2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Commutant Lifting for Linear Time-Varying Systems

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Abstract— **In this paper, we study two robust control problems for possibly infinite dimensional (i.e., systems with an infinite number of states) linear time-varying (LTV) systems using a framework based on a version of the commutant lifting theorem developed for nest algebras. The approach is purely operator theoretic and does not use any state space representation. The two problems studied include the optimal disturbance attenuation and the optimal mixed sensitivity problems for LTV systems. The proposed solutions are given in terms of projections of time-varying multiplication operators. The latter are computed explicitly.**

Definitions and Notation

- $\mathcal{B}(E,F)$ denotes the space of bounded linear operators from a Banach space E to a Banach space F, endowed with the operator norm

$$\|A\| := \sup_{x\in E,\ \|x\|\leq 1} \|Ax\|, \ \ A \in \mathcal{B}(E,F)$$

- ℓ^2 denotes the usual Hilbert space of square summable sequences with the standard norm

$$\|x\|_2^2 := \sum_{j=0}^{\infty} |x_j|^2, \ \ x := \big(x_0, x_1, x_2, \cdots\big) \in \ell^2$$

- P_k the usual truncation operator for some integer k, which sets all outputs after time k to zero.
- An operator $A \in \mathcal{B}(E,F)$ is said to be causal if it satisfies the operator equation:

$$P_k A P_k = P_k A, \ \forall k \ \text{ positive integers}$$

- $tr(\cdot)$ denotes the trace of its argument.

The subscript "$_c$" denotes the restriction of a subspace of operators to its intersection with causal operators (see [29], [7] for the definition). "$\oplus$" denotes for the direct sum of two spaces. "*" stands for the adjoint of an operator or the dual space of a Banach space depending on the context [5].

I. Introduction

Linear Time varying (LTV) systems are becoming of more and more interest not only because they model time varying practical processes, but also they provide good approximations for time invariant nonlinear systems [1]. Indeed, There have been numerous papers on controlling time-varying systems (for e.g. [13], [14], [7], [10], [31], [33], [15], [11], [16], [26], [27] and references therein). In [13], [14] and more recently [7] the authors studied the optimal weighted sensitivity minimization problem, the two-block problem, and the model-matching problem for LTV systems using inner-outer factorization for positive operators. They obtained abstract solutions involving the computation of norms of certain operators. In [10], [15], [16] the authors use state space models for LTV systems and algorithms are given in terms of infinite dimensional Riccati equations or inequalities.

In [31], [34] a geometric framework for robust stabilization of infinite-dimensional time-varying systems was developed. The uncertainty was described in terms of its graph and measured in the gap metric. Several results on the gap metric and the gap topology were established

For linear time invariant (LTI) systems the commutant lifting theorem has played an important role in solving several robust control problems. In [22], [23] the theorem has been used in solving classical interpolation problems related to H^∞ control problems. In [6] the commutant lifting theorem was used to solve a MIMO two block H^∞ problem. In [30], [28] the commutant lifting theorem was used in conjunction of a particular two block H^∞ that is related to computing the gap metric for LTI systems. The latter was introduced to study stability robustness of feedback systems and induces the weakest topology in which feedback stability and performance is robust [2], [30], [28], [32]. In [7], some of the results obtained in [30] were generalized, in particular, the gap metric for time-varying systems was generalized to a two-block time varying optimization analogous to the two-block H^∞-optimization proposed in [30]. This was achieved by introducing a metric which is the supremum of the sequence of gaps between the plants measured at every instant of time. The latter reduces to the standard gap metric for LTI systems. In [8], [7], [12] using the time-varying gap metric it is shown that the ball of uncertainty in the time-varying gap metric of a given radius is equal to the ball of uncertainty of the same radius defined by perturbations of a normalized right coprime fraction, provided the radius is smaller than a certain quantity. In [27], the authors showed that the TV directed gap reduces to the computation of an operator with a TV Hankel plus Toeplitz structure. Computation of the norm of such an operator can be carried out using an iterative scheme. The minimization in the TV directed gap formula was shown to be a minimum using duality theory.

In this paper we adapt a specific generalization of the commutant lifting theorem to nest algebras [4] (Chapter 20), in our context, the algebra of lower triangular operators

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representing causal LTV systems, to solve two important robust control problems: The optimal TV sensitivity and optimal mixed sensitivity problems. These problems were studied in terms of Banach space duality theory in [26], but several assumptions related to the plant and weights were made. The approach taken here is completely different in that it relies solely on the commutant lifting theorem, and does not require any computation of dual spaces and annihilators (see [26]). Moreover, unlike duality theory, the commutant lifting approach can be extended to continuous time-varying systems by working in the so-called Macaev ideal (see [4] for the definition). We point out that the results obtained for the mixed sensitivity apply to computing the TV gap metric as well without resorting to duality theory.

