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Robust Controller Interpolation via Convex Optimization

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Abstract—**Arbitrarily fast switching or blending among controllers often leads to reduced performance and possible instability. This paper introduces a controller interpolation framework for optimizing the dynamic transition among controllers with respect to the H_∞ norm. Based on the framework, necessary and sufficient conditions are developed in terms of linear matrix inequalities (LMIs). In addition, the framework is also amenable to optimizing the controller interpolation for other control design objectives expressible via LMIs.**

I. Introduction

In many applications, a controller must accommodate a plant with changing objectives or operating conditions. Generally, a fixed controller cannot accommodate such changes without making significant tradeoffs among the objectives. A practical alternative involves switching or blending among a family of controllers in response to changing objectives, thereby allowing improved performance over a fixed controller. Bumpless-transfer [1], gain-scheduling [2], and switched control [3] all invoke variations of this design philosophy. Beyond the potential for improved performance, the broad appeal of such techniques arises from addressing each situation individually rather than the entire set simultaneously.

We refer to act of switching or blending among a set of *a priori* designed controllers as controller interpolation. The collective behavior of the individual controllers commanded by an interpolation signal (i.e. switching or blending signals) is described by the interpolated controller. Improper interpolated controller design can lead to loss of stability for a worst case interpolation signal. Lyapunov-based stability methods have been used to verify performance and stability for a given interpolated controller [3],[4]. However, relatively little research has been done in terms of interpolated controller design to guarantee closed loop stability, let alone closed loop performance.

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Controller interpolation is not to be confused with related, however different, self-scheduled controller approaches [12],[13]. Self-scheduled approaches ensure stability and performance by simultaneously designing the individual controllers comprising the self-scheduled controller, whereas controller interpolation seeks to integrate a given set of separately designed controllers.

Youla parameterization has been proven to be a particularly useful tool for designing interpolated controllers with stability guarantees. Hespanha [5] used Youla parameterization in a switched system framework to show there always exists a switched controller, composed of a given set of linear time-invariant controllers, that stabilizes a linear time-invariant (LTI) plant for any piecewise continuous switching signal. More recently, the approach has been applied to linear parameter-varying systems [11].

Although the nominal stability problem for controller interpolation has been addressed, design techniques guaranteeing interpolated controller performance have not been extensively investigated. The robust controller interpolation problem seeks to optimize interpolated controller performance by minimizing the closed loop H_∞ norm. A sub-optimal robust controller interpolation solution was first investigated via an iterative LMI algorithm in [9]. Recently, a controller parameterization approach, similar to the Youla parameterization has been shown to be non-conservative in [10]. Practical examples in [9],[10] illustrate the benefits of robust controller interpolation over stability oriented controller interpolation techniques.

Nevertheless, such parameterization approaches are limited to cases where there exists an appropriate controller parameterization. This paper seeks to close the perceived gap in controller interpolation techniques by developing a more versatile framework using linear matrix inequality (LMI) based convex optimization. Numerous control synthesis problems have been stated in the form of LMIs [8], and LMIs solvers are known to be computationally efficient. This paper presents a LMI-based robust controller interpolation synthesis technique that is shown to be non-conservative with respect to the H_∞ norm.

The robust controller interpolation problem addressed in this paper is defined in Section II. In Section III, the main result is presented in terms of necessary and sufficient LMI conditions. Finally, Section V discusses implications of the main result and next steps.

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II. Problem Definition

For nomenclature, let the H_∞ norm $\|\cdot\|_\infty$ denote the induced L_2 norm. The notation $Q_1 \sim Q_2$ denotes that systems Q_1 and Q_2 are input-output equivalent. The two systems Q_1 and Q_2 are input-output equivalent if $\|Q_1 - Q_2\|_\infty = 0$. The transpose of the matrix A is denoted A^*. Let $\mathrm{N}(B)$ denote the null space of the matrix B. The space of symmetric matrices of dimension n is denoted as $\mathbb{H}^n$.

In the context of our investigation, the closed loop system T_{zw} is described by the interconnection of the LTI plant

$$P = \begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{cases} \qquad (1)$$

and a LTI controller

$$K = \begin{cases} \dot{x}_K = A_K x_K(t) + B_K y(t) \\ u = C_K x_K(t) + D_K y(t) \end{cases}, \qquad (2)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$, $y \in \mathbb{R}^{n_y}$, and $x_K \in \mathbb{R}^{n_K}$. Without loss of generality, it is assumed that $D_{22} = 0$. The interconnection in Figure 1 is described in shorthand via the lower linear fractional transformation $T_{zw}(K)$=LFT(P,K).

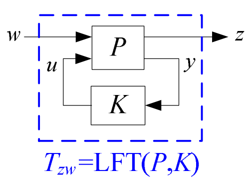

Figure 1: Closed loop system $T_{zw}(K)$=LFT(P,K) formed via lower linear fractional transformation

In this paper, we consider interpolating a set of controllers K_i for i=1,…,r. The *interpolation signal* describes to what degree each controller in the family of controllers is active. Let the class of piecewise continuous *arbitrary interpolation signals* be defined as

$$\mathcal{A} = \left\{ \alpha(t) : \sum_{i=1}^{r} \alpha_i(t) = 1,\ \alpha_i(t) \geq 0 \right\}. \qquad (3)$$

The class of interpolation signals $\alpha \in \mathcal{A}$ includes prevalent concepts in the literature such as controller switching [5] and controller blending [3]. It is assumed that the interpolation signal is measurable but unknown *a priori*. Throughout this paper, $\alpha(t)$ refers to the current value of the interpolation signal, whereas α refers to the entire trajectory of the interpolation signal.

