2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Overlapping Control Systems with Delayed Communication Channels: Stability Analysis and Controller Design

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Abstract— **In this paper, a decentralized overlapping static output feedback law is proposed to control a linear time-invariant (LTI) interconnected system. It is assumed that an overlapping information flow structure is given which determines which output measurements are available for any local control agent. Uncertain transmission delay is also considered in communication links among different subsystems. Each subsystem is assumed to be subject to input disturbances with finite energy. A necessary condition for the existence of a stabilizing overlapping controller is obtained which is easy to check. Furthermore, a linear matrix inequality (LMI) based design methodology is proposed to achieve internal stability and H_∞ disturbance attenuation. Simulations are presented to demonstrate the efficacy of the developed results.**

I. Introduction

Design of overlapping control systems has been of special interest recently and various aspects of it have been vastly studied in the literature [1], [2], [3]. This type of systems have a wide range of real-world applications, e.g., in multi-agent systems [4]. The cooperative nature of control paradigm in such systems is characterized based on the topology of communication between control agents. Typically, it is not realistic to assume each control agent can use all the measurement signals of the system to generate its local control input. In other words, some kind of constraint on the information flow between different control agents is inevitable.

In a geographically distributed large-scale system such as coordinated vehicles, a decentralized structure is often more desirable in control [2]. Decentralized control theory has attracted several researchers in the past three decades [5], [1], [6]. Particularly, overlapping control has been studied more recently and has found applications in various areas [1], [4], [7]. In [1], an expansion transformation is used to convert the original overlapping control problem into a decentralized one. The contraction procedure is applied consequently to provide an appropriate controller for the original system. It is shown that such an approach is more efficient if the system structure itself is overlapping too. The work [6] introduces the notion of a decentralized overlapping fixed mode (DOFM) to characterize the fixed modes of an interconnected system with respect to the class of linear time-invariant (LTI) structurally constrained controllers.

In a physical large-scale control system, on the other hand, communication delays inherently exist in information exchange between different control agents. Time-delay in system dynamics has a significant impact on the stability and performance of the system, and needs to be taken into account in controller design. This problem has been investigated intensively in the control literature, e.g. see [8], [9], [10], [11].

Some of the recent developments in delay-dependent stability analysis have been reported in [8], [12]. Different approaches are proposed for designing a proper feedback controller which satisfies prescribed performance requirements, such as H_∞ disturbance attenuation [13].

In this manuscript, an overlapping control strategy is proposed for interconnected systems consisting of a number of interacting subsystems. Each local controller is assumed to share its local measurements with some of the others local controllers (which are known *a priori*). The signal transmission between different control agents is assumed to be subject to uncertain delay. Furthermore, all actuators are exposed to disturbances, affecting the resultant control signals. The main contributions of this paper are as follows. A necessary condition for the stabilizability of interconnected systems by means of overlapping output feedback controllers is derived first. A methodology is then proposed using linear matrix inequalities (LMI) to design an overlapping static output feedback controller which stabilizes the system and attenuates the effect of disturbances on the regulated signal. It is assumed in this paper that the interconnected system possesses a LTI state space representation. The control gain is then decomposed into diagonal and off-diagonal components. A description of the resultant closed-loop system dynamics is presented through the above gain decomposition procedure. This results in a LTI system with an uncertain state-delay. A graph-based algorithm is utilized subsequently to transform the overlapping gain matrix into a block-diagonal form.

This paper is organized as follows. The problem is formulated in Section II, and the main objectives of the work are presented. In Section III, the closed-loop dynamics of the system under overlapping static output feedback control law is obtained and the matrix block diagonalization procedure is reviewed. Then in Section IV, the stability analysis and

This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant STPGP-364892-08, and in part by Motion Metrics International Corp.

H_∞ control synthesis are addressed. Section V presents some simulations which support the theoretical results of the paper. Finally, concluding remarks are given in Section VI.

II. Problem Formulation

A. Problem Statement

Consider a LTI interconnected system $\mathbf{S}$ consisting of ν subsystems. Assume that the state-space model for the i-th subsystem is described by

$$\begin{aligned}\dot{x}_i(t) &= A_{ii}x_i(t) + \sum_{\substack{j=1,\\ j\neq i}}^{\nu} A_{ij}x_j(t) + B_i u_i(t) + E_i w_i(t),\\ i &\in \bar{\nu} := \{1, 2, \ldots, \nu\}\end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and input for the i-th subsystem, respectively. In (1), the term $A_{ij}x_j$, $j \in \bar{\nu}$, represents the effect of the j-th subsystem on the dynamics of subsystem i. The system matrices A_i, B_i, E_i and A_{ij}, $i, j \in \bar{\nu}$ are constant and have appropriate dimensions. Furthermore, $w_i \in \mathbb{R}^{p_i}$ is the disturbance affecting the input of subsystem i, with the property $w_i(t) \in \mathscr{L}_2[0, \infty)$.