Our approach is purely input-output and does not use any state space realization, therefore the results derived here apply to *infinite dimensional LTV systems*, i.e., TV systems with an *infinite number of state variables* [31]. Although the theory is developed for causal stable system, it can be extended in a straightforward fashion to the unstable case using coprime factorization techniques for LTV systems developed in [33], [14], [7].

The rest of the paper is organized as follows. In section II the commutant lifting theorem for nest algebras is introduced. In section III the solution of the optimal disturbance rejection problem for LTV plants is derived. Section IV discusses the mixed sensitivity problem for LTV systems and its solution using commutant lifting. Section V contains a summary of the paper contributions.

II. Mathematical Background

In this section we introduce the mathematical framework needed in the sequel. For more details refer to [4] Chapter 20. Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces, and let T and T' be contractions on $\mathcal{H}$ and $\mathcal{H}'$, respectively. The Sz. Nagy dilation Theorem asserts that T and T' have unitary dilations U and U', respectively, on Hilbert spaces $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{H}' \subset \mathcal{K}'$, such that [4]

$$\begin{aligned} T^n &= P_{\mathcal{H}} U^n \mid_{\mathcal{H}}, \;\; n = 1,\, 2,\, \cdots && (1) \\ T'^n &= P_{\mathcal{H}'} U'^n \mid_{\mathcal{H}'}, \;\; n = 1,\, 2,\, \cdots && (2) \end{aligned}$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, likewise for $P_{\mathcal{H}'}$ is the orthogonal projection of $\mathcal{K}'$ onto $\mathcal{H}'$.
In addition, $\mathcal{H}$ and $\mathcal{H}'$ can be expressed as orthogonal differences of subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ of $\mathcal{K}$, and subspaces $\mathcal{H}'_1$ and $\mathcal{H}'_2$ of $\mathcal{K}'$ as follows

$$\mathcal{H} = \mathcal{H}_1 \ominus \mathcal{H}_2, \quad \mathcal{H}' = \mathcal{H}'_1 \ominus \mathcal{H}'_2 \qquad (3)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant under U, i.e., $U\mathcal{H}_1 \subset \mathcal{H}_1$ and $U\mathcal{H}_2 \subset \mathcal{H}_2$. Similarly for $\mathcal{H}'_1$ and $\mathcal{H}'_2$.

Now Suppose that $\mathcal{N}$ is a nest of subspaces of a Hilbert space $\mathcal{K}$, i.e., $\mathcal{N}$ is a collection of closed subspaces of $\mathcal{K}$ ordered under inclusion.

Let $\mathcal{T}(\mathcal{N})$ be the algebra of all bounded linear operators of a Hilbert space $\mathcal{L}$ which leave invariant every subspace N in $\mathcal{N}$, i.e., $A \in \mathcal{T}(\mathcal{N})$, $AN \subset N$.

By a representation of $\mathcal{T}(\mathcal{N})$ we mean an algebra homomorphism h from $\mathcal{T}(\mathcal{N})$ into the algebra $\mathcal{B}(\mathcal{H},\, \mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Such a representation is contractive if $\|h(A)\| \leq \|A\|$ for all $A \in \mathcal{T}(\mathcal{N})$. The representation h is weak* continuous if $h(A_\alpha)$ converges to zero in the weak* topology of $\mathcal{B}(\mathcal{H},\, \mathcal{H})$ whenever the net $\{A_\alpha\}$ converges to zero in the weak* topology of $\mathcal{B}(\mathcal{L}, \mathcal{L})$, and that it is unital if it maps the identity operator on $\mathcal{L}$ to the identity operator on $\mathcal{H}$. The Sz. Nagy dilation theorem asserts that there is a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a positive representation h' of $\mathcal{B}(\mathcal{L}, \mathcal{L})$ such that [4]

$$P_{\mathcal{H}} h'(A) f = h(A) f, \quad \forall f \in \mathcal{H}, \quad \forall \mathcal{A} \in \mathcal{T}(\mathcal{N}) \qquad (4)$$

The following result is the analogue of the commutant lifting theorem for representations of nest algebras.

Theorem 1: (Theorem 20.22 and Remark 20.24 in [4]) Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces, and $h : \mathcal{T}(\mathcal{N}) \longmapsto B(\mathcal{H}, \mathcal{H})$ and $h' : \mathcal{T}(\mathcal{N}) \longmapsto \mathcal{B}(\mathcal{H}', \mathcal{H}')$ be two unital, weak*, contractive representations with $\mathcal{B}(\mathcal{L}, \mathcal{L})$-dilations $H : \mathcal{B}(\mathcal{L}, \mathcal{L}) \longmapsto \mathcal{B}(\mathcal{K}, \mathcal{K})$ and $H' : \mathcal{B}(\mathcal{L}, \mathcal{L}) \longmapsto \mathcal{B}(\mathcal{K}', \mathcal{K}')$, respectively, and assume that Ξ is a linear operator from $\mathcal{H}$ into $\mathcal{H}'$ with $\|\Xi\| \leq 1$, which intertwines h and h', i.e., $\Xi h(A) = h'(A)\Xi$, $\forall A \in \mathcal{T}(\mathcal{N})$. Then there is an operator $\Upsilon : \mathcal{K} \longmapsto \mathcal{K}'$ such that

1. Υ is a contraction.
2. Υ intertwines the $\mathcal{B}(\mathcal{L})$-dilations H and H' of h and h'
3. Υ dilates Ξ.