Definition 1: For a family of r controllers K_i i=1,…,r, the interpolated controller $K(\alpha)$ is admissible for an H_∞ performance level γ if the *robust controller interpolation criteria* are satisfied:

- **A1)** The closed loop system $T_{zw}(K(\alpha))$ has H_∞ norm less than γ for any admissible interpolation signal $\alpha \in \mathcal{A}$.
- **A2)** The local controller K_i is input-output equivalent to $K(\alpha)$ when $\alpha_i(t) \equiv 1$.
- **A3)** The interpolated controller is a continuous function of $\alpha(t)$. ◊

For a given interpolation signal trajectory α, $T_{zw}(K(\alpha))$ may be treated as a linear time-varying system. Criterion **(A1)** stipulates closed loop system has an H_∞ norm less than γ for all arbitrary interpolation signals. Criterion **(A2)** enforces that each of the *a priori* designed controllers may be recovered from the interpolated controller. Lastly, criterion **(A3)** ensures there are no discontinuities in the interpolated controller behavior with respect to the interpolation signal, thereby avoiding discontinuities in the control signal for a continuous interpolation signal.

III. Controller Interpolation Synthesis

A. System Theoretic Context

In order to place the following results in context, let us discuss necessary and sufficient LMI conditions for verifying the closed loop system H_∞ norm and the existence of an LTI controller K such that $\|T_{zw}(K)\|_\infty < \gamma$. First recall the well known LMI formulation of the bounded real lemma for continuous-time systems (see [6] for a complete proof).

Lemma 1: Consider the linear time-invariant system

$$T_{zw}(K) = \begin{cases} \dot{x}_{cl}(t) = A_{cl} x_{cl}(t) + B_{cl} w(t) \\ z(t) = C_{cl} x_{cl}(t) + D_{cl} w(t) \end{cases}. \qquad (4)$$

The system $T_{zw}(K)$ is exponentially stable and $\|T_{zw}(K)\|_\infty < \gamma$ if and only if there exists positive definite X satisfying

$$\Phi(X, T_{zw}(K), \gamma) \triangleq \begin{bmatrix} XA_{cl} + A_{cl}^* X & XB_{cl} & C_{cl}^* \\ B_{cl}^* X & -\gamma I & D_{cl}^* \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0. \qquad (5)$$

From *Lemma 1*, necessary and sufficient conditions have been developed for the existence of controller K such that $\|T_{zw}(K)\|_\infty < \gamma$. See [6] and [7] for further details.

Lemma 2: There exists a linear time-invariant controller K ensuring $\|T_{zw}(K)\|_\infty < \gamma$ for the plant P in (1) if and only if there exist symmetric R_0 and S_0 satisfying the conditions

$$\Gamma(S_0, P, \gamma) \triangleq N_S^* \Phi(S_0, P_{zw}, \gamma) N_S < 0, \qquad (6)$$

$$\Gamma'(R_0, P, \gamma) \triangleq N_R^* \Phi(R_0, P_{zw}^T, \gamma) N_R < 0, \qquad (7)$$

$$\text{and} \begin{bmatrix} R_0 & I \\ I & S_0 \end{bmatrix} \geq 0, \qquad (8)$$

where $N_S = \mathrm{N}([C_2 \quad D_{21} \quad 0])$ and $N_R = \mathrm{N}([B_2^* \quad D_{12}^* \quad 0])$.

Definition 2: Given P in (1) and γ>0, let the pair $(\hat{R}_0, \hat{S}_0)$ denote the *maximal elements* for the set of all (R_0, S_0) satisfying the conditions of *Lemma 2*.

The existence of the maximal elements is guaranteed by the well-ordered property [7] of the associated algebraic Riccati inequalities. That is, $\hat{R}_0 \geq R_0$ and $\hat{S}_0 \geq S_0$ for all (R_0, S_0) satisfying *Lemma 2*.

B. Interpolated Controller Synthesis Conditions

The following theorem, representing our main result, provides necessary and sufficient conditions for the existence of an interpolated controller that satisfies the robust controller interpolation criteria. Let

$$S_i = \begin{bmatrix} S_{i,11} & S_{i,12} \\ S_{i,12}^* & S_{i,22} \end{bmatrix} = \begin{bmatrix} R_{i,11} & R_{i,12} \\ R_{i,12}^* & R_{i,22} \end{bmatrix}^{-1}, \qquad (9)$$

where $S_{i,11} \in \mathbb{H}^n$, $S_{i,22} \in \mathbb{H}^{n_{Ki}}$.

Theorem 1: Given the plant P in (1), r controllers K_i in (2), and a scalar $\gamma \geq 0$. Then there exists an admissible interpolated controller $K(\alpha)$ satisfying $\| T_{zw}(K(\alpha)) \|_\infty < \gamma$ if and only if there exists positive definite matrices $R_0, S_0 \in \mathbb{H}^n$ and $S_i \in \mathbb{H}^{n+n_{Ki}}$ satisfying (6), (7),

$$\Phi(S_i, T_{zw}(K_i), \gamma) < 0, \qquad (10)$$

$$\begin{bmatrix} R_0 & I & 0 \\ I & S_{i,11} & S_{i,12} \\ 0 & S_{i,12}^* & S_{i,22} \end{bmatrix} \geq 0, \text{ and } S_0 - S_{i,11} \geq 0 \qquad (11)$$

for i=1,…,r.