By putting together the state-space representations of all ν subsystems, the overall dynamics of the interconnected system $\mathbf{S}$ can be expressed as

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{2}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t)^T & x_2(t)^T & \ldots & x_\nu(t)^T \end{bmatrix}^T\\ u(t) &= \begin{bmatrix} u_1(t)^T & u_2(t)^T & \ldots & u_\nu(t)^T \end{bmatrix}^T\\ w(t) &= \begin{bmatrix} w_1(t)^T & w_2(t)^T & \ldots & w_\nu(t)^T \end{bmatrix}^T \end{aligned}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\nu}\\ A_{21} & A_{22} & \cdots & A_{2\nu}\\ \vdots & \vdots & \ddots & \vdots\\ A_{\nu 1} & A_{\nu 2} & \cdots & A_{\nu\nu} \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & & & 0\\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_\nu \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & & & 0\\ & E_2 & & \\ & & \ddots & \\ 0 & & & E_\nu \end{bmatrix}$$

The local measurement signal for the i-th local controller is represented by

$$y_i(t) = C_i x_i(t) \tag{3}$$

where $y_i \in \mathbb{R}^{r_i}$, and C_i is a given constant matrix with appropriate dimension.

Assumption 1: For the sake of non-triviality (i.e., to avoid the exact decentralized structure with no overlapping), it is assumed that at least one of the local controllers has access to at least one of the other subsystems' measurement signal through a communication link.

Let the measurement signal y_j of the j-th subsystem be transmitted to the control agent of subsystem i to construct the local control input u_i, $i, j \in \bar{\nu}, i \neq j$. Denote the received signal with s_j, which can be represented by

$$s_j(t) = y_j(t-h) = C_j x_j(t-h) \tag{4}$$

In the above equation, h is the communication delay which is uncertain, but is known to be strictly positive with finite magnitude. For simplicity and without loss of generality, it is assumed here that the communication delay is identical for all channels.

B. Control Objectives

To control the system $\mathbf{S}$, let the following local static output feedback controller be considered for the i-th subsystem

$$u_i(t) = K_i s^i(t) \tag{5}$$

where $K_i \in \mathbb{R}^{m_i \times r}$, $r := \sum_{i=1}^{\nu} r_i$ and

$$s^i(t) = \begin{bmatrix} s_1(t)^T & \ldots & s_{i-1}(t)^T & y_i(t)^T & s_{i+1}(t)^T & \ldots & s_\nu(t)^T \end{bmatrix}^T$$

In other words, $s^i(t)$ is obtained by replacing $s_i(t)$ with $y_i(t)$ in the vector $s(t)$. Let K_i be written as

$$K_i = \begin{bmatrix} K_{i1} & K_{i2} & \cdots & K_{i\nu} \end{bmatrix} \tag{6}$$

where $K_{ij} \in \mathbb{R}^{m_i \times r_j}$ for $i, j \in \bar{\nu}$. Assumption 1 implies that there exist distinct integers $i, j \in \bar{\nu}$, for which the gain matrix K_{ij} is nonzero. Note that the local controller for the i-th subsystem is characterized by the set of given K_{ij}'s, where K_{ii} is the control coefficient for the instantaneous local output, and K_{ij}'s, $j \neq i$, are the coefficients of the non-local output signals which are subject to the communication delay. Define $\mathbf{K}$ as an overlapping static output controller whose (i, j) block entry is K_{ij}.

Let the regulated signal be represented by

$$z(t) = \Gamma x(t)$$

where $z \in \mathbb{R}^{\xi}$ and $\Gamma \in \mathbb{R}^{\xi \times n}$ ($n := \sum_{i=1}^{\nu} n_i$). In this paper:

i) It is desired to find a necessary condition for the existence of a stabilizing overlapping controller $\mathbf{K}$ for the interconnected system $\mathbf{S}$.
ii) A set of distributed overlapping output feedback gains K_i, $i \in \bar{\nu}$, is sought such that for any delay h with a known upper bound,
   - the internal stability of the closed-loop system is achieved.
   - the ∞-norm of the closed-loop gain from $w(t)$ to $z(t)$ is less than a prescribed value γ, i.e.

$$\|T_{zw}\|_\infty := \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma$$

III. Preliminaries

A. Closed-loop Dynamics under the Controller **K**

Consider a distributed overlapping control gain **K** with the i-th local output feedback gain denoted by K_i, $i \in \bar{\nu}$, as given in (5) and (6).

Definition 1: $\mathbb{K}_D$ is the set of all block-diagonal matrices which have ν diagonal entries, where the i-th block entry on the main diagonal is a $m_i \times r_i$ matrix, for all $i \in \bar{\nu}$.

Definition 2: The decentralized gain matrix $\bar{K}$ is defined as

1) $\bar{K} \in \mathbb{K}_D$.
2) The (i,i) block entry of $\bar{K}$, is equal to K_{ii}.