This Theorem will be used to solve two robust control problems for LTV systems. We start with the optimal disturbance rejection problem for LTV systems which corresponds to the optimal H^∞ problem for LTI systems.

III. Optimal LTV Disturbance Rejection

In this section we consider the problem of optimizing performance for causal linear time varying systems. Let P denote a causal stable linear time varying plant, and K denotes a time varying controller. The closed-loop transmission from external disturbances w to controlled output z is denoted by T_{zw}. Using the standard Youla parametrization of all stabilizing controllers the closed loop operator T_{zw} can be written as [7],

$$T_{zw} = T_1 - T_2 Q T_3$$

where T_1, T_2 and T_3 are stable causal time-varying operators, that is, T_1, T_2 and $T_3 \in B_c(\ell^2, \ell^2)$. In this paper we assume without loss of generality that P is stable, the Youla parameter $Q := K(I+PK)^{-1}$ is then an operator belonging to $B_c(\ell^2, \ell^2)$, and is related univoquely to the controller K [29]. Note that Q is allowed to be *time-varying*. If P is unstable it suffices to use the coprime factorization techniques in [33], [7] which lead to similar results. The magnitude of the signals w and z is measured in the ℓ^2-norm. Two problems are considered here optimal disturbance rejection which corresponds to the optimal standard H^∞ problem in the LTI case, and the mixed sensitivity problem for LTV systems which includes a robustness problem in the gap metric studied in [7], [31]. Note that for the latter problem P is assumed to be unstable and we have to use coprime factorizations. The performance index can be written in the following form

$$\begin{aligned} \mu &:= \inf\{\|T_{zw}\| : K \text{ being stabilizing} \\ &\qquad \text{linear time-varying controller}\} \\ &= \inf_{Q\in\mathcal{B}_c(\ell^2,\ell^2)} \|T_1 - T_2QT_3\| \end{aligned} \tag{5}$$

define a nest $\mathcal{N}$ as a family of closed subspaces of the Hilbert space ℓ^2 containing $\{0\}$ and ℓ^2 which is closed under intersection and closed span. Let $Q_n := I - P_n$, for $n = -1, 0, 1, \cdots$, where $P_{-1} := 0$ and $P_\infty := I$. Then Q_n is a projection, and we can associate to it the following nest $\mathcal{N} := \{Q_n\ell^2,\ n = -1, 0, 1, \cdots\}$. The triangular or nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$. That is

$$\mathcal{T}(\mathcal{N}) = \{A \in \mathcal{B}(\ell^2,\ell^2) :\ (I - Q_n)AQ_n = 0,\ \forall\, n\}$$

Note that the Banach space $\mathcal{B}_c(\ell^2,\ell^2)$ is identical to the nest algebra $\mathcal{T}(\mathcal{N})$. For N belonging to the nest $\mathcal{N}$, N has the form $Q_n\ell^2$ for some n.

An operator $A \in \mathcal{T}(\mathcal{N})$ is outer if A commutes with each Q_n and $AQ_n\ell^2$ is dense in $Q_n\ell^2 \cap A\ell^2$. $U \in \mathcal{B}_c(\ell^2,\ell^2)$ is inner if U is a partial isometry and $U^\star U$ commutes with every Q_n [4]. Applying these notions to the time-varying operator $T_2 \in \mathcal{B}_c(\ell^2,\ \ell^2)$, we get $T_2 = T_{2i}T_{2o}$, where T_{2i} and T_{2o} are inner outer operators in $\mathcal{B}_c(\ell^2,\ell^2)$, respectively [4]. Similarly, co-inner-co-outer factorization can be defined and the operator T_3 can be factored as $T_3 = T_{3co}T_{3ci}$ where $T_{3ci} \in \mathcal{B}_c(\ell^2,\ell^2)$ is co-inner that is $T_{3ci}^\star$ is inner, $T_{3co} \in \mathcal{B}_c(\ell^2,\ell^2)$ is co-outer, that is, $T_{3co}^\star$ is outer. The performance index μ in (5) can then be written as

$$\mu = \inf_{Q\in\mathcal{B}_c(\ell^2,\ell^2)} \|T_1 - T_{2i}T_{2o}QT_{3co}T_{3ci}\| \tag{6}$$