Proof: See Section IV for the proof.

Remark 1: For the single controller case (r=1), no controller interpolation is necessary. By constraining $S_0 = S_{1,11}$ and $R_0 = R_{1,11}$, the conditions of *Theorem 1* the conditions expressed in *Lemma 1*. ○

Remark 2: The controller synthesis conditions of *Lemma 2* are reflected in *Theorem 1* when $S_{i,11}$=S_0. ○

Remark 3: The robust controller interpolation in [10] addresses the regular case, i.e. $D_{12}^* D_{12} > 0$ and $D_{21} D_{21}^* > 0$, whereas *Theorem 1* also addresses the singular case, i.e. $D_{12}^* D_{12} \geq 0$ and $D_{21} D_{21}^* \geq 0$. ○

Remark 4: The H_∞ performance lower bound is $\gamma^* \triangleq \max_i \| T_{zw}(K_i) \|_\infty$. That is, the interpolated controller performance is limited by the the *worst* performing controller K_j, where $j \triangleq \arg\max_i \| T_{zw}(K_i) \|_\infty$. *Theorem 1* guarantees for any $\gamma > \gamma^*$ there exists an admissible interpolated controller. Hence, the conditions of *Theorem 1* are not conservative with respect to the robust controller interpolation criteria. ○

IV. Proof of Main Result

In the process of developing a proof for *Theorem 1*, the following section offers two complementary perspectives towards a framework in interpolated controller design. The perspectives address two critical aspects: enabling controller state information to be shared among controllers and mediating controller fighting.

A. Information Sharing Perspective

In a switched controller, it is desirable to share state information among the controllers. For example, an offline controller K_j is updated with information from an online controller K_i as it comes online in an effort to avoid undesirable switching transients. In effect, we seek functional relationships $x_{Kj} = f_{ji}(x_{Ki})$ which translate the information stored in the online controller states x_{Ki} into a form that is useful to the offline controllers. Moreover, we desire functional relationships that admit an interpolated controller satisfying the robust controller interpolation criteria. The following theorem searches for a linear function of the form $x_{Kj} = T_{ji} x_{Ki}$. Let $X_i \in \mathbb{H}^{n+n_K}$ take the block structure

$$X_i = \begin{bmatrix} S_0 & X_{i,12} \\ X_{i,12}^* & X_{i,22} \end{bmatrix} = \begin{bmatrix} R_0 & Y_{i,12} \\ Y_{i,12}^* & Y_{i,22} \end{bmatrix}^{-1}. \qquad (12)$$

Theorem 2: Given controllers K_i for i=1,…,r in (2) of order n_K, there exists an admissible interpolated controller $K(\alpha)$ satisfying $\| T_{zw}(K(\alpha)) \|_\infty < \gamma$ for all $\alpha \in \mathcal{A}$ if there exists positive definite $X_i \in \mathbb{H}^{n+n_K}$ of the form (12) satisfying $\Phi(X_i, T_{zw}(K_i), \gamma) < 0$ for i=1,…,r.

Proof: First, form the singular value decomposition $V \Sigma V^* = S_0 - R_0^{-1}$, where $V = [V_1 \;\; V_2]$, $\Sigma = \mathrm{diag}(\Sigma_1, 0)$, $\Sigma_1 = V_1^*(S_0 - R_0^{-1})V_1$. Consider the alternate controller realization

$$K_i' = \begin{cases} \dot{x}_{Ki}' = T_{Ki} A_{Ki} T_{Ki}^{-1} x_{Ki}' + T_{Ki} B_{Ki} y \\ u_i = C_{Ki} T_{Ki}^{-1} x_{Ki}' + D_{Ki} y \end{cases}, \qquad (13)$$

where

$$T_{Ki}^{-1} = \left[X_{i,22}^{-1} X_{i,12}^* V_1 \quad \mathrm{N}(X_{i,12}) \left(\mathrm{N}(X_{i,12})^* X_{i,22} \mathrm{N}(X_{i,12}) \right)^{-1/2} \right], \qquad (14)$$

Consequently, $\Phi(X, T_{zw}(K_i'), \gamma) = U_{T_i}^{-*} \Phi(X_i, T_{zw}(K_i), \gamma) U_{T_i}^{-1}$, where $U_{T_i} = \mathrm{diag}(T_i, I_{n_w + n_z})$, $T_i = \mathrm{diag}(I_n, T_{Ki})$,

$$X = \begin{bmatrix} S_0 & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} = T_i^{-*} X_i T_i^{-1},$$

$X_{12} = \begin{bmatrix} V\Sigma_1 & 0 \end{bmatrix}$ and $X_{22} = \mathrm{diag}(\Sigma_1, 0)$.