Definition 3: Define the overlapping gain matrix $\tilde{K}$ as a matrix of the following form:

1) Its (i,j) block entry, $i \neq j$, is K_{ij} if the output of subsystem j is available to local controller i, and is a $m_i \times r_j$ zero matrix otherwise.
2) Its (i,i) block entry is a $m_i \times r_i$ zero matrix.

Consider the interconnected system **S** given by (2), and let the overlapping static output feedback control law **K** be applied to **S**. The input u_i in (5) can then be rewritten as

$$u_i(t) = \sum_{j=1, j\neq i}^{\nu} K_{ij} s_j + K_{ii} y_i$$

From (3) and (4), it follows that

$$u_i(t) = \sum_{j=1, j\neq i}^{\nu} K_{ij} C_j x_j(t-h) + K_{ii} C_i x_i(t)$$

This leads to the following expression for the input

$$u(t) = \tilde{K} C x(t-h) + \bar{K} C x(t) \tag{7}$$

where

$$C = \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_\nu \end{bmatrix}$$

and $\bar{K}$ and $\tilde{K}$ are introduced in Definitions 2 and 3. By Substituting (7) into (2), the closed-loop dynamics of the system **S** under the overlapping static output feedback **K** is obtained as follows

$$\dot{x}(t) = (A + B\bar{K}C)x(t) + B\tilde{K}Cx(t-h) + Ew(t)$$

B. Matrix Block Diagonalization Procedure

Inspired by [6], the following graph-theoretic algorithm is presented to convert $\tilde{K}$ to a block diagonal matrix H using a single transformation matrix. This diagonalization procedure is used in developing the main results of the paper.

Algorithm 1:

Step 1- Construct the overlapping graph **G** as follows:

a. Consider two sets of ν vertices denoted by **I** and **J**. Label the vertices in **I** and **J** as vertex 1 to vertex ν.
b. For any $i, j \in \bar{\nu}$, $i \neq j$, if there exists a communication link from local controller j to local controller i, connect vertex $i \in \mathbf{I}$ to vertex $j \in \mathbf{J}$ with an edge. The gain of this edge is K_{ij}.

Step 2- Consider the i-th vertex in **I** and define a new graph $\mathbf{G}_i$ which includes all the edges connected to this vertex. Thus, the graph **G** is partitioned into ν subgraphs $\mathbf{G}_1$, $\mathbf{G}_2$, $\ldots$, $\mathbf{G}_\nu$.

Step 3- Consider the subgraph $\mathbf{G}_i$, $i \in \bar{\nu}$, and denote the set of all vertices of **I** which appear in $\mathbf{G}_i$ with $\mathbf{I}_i$. Note that $|\mathbf{I}_i| = 1$, where $|.|$ is the cardinality of a set. In addition, let the set of all vertices of **J** which appear in $\mathbf{G}_i$ be denoted by $\mathbf{J}_i$. Suppose that $|\mathbf{J}_i| = \delta_i$, $i \in \bar{\nu}$; define H as a block-diagonal matrix where its (i,i) block entry, $i \in \bar{\nu}$, is a block row whose j-th block entry, $j = 1, \ldots, \delta_i$, is the gain of the edge connecting the only vertex in $\mathbf{I}_i$ to the j-th vertex in $\mathbf{J}_i$.

Remark 1: In step 2 of Algorithm 1, some vertices of **J** might appear in more than one subgraph $\mathbf{G}_i$, $i \in \bar{\nu}$. In other words, for some distinct $i, j \in \bar{\nu}$, $\mathbf{J}_i \bigcap \mathbf{J}_j$ might be non-empty; however, $\mathbf{I}_i \bigcap \mathbf{I}_j = \emptyset$, $\forall i, j \in \bar{\nu}$, $i \neq j$.

Definition 4: Let $\mathscr{C}_i = \mathbf{J}_i \bigcup \{i\}$, for any $i \in \bar{\nu}$. Define $\mathbb{H}_D$ as the set of all block-diagonal matrices which have ν diagonal block entries, where the i-th block entry, $i \in \bar{\nu}$, is a $m_i \times \mu_i$ matrix itself, and

$$\mu_i = \sum_{j \in \mathscr{C}_i} r_j \tag{8}$$

The following lemma relates the matrix H, obtained from step 3 of Algorithm 1, to $\tilde{K}$.

Lemma 1: Assume the block-diagonal matrix $H \in \mathbb{H}_D$ is obtained from $\tilde{K}$ using Algorithm 1. One can find a matrix T such that

$$\tilde{K} = HT \tag{9}$$

Proof: Following an approach similar to [14], it is straightforward to show that the matrix H can be derived through only a finite sequence of operations on the columns of $\tilde{K}$, and therefore a unique transformation matrix T can be obtained such that (9) holds. ■