Following the classical H^∞ control theory [19], [6], [35],we assume **(A1)** that T_{2o} and T_{3co} are invertible both in $\mathcal{B}_c(\ell^2,\ell^2)$. This assumption guarantees that the map $Q \longrightarrow T_{2o}\mathcal{B}_c(\ell^2,\ell^2)T_{3co}$ is one-one onto. In the time-invariant case this assumption means essentially that the outer factor of the plant P is invertible [19], [21]. Under this assumption we have $T_{2i}^\star T_{2i} = I$ and $T_{3ci}T_{3ci}^\star = I$. By "absorbing" the operators T_{2o} and T_{3co} into the "free" operator Q, expression (6) is then equivalent to

$$\mu = \inf_{Q\in\mathcal{B}_c(\ell^2,\ell^2)} \|T_{2i}^\star T_1 T_{3ci}^\star - Q\| \tag{7}$$

Let $\mathcal{C}_2$ denote the class of compact operators on ℓ^2 called the Hilbert-Schmidt or Schatten 2-class [17], [4] under the norm,

$$\|A\|_2 := \Big(tr(A^\star A)\Big)^{\frac{1}{2}} \tag{8}$$

Note that $\mathcal{C}_2$ is a Hilbert space with inner product

$$< A,\ B >:= tr(B^\star A) \tag{9}$$

Define the space $\mathcal{A}_2 := \mathcal{C}_2 \cap \mathcal{B}_c(\ell^2,\ell^2)$, then $\mathcal{A}_2$ is the space of causal Hilbert-Schmidt operators.
Define the orthogonal projection $\mathcal{P}$ of $\mathcal{C}_2$ onto $\mathcal{A}_2$. $\mathcal{P}$ is the lower triangular truncation. Following [24] an operator X in $\mathcal{B}(\ell^2,\ell^2)$ determines a TV Hankel operator H_X on $\mathcal{A}_2$ if

$$\begin{aligned} H_X &: \mathcal{A}_2 \longmapsto \mathcal{A}_2^\perp \\ H_XA &= (I - \mathcal{P})XA, \quad \text{for } A \in \mathcal{A}_2 \end{aligned} \tag{10}$$

where $\mathcal{A}_2^\perp$ is the orthogonal complement of $\mathcal{A}_2$ in $\mathcal{C}_2$.

We have then the following Theorem which relates the optimal performance μ to the induced norm of the Hankel operator $H_{T_{2i}^\star T_1 T_{3ci}^\star}$. We use the commutant lifting theorem to prove it.

Theorem 2: Under assumptions (A1) the following holds:

$$\mu = \|H_{T_{2i}^\star T_1 T_{3ci}^\star}\| \tag{11}$$

$$\quad = \|(I - \mathcal{P})T_{2i}^\star T_1 T_{3ci}^\star\| \tag{12}$$

Proof. To apply Theorem 1 to our setting let $\mathcal{H} = \mathcal{A}_2$ and $\mathcal{H}' = \mathcal{A}_2^\perp$, and define the representations h and h' of $\mathcal{T}(\mathcal{N}) = \mathcal{B}_c(\ell^2,\ell^2)$ by

$$\begin{aligned} h : &\quad \mathcal{B}_c(\ell^2,\ell^2) \longmapsto B(\mathcal{A}_2,\mathcal{A}_2) \\ h(A) &:= R_A,\ \ A \in \mathcal{B}_c(\ell^2,\ell^2) \end{aligned} \tag{13}$$

and

$$h' : \quad \mathcal{B}_c(\ell^2,\ell^2) \longmapsto \mathcal{B}(\mathcal{A}_2^\perp,\mathcal{A}_2^\perp) \tag{14}$$

$$h'(A) := P_{\mathcal{H}'}R_A,\ \ A \in \mathcal{B}_c(\ell^2,\ell^2) \tag{15}$$

where R_A denotes the right multiplication associated to the operator A defined on the specified Hilbert space. By the Sz. Nagy dilation Theorem there exist dilations H (respectively H') for h (respectively h'), $H = H'$ given by

$$H(A) = R_A \quad \text{on } \mathcal{C}_2, \quad \text{for } A \in \mathcal{B}(\mathcal{C}_2,\mathcal{C}_2) \tag{16}$$