Consider the interpolated controller

$$K(\alpha) = \begin{cases} \dot{x}_K = A_K(\alpha) x_K + B_K(\alpha) y \\ u = C_K(\alpha) x_K + D_K(\alpha) y \end{cases}$$

where

$$\begin{bmatrix} A_K(\alpha) & B_K(\alpha) \\ C_K(\alpha) & D_K(\alpha) \end{bmatrix} = \sum_{i=1}^r \alpha_i \begin{bmatrix} T_{Ki} A_{Ki} T_{Ki}^{-1} & T_{Ki} B_{Ki} \\ C_{Ki} T_{Ki}^{-1} & D_{Ki} \end{bmatrix}.$$

Given $\Phi(X, T_{zw}(K_i'), \gamma) < 0$, then summing over α_i yields

$$\sum_{i=1}^r \alpha_i \Phi(X, T_{zw}(K_i'), \gamma) = \Phi(X, T_{zw}(K(\alpha)), \gamma) < 0,$$

thus guaranteeing $\| T_{zw}(K(\alpha)) \|_\infty < \gamma$ for all $\alpha \in \mathcal{A}$. □

In the proof of *Theorem 2*, the linear translation function takes the form $x_{Kj} = T_{Kj}^{-1} T_{Ki} x_{Ki}$, and a mapping to a shared controller state $x_K = T_{Ki} x_{Ki}$ was used to prove existence of $K(\alpha)$. Although *Theorem 2* provides sufficient conditions for information sharing, the conditions are not convex and may be quite conservative.

B. Stabilizing Signals Perspective

Suppose each controller K_i is designed for a unique objective. *Controller fighting* arises as the controllers interfere with one another as the controllers attempt to simultaneously achieve each objective. We propose injecting stabilizing signals into the controllers in order to mediate the interaction among controllers.

The proposed controller interpolation framework injects a stabilizing signal $\zeta_i(t)=[\zeta_{i1}(t)\ \ \zeta_{i2}(t)]$ into each controller

$$\hat{K}_i \triangleq \begin{cases} \dot{x}_{Ki}(t)=A_{Ki}x_{ki}(t)+B_{Ki}y(t)+\zeta_{i1}(t) \\ u_i(t)=C_{Ki}x_K(t)+D_{Ki}y(t)+\zeta_{i2}(t) \end{cases}, \tag{15}$$

where $x_{Ki}\in\mathbb{R}^{n_{Ki}}$. This has similarities to the notion of injecting signals into a controller in an effort to smooth the transition between controllers used in bumpless transfer [1]. Drawing from such bumpless transfer techniques, one way of defining the stabilizing signals is to dynamically generate $\zeta_i(t)$ via a *stabilizing compensator* Λ_i, as shown in Figure 2. Let the interconnection of $\hat{K}_i$ and Λ_i be represented by the augmented controller $\tilde{K}_i(\Lambda_i)$.

It is critical that each Λ_i is designed to ensure the robust controller interpolation criteria are satisfied. The augmented controller must satisfy $\tilde{K}_i(\Lambda_i)\sim K_i$ when $\alpha_i(t)=1$, i.e. $\Lambda_i\sim 0$ when $\alpha_i(t)=1$. The strategy is to search for Λ_i that both enables information sharing and satisfies $\tilde{K}_i(\Lambda_i)\sim K_i$, for i=1,…,r.

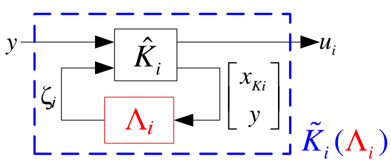

Figure 2: Stabilizing signals ζ_i generated by stabilizing compensator Λ_i

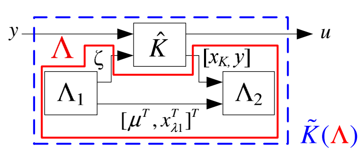

Figure 3: Λ_1 and Λ_2 in augmented controller $\tilde{K}(\Lambda)$

In order to simplify notation, we temporarily focus on constructing a single augmented controller $\tilde{K}(\Lambda)\sim K$ and $\tilde{X}\in\mathbb{H}^{n+n_K+n_\lambda}$ of the form (12), where n_λ is the number of additional states introduced by Λ. An explicit parameterization of all linear time-invariant Λ satisfying $\tilde{K}(\Lambda)\sim K$ is necessary. All such Λ are completely parameterized by Λ of the form shown in Figure 3, where

$$\Lambda_1=\begin{cases} \dot{x}_{\lambda 1}=A_{\lambda 1}x_{\lambda 1} \\ \mu=C_{\lambda 11}x_{\lambda 1} \\ \zeta=C_{\lambda 21}x_{\lambda 1} \end{cases}, \tag{16}$$

$$\Lambda_2=\left\{\dot{x}_{\lambda 2}=A_{\lambda 2}x_{\lambda 2}+B_{\lambda 21}x_{\lambda 1}+B_{\lambda 22}[x_K^*,y^*]^*+\mu\right., \tag{17}$$

$x_{\lambda 1}\in\mathbb{R}^{n_{\lambda 1}}$, and $x_{\lambda 2}\in\mathbb{R}^{n_{\lambda 2}}$. This may be shown by introducing Λ_1 and Λ_2 as the uncontrollable and unobservable modes, respectively, in the Kalman canonical decomposition of $\tilde{K}(\Lambda)$.