As an illustrative example, consider a vehicle formation system **F** consisting of 3 vehicles with the i-th input and output ($i = 1,2,3$) denoted by $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}^2$, respectively. Suppose that vehicle 2 has access to the local measurements of the other 2 vehicles while vehicles 1 and 3 receive the measurement of vehicle 2 only, and there is no communication link between them (this formation topology is referred to as leader-follower in the literature, where vehicle 1 is the leader and vehicles 2 and 3 are followers [3]). In this case, the structure of the gain matrix $\tilde{K}$ is as follows

$$\tilde{K} = \begin{bmatrix} 0_{1\times 2} & K_{12} & 0_{1\times 2} \\ K_{21} & 0_{1\times 2} & K_{23} \\ 0_{1\times 2} & K_{32} & 0_{1\times 2} \end{bmatrix}$$

where $K_{12}, K_{21}, K_{23}, K_{32} \in \mathbb{R}^{1\times 2}$. Following the procedure given in Algorithm 1, the overlapping graph **G** corresponding to the matrix $\tilde{K}$ is obtained as shown in Figure 1. Furthermore, following step 2 of the algorithm, one can find the subgraphs $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ depicted in Figure 2.

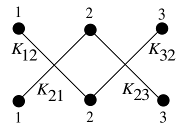

Fig. 1. The overlapping graph $\mathbf{G}$ for the formation $\mathbf{F}$

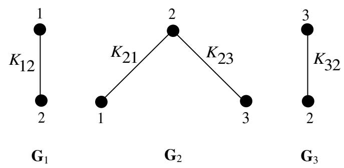

Fig. 2. The subgraphs obtained from the graph $\mathbf{G}$ using step 2 of Algorithm 1

Using step 3 of the algorithm, the block diagonal matrix H is obtained as

$$H = \begin{bmatrix} K_{12} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} \\ 0_{1\times 2} & K_{21} & K_{23} & 0_{1\times 2} \\ 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & K_{32} \end{bmatrix}$$

Moreover, the transformation matrix T (defined in (9)) for this example is

$$T = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & I_2 & 0_2 \end{bmatrix}$$

where I_2 and 0_2 denote the 2×2 identity matrix and the 2×2 zero matrix, respectively.

IV. Main Results

A. Stabilizability Conditions for Overlapping Control Systems

Definition 5: $\mathbb{G}_D$ is defined as the set of all block-diagonal matrices with ν diagonal entries, where the i-th block entry on the main diagonal is a $m_i \times (r_i + \mu_i)$ matrix itself (μ_i is given by (8)).

It is easy to verify that the block matrix

$$\begin{bmatrix} \bar{K} & H \end{bmatrix}$$

can be converted to a proper block-diagonal matrix of the form given in the above definition by a finite number of elementary column operations. Therefore, let

$$\begin{bmatrix} \bar{K} & H \end{bmatrix} = \hat{K}\Omega \tag{10}$$

where $\hat{K} \in \mathbb{G}_D$, and Ω is a proper transformation matrix (associated with the above-mentioned elementary operations).

Definition 6: The system $\hat{\mathbf{S}}$ is defined by the following state-space equations

$$\dot{x}(t) = Ax(t) + \hat{B}u(t)$$
$$q(t) = \hat{C}^0 x(t) + \hat{C}^1 x(t-h)$$

where

$$\hat{B} = B, \quad \hat{C}^0 = \Omega \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix}, \quad \hat{C}^1 = \Omega \begin{bmatrix} 0_{r\times n} \\ TC \end{bmatrix}$$

(note that $q(t) \in \mathbb{R}^{2r}$, and Ω is given in (10)).
Now, let the matrices $\hat{B}$, $\hat{C}^0$ and $\hat{C}^1$ be partitioned as

$$\hat{B} = \begin{bmatrix} \hat{B}_1 & \hat{B}_2 & \cdots & \hat{B}_\nu \end{bmatrix}$$

$$\hat{C}^0 = \begin{bmatrix} \hat{C}^0_1 \\ \hat{C}^0_2 \\ \vdots \\ \hat{C}^0_\nu \end{bmatrix}, \quad \hat{C}^1 = \begin{bmatrix} \hat{C}^1_1 \\ \hat{C}^1_2 \\ \vdots \\ \hat{C}^1_\nu \end{bmatrix}$$

where $\hat{B}_i \in \mathbb{R}^{n\times m_i}$ and $\hat{C}^\sigma_i \in \mathbb{R}^{(r_i+\mu_i)\times n}$, for $i \in \bar{\nu}$ and $\sigma = 0,1$.

The following two lemmas play key roles in obtaining a necessary condition for the stabilizability of the system $\mathbf{S}$ with respect to the overlapping controller $\mathbf{K}$ (the first one is borrowed from [15]).