The spaces $\mathcal{A}_2$ and $\mathcal{A}_2^\perp$ can be written as orthogonal differences of subspaces invariant under H and H', respectively, as

$$\mathcal{H} = \mathcal{A}_2 \ominus 0, \qquad \mathcal{H}' = \mathcal{C}_2 \ominus \mathcal{A}_2 \tag{17}$$

and we see that h and h' are representations of $\mathcal{T}(\mathcal{N})$.
Now we define the operator $\Gamma := P_{\mathcal{A}_2^\perp}L_{T_{2i}^\star T_1 T_{3ci}^\star}$ acting from $\mathcal{H}$ into $\mathcal{H}'$, where $L_{T_{2i}^\star T_1 T_{3ci}^\star}$ is the left multiplication operator associated to $T_{2i}^\star T_1 T_{3ci}^\star$, i.e., $L_{T_{2i}^\star T_1 T_{3ci}^\star}f = T_{2i}^\star T_1 T_{3ci}^\star f,\ f \in \mathcal{H}$. Now we have to show that the operator

Γ intertwines h and h', that is, $h'(A)\Gamma = \Gamma h(A)$ for all $A \in \mathcal{B}_c(\ell^2, \ell^2)$, and for all $f \in \mathcal{H}$,

$$\begin{aligned} h'(A)\Gamma f &= P_{\mathcal{H}'} R_A P_{\mathcal{H}'} L_{T_{2i}^\star T_1 T_{3ci}^\star} f && (18)\\ &= P_{\mathcal{H}'} P_{\mathcal{H}'} T_{2i}^\star T_1 T_{3ci}^\star f A && (19)\\ &= P_{\mathcal{H}'} R_A L_{T_{2i}^\star T_1 T_{3ci}^\star} f && (20)\\ &= P_{\mathcal{H}'} R_A L_{T_{2i}^\star T_1 T_{3ci}^\star} f && (21)\\ &= P_{\mathcal{H}'} L_{T_{2i}^\star T_1 T_{3ci}^\star} R_A f && (22)\\ &= P_{\mathcal{H}'} L_{T_{2i}^\star T_1 T_{3ci}^\star} P_{\mathcal{H}} R_A f && (23)\\ &= \Gamma h(A) && (24) \end{aligned}$$

Applying Theorem 1 implies that Γ has a dilation Γ' that intertwines H and H', i.e., $\Gamma' H(A) = H'(A)\Gamma'$, $\forall A \in B(\ell^2, \ell^2)$. By a result in [25] Γ' is a left multiplication operator acting from $\mathcal{A}_2$ into $\mathcal{B}(\ell^2,\ \ell^2)$. That is, $\Gamma' = L_F$ for some $F \in \mathcal{B}_c(\mathcal{A}_2,\ \mathcal{C}_2 \oplus \mathcal{C}_2)$, with $\|F\| = \|\Gamma'\| = \|\Gamma\|$. Now $\Gamma = P_{\mathcal{H}'} L_{T_{2i}^\star T_1 T_{3ci}^\star} = P_{\mathcal{H}'} L_F$, which implies $P_{\mathcal{H}'} L_{T_{2i}^\star T_1 T_{3ci}^\star - F} = 0$. Hence, $(T_{2i}^\star T_1 T_{3ci}^\star - F) f \in \mathcal{A}_2$ for all $f \in \mathcal{A}_2$. That is,

$$(T_{2i}^\star T_1 T_{3ci}^\star - F) f = g,\ \exists g \in \mathcal{A}_2 \tag{25}$$

In particular,

$$(T_{2i}^\star T_1 T_{3ci}^\star - F) f \in \mathcal{T}(\mathcal{N}) \tag{26}$$

for all $f \in \mathcal{T}(\mathcal{N})$ of finite rank. By Theorem 3.10 [4] there is a sequence F_n of finite rank contractions in $\mathcal{B}_c(\ell^2, \ell^2)$ which converges to the identity operator in the strong *-topology. By an approximation argument it follows that $(T_{2i}^\star T_1 T_{3ci}^\star - F) \in \mathcal{B}_c(\ell^2,\ \ell^2)$. Letting $Q := T_{2i}^\star T_1 T_{3ci}^\star - F$ we have $g = Qf$. We conclude that $T_{2i}^\star T_1 T_{3ci}^\star - F = Q$, that is, $T_{2i}^\star T_1 T_{3ci}^\star - Q = F$, with $\|F\| = \|\Gamma\|$ as required.

As a consequence there exists an optimal $Q_o \in \mathcal{B}_c(\ell^2, \ell^2)$ such that the infimum in (5) is achieved. By Theorem 2.1. [24] a Hankel operator H_B is a compact operator if and only if B belongs to the space $\mathcal{B}_c(\ell^2, \ell^2) + \mathcal{C}$, where $\mathcal{C}$ is the space of compact operators from ℓ^2 into ℓ^2, that is, B is the sum of a causal bounded linear operator and a compact operator both defined on ℓ^2. It follows in our case that $H_{T_{2i}^\star T_1 T_{3ci}^\star}$ is a compact operator on $\mathcal{A}_2$ iff $T_{2i}^\star T_1 T_{3ci}^\star \in \mathcal{B}_c(\ell^2, \ell^2) + \mathcal{C}$. In this case there exists an $A \in \mathcal{A}_2$, $\|A\|_2 = 1$ such that

$$\|H_{T_{2i}^\star T_1 T_{3ci}^\star}\| = \|H_{T_{2i}^\star T_1 T_{3ci}^\star} A\|_2$$

that is, a A achieves the norm of $H_{T_{2i}^\star T_1 T_{3ci}^\star}$, and we necessarily have

$$Q_o A = T_{2i}^\star T_1 T_{3ci}^\star A - H_{T_{2i}^\star T_1 T_{3ci}^\star} A \tag{27}$$

which gives the optimal Q_o as the solution of the operator identity (27).