Theorem 3: Given P of order n in (1), controller K of order n_K, scalar $\gamma\geq 0$, and $R_0,S_0\in\mathbb{H}^n$. Then there exists $\tilde{K}(\Lambda)\sim K$ and positive definite $\tilde{X}$ of the form (9) satisfying $\Phi(\tilde{X},T_{zw}(\tilde{K}(\Lambda)),\gamma)<0$ if and only if (6), (7), and there exists a positive definite $S\in\mathbb{H}^{n+n_K}$ satisfying

$$\Phi(S,T_{zw}(K),\gamma)<0, \tag{18}$$

$$\begin{bmatrix} R_0 & I & 0 \\ I & S_{11} & S_{12} \\ 0 & S_{12}^T & S_{22} \end{bmatrix}\geq 0, \tag{19}$$

and

$$S_0-S_{11}\geq 0. \tag{20}$$

Proof: Please see the Appendix for the proof.

The following algorithm presents the construction of an augmented controller $\tilde{K}(\Lambda)\sim K$ positive definite $\tilde{X}$ of the form (9) satisfying $\Phi(\tilde{X},T_{zw}(\tilde{K}(\Lambda)),\gamma)<0$.

***Algorithm 1*:** (Construction of an Augmented Controller)

Step 1: Given the plant P, controller K, scalar $\gamma\geq 0$, and $R_0,S_0\in\mathbb{H}^n$ satisfying (6)-(7), determine S satisfying (18)-(20).

Step 2: Construct positive definite matrix $\tilde{X}$. Define the matrix $N_1\in\mathbb{R}^{n\times n_{\lambda 1}}$ as a solution to

$$N_1N_1^*=(S_{11}-S_{12}S_{22}^{-1}S_{12})-R_0^{-1},$$

and define the matrix $N_2\in\mathbb{R}^{n\times n_{\lambda 2}}$ as a solution to

$$N_2N_2^*=S_0-S.$$

Define the matrix $\tilde{X}\in\mathbb{H}^{n+n_K+n_\lambda}$ as

$$\tilde{X}=\begin{bmatrix} S_0 & S_{12} & N_1 & N_2 \\ S_{12}^* & S_{22} & 0 & 0 \\ N_1^* & 0 & I_{n_{\lambda 1}} & 0 \\ N_2^* & 0 & 0 & I_{n_{\lambda 2}} \end{bmatrix}.$$

Step 3:Parameterize augmented controller closed loop system. The two static output feedback gains $\Theta_1=[[A_{\lambda 1}^*\ \ C_{\lambda 11}^*]^*\ \ C_{\lambda 21}^*]^*$ and $\Theta_2=[[B_{\lambda 21}\ \ A_{\lambda 2}]\ \ B_{\lambda 22}]$ parameterize the closed loop system $T_{zw}(\tilde{K}(\Lambda))$

$$\begin{pmatrix} \dot{\tilde{x}} \\ z \end{pmatrix}=\begin{bmatrix} \bar{A}_{cl} & \bar{B}_{cl} \\ \bar{C}_{cl} & \bar{D}_{cl} \end{bmatrix}\begin{pmatrix} \tilde{x} \\ w \end{pmatrix}+\begin{bmatrix} \bar{B}_{\lambda 1} \\ \bar{D}_{\lambda 1} \end{bmatrix}\Theta_1\begin{bmatrix} \bar{C}_{\lambda 1} & 0 \end{bmatrix}\begin{pmatrix} \tilde{x} \\ w \end{pmatrix} +\begin{bmatrix} \bar{B}_{\lambda 2} \\ 0 \end{bmatrix}\Theta_2\begin{bmatrix} \bar{C}_{\lambda 2} & \bar{D}_{\lambda 2} \end{bmatrix}\begin{pmatrix} \tilde{x} \\ w \end{pmatrix}, \tag{21}$$

where $\tilde{x}=[x_{cl}^*,x_\lambda^*]^*$, $x_\lambda=[x_{\lambda 1}^*,x_{\lambda 2}^*]^*$, $\bar{D}_{cl}=D_{cl}$,

$$\bar{A}_{cl} = \begin{bmatrix} A_{cl} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{cl} = \begin{bmatrix} B_{cl} \\ 0 \end{bmatrix}, \bar{C}_{cl} = \begin{bmatrix} C_{cl} & 0 \end{bmatrix},$$

$$\begin{bmatrix} \bar{B}_{\lambda 1}^{*} & \bar{D}_{\lambda 1}^{*} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & I_{n_{\lambda 1}+n_{\lambda 2}} & 0 \\ \begin{bmatrix} 0 & I_{n_K} \\ B_2^T & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ D_{12}^{*} \end{bmatrix} \end{array}\right],$$

$$\begin{bmatrix} \bar{C}_{\lambda 2} & \bar{D}_{\lambda 2} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & I_{n_{\lambda 1}+n_{\lambda 2}} & 0 \\ \begin{bmatrix} 0 & I_{n_K} \\ C_2 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \end{array}\right],$$

$$\bar{C}_{\lambda 1}^{*} = \begin{bmatrix} 0 \\ \begin{bmatrix} I_{n_{\lambda 1}} \\ 0 \end{bmatrix} \end{bmatrix}, \quad \bar{B}_{\lambda 2} = \begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ I_{n_{\lambda 2}} \end{bmatrix} \end{bmatrix},$$

and $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ are the state space matrices of $T_{zw}(K)$.