Lemma 2: Consider the matrices M_i and N_i, $i = 1,2,\ldots,\eta$, where $M_i \in \mathbb{C}^{\rho\times\gamma_i}$ and $N_i \in \mathbb{C}^{\rho\times\nu_i}$. A necessary and sufficient condition for the following inequality

$$\operatorname{rank}\begin{bmatrix} M_1 + N_1K_1 & M_2 + N_2K_2 & \cdots & M_\eta + N_\eta K_\eta \end{bmatrix} < \min\left\{\rho, \sum_{i=1}^{\eta}\gamma_i\right\}$$

to hold for all $K_i \in \mathbb{C}^{\nu_i\times\gamma_i}$, $i = 1,2,\ldots,\eta$, is that there exists a non-empty subset $\Phi = \{i_1, i_2, \cdots, i_j\}$ of the index set $\{1,2,\cdots,\eta\}$ for which

$$\operatorname{rank}\begin{bmatrix} M_{i_1} & N_{i_1} & \cdots & M_{i_j} & N_{i_j} \end{bmatrix} < \min\left\{\rho - \sum_{i\notin\Phi}\gamma_i, \sum_{i\in\Phi}\gamma_i\right\}$$

The following lemma follows from the result obtained originally in [15]. In this lemma, $\mathbb{C}$ denotes the set of the complex numbers and $A(e^{-sh})$, $B(e^{-sh})$, $C(e^{-sh})$ are quasi-polynomials matrices corresponding to $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ of a LTI time-delay system (see [16]).

Lemma 3: Let the matrices $A(e^{-sh}) \in \mathbb{C}^{n\times n}$, $\hat{B}_i(e^{-sh}) \in \mathbb{C}^{n\times\pi_i}$, and $\hat{C}_i(e^{-sh}) \in \mathbb{C}^{\pi_i\times n}$, $i \in \bar{\nu}$, be given. For any $s \in \mathbb{C}$, the matrix

$$sI - A(e^{-sh}) - \sum_{i=1}^{\nu}\hat{B}_i(e^{-sh})\hat{K}_i\hat{C}_i(e^{-sh})$$

is not full-rank for all $\hat{K}_i \in \mathbb{R}^{\pi_i\times\pi_i}$ if and only if

$$\begin{bmatrix} sI - A(e^{-sh}) & \hat{B}_1(e^{-sh}) & \hat{B}_2(e^{-sh}) & \cdots & \hat{B}_\nu(e^{-sh}) \\ \hat{C}_1(e^{-sh}) & L_1 & 0 & \cdots & 0 \\ \hat{C}_2(e^{-sh}) & 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_\nu(e^{-sh}) & 0 & 0 & \cdots & L_\nu \end{bmatrix}$$

is not full-rank for all $\pi_i \times \pi_i$ real matrices L_i, $i \in \bar{\nu}$.

Theorem 1: A necessary condition for the existence of a stabilizing overlapping controller $\mathbf{K}$ for the system $\mathbf{S}$ is that for any $s \in \operatorname{sp}(A)$, $\operatorname{Re}\{s\} \geq 0$, all of the following matrices are full-rank

$$\begin{bmatrix} sI - A & \hat{B}_{i_1} & \cdots & \hat{B}_{i_l} \\ \hat{C}^0_{i_{l+1}} + \hat{C}^1_{i_{l+1}}e^{-sh} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}^0_{i_\nu} + \hat{C}^1_{i_\nu}e^{-sh} & 0 & \cdots & 0 \end{bmatrix}$$

where i_1, i_2, ..., i_ν are distinct integers representing any permutation of the set $\bar{\nu}$. Furthermore, $l = 0, \ldots, \nu+1$ and $\hat{B}_{i_0} = \hat{C}^{\sigma}_{i_{\nu+1}} = \emptyset$, $\sigma = 0,1$.

Proof: Equation (7) can be expressed as

$$u(t) = \begin{bmatrix} \bar{K} & \tilde{K} \end{bmatrix} \begin{bmatrix} Cx(t) \\ Cx(t-h) \end{bmatrix}$$

It follows from Lemma 1 and equation (9) that

$$u = \begin{bmatrix} \bar{K} & H \end{bmatrix} \begin{bmatrix} Cx(t) \\ TCx(t-h) \end{bmatrix}$$

Using (10), one will obtain

$$u = \hat{K}\left(\Omega \begin{bmatrix} C \\ 0 \end{bmatrix} x(t) + \Omega \begin{bmatrix} 0 \\ TC \end{bmatrix} x(t-h)\right)$$

From the above equation, it is inferred that the system **S** is stabilizable by an overlapping static controller of the form **K** if and only if $\hat{\mathbf{S}}$ is stabilizable by a decentralized static output feedback controller with the output feedback gain $\hat{K} \in \mathbb{G}_D$. A necessary condition for the latter statement to hold is that for any $s \in \mathrm{sp}(A)$ with $\mathrm{Re}\{s\} \geq 0$, there exists a $\hat{K}^* \in \mathbb{G}_D$ such that [17]

$$\det\left(sI - A - \sum_{i=1}^{\nu} \hat{B}_i \hat{K}_i^* \left(\hat{C}_i^0 + \hat{C}_i^1 e^{-sh}\right)\right) \neq 0$$

$\hat{K}_i^* \in \mathbb{R}^{m_i \times (r_i + \mu_i)}$, and

$$\hat{K}^* = \text{block diagonal}\left[\hat{K}_1^*, \hat{K}_2^*, \ldots, \hat{K}_\nu^*\right]$$

Using Lemmas 2 and 3, it can be shown in a manner similar to the techniques used in [15] that all the rank conditions provided in this theorem must hold for the system **S** to be stabilizable with respect to an overlapping controller of the form **K**. This completes the proof. ■

B. H$_\infty$ *Decentralized Overlapping Control Synthesis*

Definition 7: $\mathbb{Q}_D$ is the set of all block diagonal matrices which have ν block-diagonal entries, where the i-th block entry of the main diagonal, $i \in \bar{\nu}$, is a $m_i \times m_i$ matrix itself.