IV. The Optimal TV Mixed Sensitivity Problem

The mixed sensitivity problem for stable plants [21], [35] involves the sensitivity operator $T_1 := \begin{pmatrix} W \\ 0 \end{pmatrix}$, the complementary sensitivity operator $T_2 = \begin{pmatrix} W \\ V \end{pmatrix} P$ and $T_3 := I$ which are all assumed to belong to $\mathcal{B}_c(\ell^2, \ell^2 \times \ell^2)$, and is given by the optimization

$$\mu_o = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \left\| \begin{pmatrix} W \\ 0 \end{pmatrix} - \begin{pmatrix} W \\ V \end{pmatrix} PQ \right\| \tag{28}$$

where $\|\cdot\|$ stands for the operator norm in $\mathcal{B}(\ell^2,\ \ell^2 \times \ell^2)$. Assume that $W^\star W + V^\star V > 0$, i.e., $W^\star W + V^\star V > 0$ is a positive operator. Then there exists an outer spectral factorization $\Lambda_1 \in \mathcal{B}_c(\ell^2, \ell^2)$, invertible in $\mathcal{B}_c(\ell^2, \ell^2)$ such that $\Lambda_1^\star \Lambda_1 = W^\star W + V^\star V$ [3], [7]. Therefore $\Lambda_1 P_o$ as a bounded linear operator in $\mathcal{B}_c(\ell^2, \ell^2)$ has a polar decomposition $U_1 G$, where U_1 is a partial isometry and G a positive operator both defined on ℓ^2 [7]. Next we assume (**A2**) U_1 is unitary, the operator G and its inverse $G^{-1} \in \mathcal{B}_c(\ell^2, \ell^2)$. (A2) is satisfied when, for e.g., the outer factor of the plant is invertible. Let $R = T_2 \Lambda_1^{-1} U_1$, assumption (A2) implies that the operator $R^\star R \in \mathcal{B}(\ell^2, \ell^2)$ has a bounded inverse. According to Arveson (Corollary 2, [3]), the self-adjoint operator $R^\star R$ has a spectral factorization of the form: $R^\star R = \Lambda^\star \Lambda$, where $\Lambda,\ \Lambda^{-1} \in\ \mathcal{B}_c(\ell^2, \ell^2)$. Define $R_2 = R\Lambda^{-1}$, then $R_2^\star R_2 = I$. After "absorbing" Λ into the free parameter Q, the optimization problem (28) is then equivalent to:

$$\mu_o = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \|T_1 - R_2 Q\| \tag{29}$$

Let Π be the orthogonal projection on the subspace $(\mathcal{A}_2 \oplus \mathcal{A}_2) \ominus R_2 \mathcal{A}_2$ the orthogonal complement of $R_2 \mathcal{A}_2$ in the operator Hilbert space $\mathcal{A}_2 \oplus \mathcal{A}_2$ under the inner product

$$(A,\ B) := tr(B^\star A),\quad A,\ B \in \mathcal{A}_2 \oplus \mathcal{A}_2$$

In the following Lemma the orthogonal projection Π_1 is computed explicitly.

Lemma 1:

$$\Pi = I - R_2 \mathcal{P} R_2^\star \tag{30}$$

Proof. For $Z \in \mathcal{A}_2 \oplus \mathcal{A}_2$, let us compute

$$\begin{aligned} & (I - R_2\mathcal{P}R_2^\star)^2 Z = (I - R_2\mathcal{P}R_2^\star)(I - R_2\mathcal{P}R_2^\star) Z \\ = & (I - R_2\mathcal{P}R_2^\star - R_2\mathcal{P}R_2^\star + R_2\mathcal{P}R_2^\star\mathcal{P}R_2^\star) Z \\ = & (I - 2R_2\mathcal{P}R_2^\star + R_2\mathcal{P}R_2^\star) Z \\ & \text{since } R_2^\star R_2 = I \text{ and } \mathcal{P}^2 = \mathcal{P} \\ = & (I - R_2\mathcal{P}R_2^\star) Z \qquad (31) \end{aligned}$$

