^2$ with $a, b \in (0,1)$. If the initial distribution vectors are $\pi_{z_1}(0) = \begin{bmatrix} a & 1-a \end{bmatrix}$ and $\pi_{z_2}(0) = \begin{bmatrix} b & 1-b \end{bmatrix}$, then processes $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ with transition probability matrices in (5) are i.i.d. processes. In addition, Π_y has equal rows, since it follows that $\mu_1 = \mu_2$ and $\pi_y(0) = \pi_z(0) M_\phi$ give $\pi_y(0) = \begin{bmatrix} 1-\mu_1 & \mu_1 \end{bmatrix}$. Thus, $\boldsymbol{y}(k)$ is not, in general, an i.i.d process but it approximates an i.i.d. process in the limit.

IV. Dynamical Systems

In this section, exponential second moment stability of a dynamical system switched by $\boldsymbol{y}(k)$ is considered. Let $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ be a NHMC for $\pi_y(0) = \pi_0 \in \Phi$ (Φ is a subset of the set of all initial distributions) taking values in $\mathcal{I}$ with transition probability matrix $\Pi_y(k)$. Now consider the following Markov jump linear system

$$\boldsymbol{x}(k+1) = A_{\boldsymbol{y}(k)} \boldsymbol{x}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad (6)$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$ for $i \in \mathcal{I}$; and $\boldsymbol{x}_0$ is a random vector with finite second moment that is independent of $\boldsymbol{y}(k)$ for $k \geq 0$. Exponential second moment stability is defined next [13].

Definition 1: The equilibrium point at 0 of system (6) (or simply, system (6)) is called *exponentially second moment stable* if for every value of the initial condition $\boldsymbol{x}_0$ and every initial distribution $\pi_0 \in \Phi$ of $\boldsymbol{y}(k)$ there exists α and β, both positive and independent of $\boldsymbol{x}_0$ and π_0 such that $E\{\|\boldsymbol{x}(k)\|^2\} \leq \alpha \|\boldsymbol{x}_0\| e^{-\beta k}$ $\forall$ $k \geq 0$.

An exponentially second moment stability test for (6) follows.

Theorem 3: Let $\boldsymbol{y}(k) = \phi(\boldsymbol{z}(k))$ be a NHMC for $\pi_0 \in \Phi$. If $\lim_{k\to\infty} \Pi_y(k) = \Pi$, where Π is a stochastic matrix, then system (6) is exponential second moment stable if the spectral radius of $\mathcal{A}$ is less than one, where

$$\mathcal{A} \triangleq (\Pi^{\mathrm{T}} \otimes I_{n^2}) \operatorname{diag}(A_0 \otimes A_0, A_1 \otimes A_1).$$

Proof: When $\boldsymbol{y}(k)$ is a NHMC, Theorem 2 gives conditions that lead to a constant matrix approximation of the transition probability matrix $\Pi_y(k)$. In this case, the result follows from [13]. ■

V. Conclusions

A characterization of the system availability process has been done for a general transformation $\boldsymbol{y}(k)$ of Markov upset processes affecting the interconnected fault tolerant devices. Since $\boldsymbol{y}(k)$ models the jump process in many jump linear systems, it is necessary to develop analysis tools for this class of processes. An initial result in that direction is Theorem 3 that gives a sufficient condition for exponential second moment stability when $\boldsymbol{y}(k)$ is a NHMC for some initial distributions.

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