Step 4:Search over parameter space (Θ_1,Θ_2). Determine Θ_1 and Θ_2 satisfying the linear matrix inequality $\Phi(\tilde{X}, T_{zw}(\tilde{K}(\Lambda)), \gamma) < 0$.

Step 5: Construct the augmented controller. Form Λ_1 and Λ_2 from Θ_1 and Θ_2, respectively. Subsequently, construct $\tilde{K}(\Lambda)$ through the interconnection of $\hat{K}$, Λ_1, and Λ_2 via (15), (16), and (17).

C. *Proof of Theorem 1*

In order to show sufficiency, suppose the conditions of *Theorem 1* are satisfied. Then from *Theorem 3*, there exists augmented controllers $\tilde{K}_i$ of order n_K and positive definite $\tilde{X}_i \in \mathbb{H}^{n+n_K}$ of the form (12), such that $\Phi(\tilde{X}_i, T_{zw}(\tilde{K}_i)) < 0$ for i=1,…,r. By construction, the augmented controllers satisfy the conditions of *Theorem 2*, thereby ensuring the existence of an admissible interpolated controller $K(\alpha)$ satisfying $\| T_{zw}(K(\alpha)) \|_\infty < \gamma$ for all $\alpha \in \mathcal{A}$.

Concerning necessity, consider any finite $\gamma > \gamma^* \triangleq \max_i \| T_{zw}(K_i) \|_\infty$. Applying *Lemma 1*, there exists $S_i = S_i^* > 0$ of the form (9) satisfying (10) for i=1,…,r. In addition, the pair $(S_{i,11}, R_{i,11})$ also satisfies *Lemma 2* for i=1,…,r. Applying the well-ordered property of Riccati inequalities, choosing (R_0, S_0) arbitrarily close to $(\hat{R}_0, \hat{S}_0)$, proves there always exists (R_0, S_0) satisfying (6), (7), $\hat{R}_0 \ge R_0 \ge R_{i,11}$, and $\hat{S}_0 \ge S_0 \ge S_{i,11}$ for i=1,…,r. Thus condition (11) is satisfied. Since γ may not be less than the lower bound γ^*, the conditions of *Theorem 1* have been shown to be necessary. □

The following algorithm presents the construction of an admissible robust interpolated controller .

Algorithm 2: *Robust Interpolated Controller Construction*

Step 1: Given P, r controllers K_i of the order n_{Ki}, and a scalar $\gamma \ge \gamma^*$, determine R_0, S_0, and S_i satisfying (6), (7), (11) for i=1,…,r.

Step 2: Via *Algorithm 3.1*, construct augmented controllers

$$\tilde{K}_i \triangleq \begin{cases} \dot{x}_{\tilde{K}i}(t) = A_{\tilde{K}i} x_{\tilde{K}i}(t) + B_{\tilde{K}i} y(t) \\ u(t) = C_{\tilde{K}i} x_{\tilde{K}i}(t) + D_{\tilde{K}i} y(t) \end{cases}$$

and

$$\tilde{X}_i = \begin{bmatrix} S_0 & \tilde{X}_{i,12} \\ \tilde{X}_{i,12}^{*} & \tilde{X}_{i,22} \end{bmatrix} = \begin{bmatrix} R_0 & \tilde{Y}_{i,12} \\ \tilde{Y}_{i,12}^{*} & \tilde{Y}_{i,22} \end{bmatrix}^{-1}$$

satisfying $\tilde{K}_i \sim K_i$ and $\Phi(\tilde{X}_i, T_{zw}(\tilde{K}_i), \gamma) < 0$, for i=1,..,r. Note that the order of $K(\alpha)$ n_K is determined by choosing $n_{\lambda 1i}$ and $n_{\lambda 2i}$ such that $n_K = n_{Ki} + n_{\lambda 1i} + n_{\lambda 2i}$.

Step 3: Construct an alternate augmented controller realization

$$\tilde{K}_i' = \begin{cases} \dot{z}_{\tilde{K}}(t) = T_{\tilde{K}i} A_{\tilde{K}i} T_{\tilde{K}i}^{-1} z_{\tilde{K}}(t) + T_{\tilde{K}i} B_{\tilde{K}i} y(t) \\ u(t) = C_{\tilde{K}i} T_{\tilde{K}i}^{-1} z_{\tilde{K}}(t) + D_{\tilde{K}i} y(t) \end{cases},$$

where $T_{\tilde{K}i}$ is defined in (14).

Step 4: Construct the interpolated controller

$$K(\alpha) = \begin{cases} \dot{x}_K = A_K(\alpha) x_K + B_K(\alpha) y \\ u = C_K(\alpha) x_K + D_K(\alpha) y \end{cases}$$

where

$$\begin{bmatrix} A_K(\alpha) & B_K(\alpha) \\ C_K(\alpha) & D_K(\alpha) \end{bmatrix} = \sum_{i=1}^{r} \alpha_i \begin{bmatrix} T_{\tilde{K}i} A_{\tilde{K}i} T_{\tilde{K}i}^{-1} & T_{\tilde{K}i} B_{\tilde{K}i} \\ C_{\tilde{K}i} T_{\tilde{K}i}^{-1} & D_{\tilde{K}i} \end{bmatrix}. \quad \circ$$

V. Conclusion

This paper addressed the problem of interpolating among a set of LTI controllers while maintaining a certain level of H_∞ performance, which is framed in terms of the robust controller interpolation criteria. The main result provides non-conservative LMI conditions for the synthesis of an admissible robust interpolated controller. The proof of the main result was presented in the form of two controller interpolation perspectives. The results presented here are extensible to interpolated controller design for various other control problems framed in terms of LMIs [8],[13]. Consequently, the controller interpolation approach discussed here enables a new dimension on control design, while complementing the existing literature.