Theorem 2: Consider the system **S** and let the delay h be an arbitrary positive value with a known upper bound $\bar{h}$. Assume that for a given $\gamma > 0$, there exist matrices $Q_1 > 0$, $0 < Q_2 \in \mathbb{Q}_D$, $Y_1 \in \mathbb{K}_D$, $Y_2 \in \mathbb{H}_D$, $R_{11} > 0$, R_{12} and $R_{22} > 0$ which satisfy the LMIs given below

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & 0 & 0 & 0 \\ * & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} \\ * & * & Z_{33} & Z_{34} & Z_{35} & 0 & 0 & 0 \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} \\ * & * & * & * & -0.5\gamma^2 I & 0 & 0 & 0 \\ * & * & * & * & * & -0.5\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -\bar{h}R_{11} & -\bar{h}R_{12} \\ * & * & * & * & * & * & * & -\bar{h}R_{22} \end{bmatrix} < 0 \quad (11)$$

$$\begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0 \quad (12)$$

where

$$\begin{aligned} Z_{11} &= A^T Q_1 + Q_1 A + \Gamma^T \Gamma \\ Z_{12} &= Q_1 B + A^T C^T Y_1{}^T + A^T C^T T^T Y_2{}^T \\ Z_{13} &= A^T Q_1 \\ Z_{14} &= A^T C^T Y_1{}^T + A^T C^T T^T Y_2{}^T \\ Z_{15} &= Z_{35} = Q_1 E \\ Z_{22} &= B^T C^T Y_1{}^T + B^T C^T T^T Y_2{}^T + Y_1 CB + Y_2 TCB \\ Z_{23} &= B^T Q_1 \\ Z_{24} &= B^T C^T Y_1{}^T + B^T C^T T^T Y_2{}^T \\ Z_{25} &= Z_{45} = Y_1 CE \\ Z_{26} &= Z_{46} = Y_2 TCE \\ Z_{27} &= Z_{47} = \bar{h} Y_2 TCA \\ Z_{28} &= Z_{48} = \bar{h} Y_2 TCB \\ Z_{33} &= -2Q_1 + \bar{h} R_{11} \\ Z_{34} &= \bar{h} R_{12} \\ Z_{44} &= -2Q_2 + \bar{h} R_{22} \end{aligned} \quad (13)$$

Set

$$\bar{K} = Q_2{}^{-1} Y_1, \qquad \tilde{K} = Q_2{}^{-1} Y_2 T \quad (14)$$

and let the overlapping controller with the above parameters be denoted by $\mathbf{K}^*$. Then,

- the system **S** under the controller $\mathbf{K}^*$ is internally stable; and
- the ∞-norm of the closed-loop transfer function from disturbance input $w(t)$ to regulated variable $z(t)$, denoted by $\|T_{zw}\|_\infty$, is less than γ, i.e.

$$\|T_{zw}\|_\infty = \|z(t)\|_2 / \|w(t)\|_2 < \gamma \quad (15)$$

Proof: Define

$$\theta(t) = \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix}^T, \qquad v(t) = \begin{bmatrix} w(t)^T & w(t-h)^T \end{bmatrix}^T$$

It is straightforward to show that

$$\begin{aligned} \dot{\theta}(t) &= \begin{bmatrix} A & B \\ \bar{K}CA & \bar{K}CB \end{bmatrix} \theta(t) + \begin{bmatrix} 0 & 0 \\ \tilde{K}CA & \tilde{K}CB \end{bmatrix} \theta(t-h) \\ &\quad + \begin{bmatrix} E & 0 \\ \bar{K}CE & \tilde{K}CE \end{bmatrix} v(t) \\ z(t) &= \begin{bmatrix} \Gamma & 0 \end{bmatrix} \theta(t) \end{aligned} \quad (16)$$

Consider the performance index given below

$$\begin{aligned} J(v) &= \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t)\right] dt \\ &= \int_0^\infty \left[z(t)^T z(t) - \frac{\gamma^2}{2} v(t)^T v(t)\right] dt \end{aligned}$$

and let the system (16) be represented in the following descriptor form

$$\begin{aligned} \dot{\theta}(t) &= \zeta(t) \\ \zeta(t) &= \begin{bmatrix} A & B \\ \bar{K}CA & \bar{K}CB \end{bmatrix} \theta(t) + \begin{bmatrix} 0 & 0 \\ \tilde{K}CA & \tilde{K}CB \end{bmatrix} \theta(t-h) \\ &\quad + \begin{bmatrix} E & 0 \\ \bar{K}CE & \tilde{K}CE \end{bmatrix} v(t) \end{aligned}$$