so $(I - R_2\mathcal{P}R_2^\star)$ is indeed a projection.
Clearly the adjoint $(I - R_2\mathcal{P}R_2^\star)^\star$ of $(I - R_2\mathcal{P}R_2^\star)$ is equal to $(I - R_2\mathcal{P}R_2^\star)$ itself, so that $(I - R_2\mathcal{P}R_2^\star)$ is an orthogonal projection. Next we show that the null space of $(I - R_2\mathcal{P}R_2^\star)$, $Ker(I - R_2\mathcal{P}R_2^\star) = R_2\mathcal{A}_2$.
Let $Z \in Ker(I - R_2\mathcal{P}R_2^\star)$ then

$$(I - R_2\mathcal{P}R_2^\star) Z = 0 \Longrightarrow Z = R_2\mathcal{P}R_2^\star Z$$

since $R_2^\star Z \in \mathcal{C}_2$, then $\mathcal{P} R_2^\star Z \in \mathcal{A}_2$ and therefore $Z \in R_2\mathcal{A}_2$. Hence $Ker(I - R_2\mathcal{P}R_2^\star) \subset R_2\mathcal{A}_2$. Conversely, let $Z \in \mathcal{A}_2$, then

$$(I - R_2\mathcal{P}R_2^\star)R_2 Z = R_2 Z - R_2\mathcal{P}Z = KZ - KZ = 0$$

Thus $R_2 Z \in Ker(I - R_2\mathcal{P}R_2^\star)$, so $R_2\mathcal{A}_2 \subset Ker(I - R_2\mathcal{P}R_2^\star)$, and therefore $(I - R_2\mathcal{P}R_2^\star) = R_2\mathcal{A}_2$, and the Lemma is proved.

Call $S := (\mathcal{A}_2 \oplus \mathcal{A}_2) \ominus R_2\mathcal{A}_2$, and define the operator

$$\begin{aligned} \Phi \quad &: \quad \mathcal{A}_2 \longmapsto S \\ \Phi \quad &:= \quad \Pi\, T_1 \end{aligned} \tag{32}$$

then Φ is a well defined bounded linear operator.

Theorem 3: Under assumption (A2) there exists at least one optimal TV operator $Q_o \in \mathcal{B}_c(\ell^2, \ell^2)$ such that

$$\mu_o = \|T_1 - R_2 Q_o\| = \|\Phi\| \tag{33}$$

Proof. To prove the Theorem we need a representation of $\mathcal{B}_c(\ell^2, \ell^2)$, that is, an algebra homomorphism, say, $h(\cdot)$ (respectively $h'(\cdot)$), from $\mathcal{B}_c(\ell^2, \ell^2)$, into the algebra $B(\mathcal{A}_2, \mathcal{A}_2)$ (respectively $\mathcal{B}_c(S, S)$), of bounded linear operators from $\mathcal{A}_2$ into $\mathcal{A}_2$ $\big($respectively from S into $S\big)$. Define the representations h and h' by

$$h: \quad \mathcal{B}_c(\ell^2, \ell^2) \longmapsto B(\mathcal{A}_2, \mathcal{A}_2) \tag{34}$$

$$h(A) \quad := R_A, \;\; A \in \mathcal{B}_c(\ell^2, \ell^2) \tag{35}$$

and

$$h': \quad \mathcal{B}_c(\ell^2, \ell^2)) \longmapsto \mathcal{B}_c(S, S) \tag{36}$$

$$h'(A) \quad := \Pi R_A, \;\; A \in \mathcal{B}_c(\ell^2, \ell^2))$$

where R_A denotes the right multiplication associated to the operator A defined on the specified Hilbert space. By the Sz. Nagy dilation Theorem there exist dilations H (respectively H') for h (respectively h') given by

$$\begin{aligned} H(A) \;\; &= R_A \;\text{ on }\; \mathcal{A}_2 \;\text{ for }\; A \in \mathcal{B}_c(\ell^2, \ell^2) \\ H'(A) \;\; &= R_A \;\text{ on }\; \mathcal{A}_2 \oplus \mathcal{A}_2 \;\text{ for }\; A \in \mathcal{B}_c(\ell^2, \ell^2) \end{aligned}$$

The spaces $\mathcal{A}_2$ and S can be written as orthogonal differences of subspaces invariant under H and H', respectively, as

$$\mathcal{A}_2 = \mathcal{A}_2 \ominus \{0\}, \quad S = \mathcal{A}_2 \oplus \mathcal{A}_2 \ominus \begin{pmatrix} M \\ N \end{pmatrix} \mathcal{A}_2 \tag{37}$$

Now we have to show that the operator Φ intertwines h and h', that is, $h'(A)\Phi = \Phi h(A)$ for all $A \in \mathcal{B}_c(\ell^2, \ell^2)$,

$$\begin{aligned} h'(A)\Phi \;\; &= \;\; \Pi R_A \Pi T_1 \mid_{\mathcal{A}_2} = \Pi R_A \Pi_1 T_1 \mid_{\mathcal{A}_2} \\ &= \;\; \Pi R_A T_1 \mid_{\mathcal{A}_2} = \Pi T_1 R_A \mid_{\mathcal{A}_2} \\ &= \;\; \Xi h(A) \end{aligned}$$