Appendix

A. *Some Useful Results*

The following lemmas will prove useful in the following proofs. For more details, the interested reader is directed to [4] and [6].

Lemma 3: Given matrices G, H, and symmetric matrix Ψ, there exists Θ satisfying

$$\Psi + H^* \Theta G + G^* \Theta^* H < 0 \tag{22}$$

if and only if both

$$\mathrm{N}(H)^* \Psi \mathrm{N}(H) < 0 \text{ and } \mathrm{N}(G)^* \Psi \mathrm{N}(G) < 0. \tag{23}$$

For *Lemma 4* consider the matrix X of the form

$$X = \begin{bmatrix} S & N \\ N^T & X_{22} \end{bmatrix} = \begin{bmatrix} R & M \\ M^T & Y_{22} \end{bmatrix}^{-1}. \tag{24}$$

Given the submatrices (R,S), the following Matrix Completion Lemma [6] guarantees the existence of positive definite X satisfying (24).

Lemma 4: Suppose $R, S \in \mathbb{H}^n$ are positive definite and $n_1 \geq 0$ is an integer. The following statements are equivalent:

i) There exists $N, M \in \mathbb{R}^{n \times n_1}$ and positive definite $X_{22}, Y_{22} \in \mathbb{H}^{n_1}$ satisfying (24)

ii) $S - R^{-1} \geq 0$ and $\operatorname{rank}\left(S - R^{-1}\right) \leq n_1$ (25)

iii) $\begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0$ and $\operatorname{rank}\begin{bmatrix} S & I \\ I & R \end{bmatrix} \leq n + n_1$ (26)

B. Proof of Theorem 3

In order to prove *Theorem 1*, first we develop necessary and sufficient conditions for the existence of an augmented controller $\tilde{K} \sim K$ satisfying $\Phi(\tilde{X}, T_{zw}(\tilde{K}), \gamma) < 0$ for some $\tilde{X}$ of the form (12).

For the following lemmas, consider positive definite $\tilde{X} \in \mathbb{H}^{(n+n_K+n_{\lambda 1}+n_{\lambda 2})}$, $\bar{R}, \bar{S}, S \in \mathbb{H}^{(n+n_K)}$, and $R_0, S_0 \in \mathbb{H}^n$ satisfying

$$\tilde{X} = \begin{bmatrix} \bar{S} & \tilde{X}_{12} & N_3 \\ \tilde{X}_{12}^* & \tilde{X}_{22} & \tilde{X}_{23} \\ N_3^* & \tilde{X}_{23}^* & X_{33} \end{bmatrix} = \begin{bmatrix} \bar{R} & M_2 & \tilde{Y}_{13} \\ M_2^* & Y_{22} & \tilde{Y}_{23} \\ \tilde{Y}_{13}^* & \tilde{Y}_{23}^* & \tilde{Y}_{33} \end{bmatrix}^{-1} = \tilde{Y}^{-1}, \quad (27)$$

$$Y_1 = \begin{bmatrix} \bar{R} & M_2 \\ M_2^* & Y_{22} \end{bmatrix} = \begin{bmatrix} S & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}^{-1}, \quad (28)$$

$$\bar{R} = \begin{bmatrix} R_0 & \bar{R}_{12} \\ \bar{R}_{12}^* & \bar{R}_{22} \end{bmatrix}, \text{and } \bar{S} = \begin{bmatrix} S_0 & \bar{S}_{12} \\ \bar{S}_{12}^* & \bar{S}_{22} \end{bmatrix}. \quad (29)$$

For a given $\tilde{X}$, *Lemma 5* employs two consecutive applications of *Lemma 3* in order to produce necessary and sufficient conditions for the existence of Θ_1 and Θ_2 satisfying $\Phi(\tilde{X}, T_{zw}(\tilde{K}(\Lambda)), \gamma) < 0$.

Lemma 5: Suppose $\tilde{X} \in \mathbb{H}^{(n+n_K+n_{\lambda 1}+n_{\lambda 2})}$ is a positive definite matrix satisfying (27)-(29). There exists Θ_1 and Θ_2 such that $\Phi(\tilde{X}, T_{zw}(\tilde{K}(\Lambda)), \gamma) < 0$, if and only if (6), (7), and (18) are simultaneously satisfied.