Define the following Lyapunov-Krasovskii functional for the above system

$$\begin{aligned} V(t) &= \begin{bmatrix} \theta(t) & \zeta(t) \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \zeta(t) \end{bmatrix} \\ &\quad + \int_{-h}^0 \int_{t+\beta}^t \zeta(\alpha)^T R \zeta(\alpha) d\alpha \, d\beta \end{aligned}$$

where

$$P=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}>0, \qquad R=\begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix}>0$$

and $Q_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{Q}_D$. Thus, it can be concluded that if the inequality

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ * & Z_{22} & Z_{23} & Z_{24} \\ * & * & -0.5\gamma^2 I & 0 \\ * & * & * & -\bar{h}R \end{bmatrix}<0$$

holds for

$$\begin{aligned} Z_{11} &= Z_{12}+Z_{12}{}^T+\Phi_1 \\ \Phi_1 &= \begin{bmatrix} \Gamma^T\Gamma & 0 \\ 0 & 0 \end{bmatrix} \\ Z_{12} &= \begin{bmatrix} A^TQ_1 & \Phi_2 \\ B^TQ_1 & \Phi_3 \end{bmatrix} \\ \Phi_2 &= A^TC^T\bar{K}^TQ_2+A^TC^T\tilde{K}^TQ_2 \\ \Phi_3 &= B^TC^T\bar{K}^TQ_2+B^TC^T\tilde{K}^TQ_2 \\ Z_{13} &= \begin{bmatrix} Q_1E & 0 \\ Q_2\bar{K}CE & Q_2\tilde{K}CE \end{bmatrix} \\ Z_{14} &= \bar{h}\begin{bmatrix} 0 & 0 \\ Q_2\tilde{K}CA & Q_2\tilde{K}CB \end{bmatrix} \\ Z_{22} &= \begin{bmatrix} -2Q_1+\bar{h}R_{11} & \bar{h}R_{12} \\ * & -2Q_2+\bar{h}R_{22} \end{bmatrix} \\ Z_{23} &= Z_{13}=Z_{24}=Z_{14} \end{aligned} \tag{17}$$

then $\dot{V}(t)<0$ and $J(v)<0$ for all nonzero $v(t)\in\mathscr{L}_2[0,\infty)$. This implies that the two objectives proposed in this theorem are satisfied. Substitute $\tilde{K}$ in (17) with HT as noted in Lemma 1, and define

$$Y_1=Q_2\bar{K}, \qquad Y_2=Q_2H$$

Using the above relations, the LMIs introduced in (11)-(13) are obtained. On the other hand, since $Q_2\in\mathbb{Q}_D$ and $\bar{K}\in\mathbb{K}_D$, this implies that Y_1 also belongs to $\mathbb{K}_D$. Similarly, it can be concluded that $Y_2\in\mathbb{H}_D$. This completes the proof. ■

Remark 2: Unlike Theorem 1, it is required in Theorem 2 to use the transformation T to find $\tilde{K}$. One can simply choose Y_2 as $Q_2\tilde{K}$, and consider a structure similar to $\tilde{K}$ for Y_2.

V. Simulation Results

Example 1: Consider a formation flight consisting of 3 unmanned aerial vehicles (UAV) with leader-follower structure. Let UAV 1 be the leader, and UAVs 2, 3 the followers. The objective here is to control the planar motion of the formation. Assume that all UAVs are desired to fly at the same velocity (v_x, v_y) with the distance vector (d_{x_i}, d_{x_i}) between UAVs i and $i+1$, $i=1,2$. The model of the formation in the relative coordinate frame is obtained as follows [18]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}=\begin{bmatrix} 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & -I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & -I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$+\begin{bmatrix} I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 \end{bmatrix}\begin{bmatrix} u_1+w_1 \\ u_2+w_2 \\ u_3+w_3 \end{bmatrix}$$

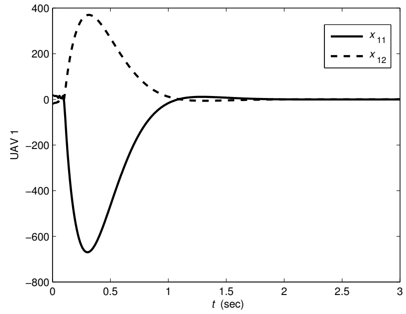

Fig. 3. The state response of vehicle 1 for $h=0.1$ in Example 1

where

$$x_1=\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T, \qquad x_1=\begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}^T$$
$$x_3=\begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}^T$$

Assume that the i-th vehicle can measure its state in the relative coordinates (i.e. x_i, $i=1,2,3$) using GPS-based sensors. Thus, $C_1=I_2$ and $C_2=C_3=I_4$. Consider the same communication topology as the one in the illustrative example of Subsection III-B, and suppose that $\Gamma=I_{10}$. The transformation matrix in this case is

$$T=\begin{bmatrix} 0_{4\times2} & I_{4\times4} & 0_{4\times4} \\ I_{2\times2} & 0_{2\times4} & 0_{2\times4} \\ 0_{4\times2} & 0_{4\times4} & I_{4\times4} \\ 0_{4\times2} & I_{4\times4} & 0_{4\times4} \end{bmatrix}$$

Using the above transformation, it is straightforward to show that the rank conditions in Theorem 1 hold for $h<1$. A proper control design technique will be employed next to achieve stability.