Applying Theorem 1 implies that Φ has a dilation Φ' that intertwines H and H', i.e., $\Phi' H(A) = H'(A)\Phi'$, $\forall A \in B(\ell^2, \ell^2)$. By Lemma 4.4. in [25] Φ' is a left multiplication operator acting from $\mathcal{A}_2$ into $\mathcal{A}_2 \oplus \mathcal{A}_2$, and causal. That is, $\Phi' = L_K$ for some $K \in \mathcal{B}_c(\mathcal{A}_2,\; \mathcal{A}_2 \oplus \mathcal{A}_2)$, with $\|K\| = \|\Phi'\| = \|\Phi\|$. Then $\Phi = \Pi T_1 = \Pi K$, which implies $\Pi(T_1 - K) = 0$. Hence, $(T_1 - K)f \in R_2\mathcal{A}_2$, for all $f \in \mathcal{A}_2$. That is, $(T_1 - K)f = R_2 g$, $\exists g \in \mathcal{A}_2$, which can be written as $R_2^\star(T_1 - K)f = g \in \mathcal{A}_2$. In particular, $R_2^\star(T_1 - K)f \in \mathcal{B}_c(\ell^2, \ell^2)$, for all $f \in \mathcal{B}_c(\ell^2, \ell^2)$ of finite rank. By Theorem 3.10 [4] there is a sequence F_n of finite rank contractions in $\mathcal{B}_c(\ell^2, \ell^2)$ which converges to the identity operator in the strong *-topology. By an approximation argument it follows that $\tilde{M}^\star(\tilde{U} - K) \in \mathcal{B}_c(\ell^2,\; \ell^2)$. Letting $Q := R_2^\star(T_1 - K)$ we have $g = Qf$. We conclude that $\tilde{U} - K = \tilde{M}Q$, that is, $T_1 - R_2 Q = K$, with $\|K\| = \|\Phi\|$ as required.

A consequence of the commutant lifting Theorem is that there exists an optimal $Q_o \in \mathcal{B}_c(\ell^2,\; \ell^2)$ such that the infimum in (29) is achieved. Moreover, Lemma 1 implies that the operator Φ is given by the following analytic expression

$$\Phi = T_1 - R_2\mathcal{P}R_2^\star \tag{38}$$

and the subspace S is given by

$$S = \Big(\mathcal{A}_2 \oplus \mathcal{A}_2\Big) - R_2\mathcal{P}R_2^\star\Big(\mathcal{A}_2 \oplus \mathcal{A}_2\Big) \tag{39}$$

If there exists a maximal vector for Φ, that is, $A \in \mathcal{A}_2$ of norm $\|A\|_2 = 1$ such that $\|\Phi A\|_2 = \|\Phi\|$, then a similar identity as (27) can be obtained for Q_o as well from

$$\Phi A = T_1 A - R_2 Q_o A \tag{40}$$

and Q_o can be computed from the operator identity

$$Q_o A = R_2^\star T_1 A - R_2^\star \Phi A \tag{41}$$

Remark. The results obtained here apply to computing the directed TV gap metric between two LTV plants G_1 and G_2 defined in [8], [12] as follows.
First, G_1 and G_2 have normalized right coprime factorizations $\begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$ and $\begin{pmatrix} M_2 \\ N_2 \end{pmatrix}$, respectively. That is, $M_i^\star M_i + N_i^\star N_i = I,\; i = 1,\; 2$. The directed time varying gap between G_1 and G_2, denoted $\overrightarrow{\alpha}(G_1, G_2)$, can be computed as [8], [12], [7]

$$\overrightarrow{\alpha}(G_1, G_2) = \inf_{Q \in \mathcal{B}_c(\ell^2, \ell^2)} \left\| \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} Q \right\| \tag{42}$$

which by Theorem 3 is given by

$$\overrightarrow{\alpha}(G_1, G_2) = \left\| \Pi \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \right\| \tag{43}$$

where Π in this case is given by $\Pi = I - \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \mathcal{P}(M_2^\star,\; N_2^\star)$, and an optimal Q that achieves the infimum in (42) exists.

V. CONCLUSION

In this paper we considered two fundamental robust control problems, the optimal disturbance attenuation and the optimal mixed sensitivity problems for LTV plants. The commutant lifting theorem for nest algebras was applied to solve these two problems in term of two operators. A generalization to computing the directed TV gap metric for LTV plants is pointed out. Although the results we derived for discrete time-varying systems, the proposed framework can be extended to continuous systems by working instead in the Macaev ideal.

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