Proof: Using the description of $T_{zw}(\tilde{K}(\Lambda))$ in (21), the matrix inequality $\Phi(\tilde{X}, T_{zw}(\tilde{K}(\Lambda)), \gamma) < 0$ can be parameterized as an affine function of Θ_1 and Θ_2,

$$\Psi_{\tilde{X}} + \sum_{i=1}^{2} H_{\tilde{X}i}^* \Theta_i G_i + G_i^* \Theta_i^* H_{\tilde{X}i} < 0. \quad (30)$$

By invoking *Lemma 3* to eliminate Θ_1, (30) yields the necessary and sufficient conditions

$$\mathbb{N}(G_1)^* \left(\Psi_{\tilde{X}} + H_{\tilde{X}2}^* \Theta_2 G_2 + G_2^* \Theta_2^* H_{\tilde{X}2}\right) \mathbb{N}(G_1) < 0 \quad (32)$$

and

$$\mathbb{N}(H_{\tilde{X}1})^* \left(\Psi_{\tilde{X}} + H_{\tilde{X}2}^* \Theta_2 G_2 + G_2^* \Theta_2^* H_{\tilde{X}2}\right) \mathbb{N}(H_{\tilde{X}1}) < 0. \quad (33)$$

Eliminating Θ_2 from (32) via *Lemma 3* produces the equivalent conditions

$$\mathbb{N}\left(G_2 \mathbb{N}(G_1)\right)^* \mathbb{N}(G_1)^* \Psi_{\tilde{X}} \mathbb{N}(G_1) \mathbb{N}\left(G_2 \mathbb{N}(G_1)\right) < 0 \quad (34)$$

and

$$\mathbb{N}\left(H_{\tilde{X}2} \mathbb{N}(G_1)\right)^* \mathbb{N}(G_1)^* \Psi_{\tilde{X}} \mathbb{N}(G_1) \mathbb{N}\left(H_{\tilde{X}2} \mathbb{N}(G_1)\right) < 0. \quad (35)$$

Matrix inequalities (33) and (34) are equivalent to (6)-(7), whereas (35) is equivalent to (18). □

Lemma 6 presents the necessary and sufficient conditions for the existence of positive definite $\tilde{X}$ satisfying (27)-(29).

Lemma 6: Given $S \in \mathbb{H}^{(n+n_K)}$ of the form (9) and $R_0, S_0 \in \mathbb{H}^n$, there exists positive definite $\tilde{X}$ satisfying (27)-(29) if and only if (19) and (20) are satisfied.

Proof: Starting with the completion of Y_1, (19) yields $\bar{R} - S^{-1} \geq 0$ by applying the Schur complement [4]. *Lemma 4* guarantees there exists $M_2, X_{12} \in \mathbb{R}^{n \times n_{\lambda 1}}$ and $Y_{22}, X_{22} \in \mathbb{H}^{n_{\lambda 1}}$ if and only if (19) is satisfied and $n_{\lambda 1}$ is greater than the rank of (19). Turning our attention to the completion of $\tilde{X}$, $\bar{S} - S \geq 0$ is equivalent to (20). *Lemma 4* subsequently ensures there exists $N_3, \tilde{Y}_{13} \in \mathbb{R}^{n \times n_{\lambda 2}}$, $\tilde{X}_{23}, \tilde{Y}_{23} \in \mathbb{R}^{n_{\lambda 1} \times n_{\lambda 2}}$, and $X_{33}, \tilde{Y}_{33} \in \mathbb{H}^{n_{\lambda 2}}$ if and only if (20) is satisfied and $n_{\lambda 2} \geq \operatorname{rank}(S_0 - S_{11})$. □

Theorem 3 may easily be shown to be true by combining the necessary and sufficient conditions for the existence of $\tilde{K}(\Lambda)$ and $\tilde{X}$ provided by *Lemma 5* and *Lemma 6*.

References

[1] Zaccarian, L. and Teel, A.R. (2002). "A common framework for anti-windup, bumpless transfer and reliable designs," *Automatica* vol. 38, pp. 1735-1744.

[2] W.J. Rugh & J.S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, pp. 1401-1425, 2000.

[3] D. Liberzon, *Switching in Systems and Control*, Boston, Birkhauser, 2003.

[4] S. Boyd, L. El Ghaoui, E. Feron, & V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia, PA: Society for Industrial and Applied Mathematics, 1994.

[5] J.P. Hespanha & S.A. Morse, "Switching between stabilizing controllers," *Automatica*, vol. 38, pp. 1905-1917, 2002.

[6] G. Dullerud & F. Paganini, *A Course in Robust Control Theory: A Convex Approach*, New York, NY: Springer Science+Business Media, Inc, 2000.

[7] Gahinet P. & Apkarian, P. (1994). "A Linear Matrix Inequality Approach to H_∞ Control," *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 421-448.

[8] I. Masubuchi, A. Ohara, & N. Suda, "LMI-based controller synthesis: a unified formulation and solution," *International Journal Robust Nonlinear Control*, vol. 8, pp. 669-686, 1998.

[9] B. Hencey and A. Alleyne, "A Robust Controller Interpolation Design Technique," *Proceedings of American Control Conference*, New York, NY, July 2007.

[10] B. Hencey and A. Alleyne "Robust Controller Interpolation via Parameterization," *Proceedings of the ASME Dynamic Systems and Control Conference*, Ann Arbor, MI, Oct. 2008.

[11] Rasmussen, B.P. & Alleyne, A., 2006. "Gain scheduled control of an air conditioning system using the Youla parameterization," in *Proceedings of American Control Conference*, Minneapolis, MN, pp. 5336-5341.

[12] A. Packard, "Gain scheduling via linear fractional transformations," *Systems and Control Letters*, vol. 22, pp. 79-92, 1993.

[13] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled H_∞ controller," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 853-864, 1995.