Consider the H_∞ control synthesis provided in Theorem 2 with $\bar{h}=0.1$ and $\gamma=0.15$, and assume that

$$w_1(t)=0, \quad w_2(t)=w_3(t)=160\times\sin(20\pi t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using the LMIs given by (11)-(13), the following overlapping static feedback control parameters are obtained

$$\bar{K}=\begin{bmatrix} \bar{K}_{11} & 0_{2\times4} & 0_{2\times4} \\ 0_{2\times2} & \bar{K}_{22} & 0_{2\times4} \\ 0_{2\times2} & 0_{2\times4} & \bar{K}_{33} \end{bmatrix}$$

$$\tilde{K}=\begin{bmatrix} 0 & 0 & -23.51 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -23.51 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.83 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} \bar{K}_{11} &= \begin{bmatrix} -11.76 & 0 \\ 0 & -11.76 \end{bmatrix} \\ \bar{K}_{22} &= \begin{bmatrix} 7.83 & 0 & -11.76 & 0 \\ 0 & 7.83 & 0 & -11.76 \end{bmatrix} \\ \bar{K}_{33} &= \begin{bmatrix} 23.51 & 0 & -11.76 & 0 \\ 0 & 23.51 & 0 & -11.76 \end{bmatrix} \end{aligned}$$

For $h=0.1$, the state variables of the system under the controller given above are depicted in Figures 3, 4 and 5. It

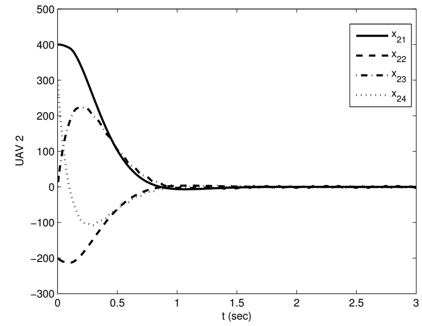

Fig. 4. The state response of vehicle 2 for $h = 0.1$ in Example 1

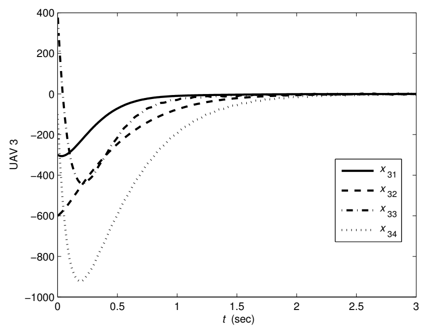

Fig. 5. The state response of vehicle 3 for $h = 0.1$ in Example 1

can be verified that the formation remains stable for all $h < 0.85$. However, the performance of the closed-loop system obtained by applying the proposed overlapping controller to the formation deteriorates as h increases. Suppose that UAVs 1, 2 and 3 are initially located in $(0,0)$, $(-450,100)$, $(-200,850)$, respectively. Let also

$$d_{x_1} = \begin{bmatrix} 50 & 100 \end{bmatrix}^T, \quad d_{x_2} = \begin{bmatrix} 50 & -150 \end{bmatrix}^T$$

and assume that the leader is moving in the $x-y$ plane with the constant velocity vector $[200 \quad 100]^T$. The trajectory of the formation under the proposed overlapping controller for $h = 0.1$ is sketched in Figure 6.

VI. Conclusions

This work deals with stability analysis and control design problem for LTI interconnected systems with a given information flow topology using decentralized overlapping controllers. The subsystems are assumed to be subject to input disturbances with finite energy. Furthermore, the information flow among different control agents is subject to transmission delay. First, some rank conditions are given which are necessary for the existence of a stabilizing overlapping output feedback controller. Then, a LMI-based design method is proposed for solving H_∞ control synthesis problem to attenuate the effect of disturbance in the regulated output. The simulation results elucidate the effectiveness of the proposed technique.

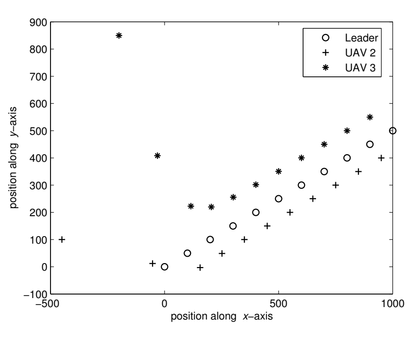

Fig. 6. Planar motion of the formation for $h = 0.1$ in Example 1

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