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An Algorithm for the Long Run Average Cost Problem for Linear Systems with Non-observed Markov Jump Parameters

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***Abstract*— This paper addresses the problem of long run average cost for linear systems with non-observed Markov jump parameters. We present an algorithm that relies on the approximation of the (infinite horizon) cost via its finite horizon version and uses an evolutionary-based algorithm for the finite horizon cost. A numerical example illustrates the proposed algorithm.**

I. Introduction

There has been a great deal of attention to linear systems with Markov jump parameters (LSMJP). LSMJP constitutes a well known class of system that can be successfully employed in applications featuring random failures, environmental changes and other phenomena that lead to random, abrupt changes of behaviour. There exist numerous results, spanning from notions of stability [13], stabilization [16], [9], basic features such as stabilizability and detectability [3], to optimal solution to finite and infinite quadratic costs [14], [7], [8] and filtering [10].

The available results for LSMJP are strong enough to allow for parallels with standard linear systems, mainly in scenarios with complete state observation (which includes the observation of the Markov state), and we mention as illustration the existence of generalized Riccati equations that characterize the optimal solution to the long run average cost (LRAC), even in the presence of additive Gaussian noise, see [6]. However, the parallel with standard linear systems is weaker when dealing with incomplete or no observation of the Markov state. For instance, in [11] authors need to consider a variational-like approach to the finite horizon quadratic cost problem with partial observation.

In the context of LRAC that we are interested in this paper, there only exists a result on bounds for the cost [19], which allows to study existence of the LRAC and convergence of the finite horizon average cost to the LRAC, assuming controls in static state feedback form. Direct extension of the above mentioned variational-like approach seems not to be viable, as it typically provides non-stationary controls, whereas the LRAC, as an infinite horizon problem, requires stationary controls. In fact, the LRAC problem for LSMJP with unobserved Markov state can be interpreted in simple terms as how to obtain a static gain that is not a function of the Markov state and minimizes the average cost incurred by the controlled Markov jump system. Such controls can be implemented when the Markov variable is not accessible and are easier to implement than non-stationary controls, and they are stable (in a certain stochastic sense) provided some mild detectability-like conditions hold, making them of much appeal for applications. However, to the extent of our knowledge there is no available study on methods for providing solutions to the LRAC problem.

In this paper, we present a simple algorithm, based on the idea of approximation via the finite horizon cost (FHC). The FHC problem with horizon is dealt with a genetic algorithm, aiming at a suboptimal static gain in the form $u_k = Kx_k$, where x_k is the so-called continuous state and u is the control. Next, we employ the algorithm for the FHC with increasing horizon T, for approximating the LRAC. The LRAC is more complex as it involves issues of stability and sensitivity to the initial condition of the system. To handle these difficulties, we employ a modified cost (by including a parameter $\varepsilon > 0$), and analyse the impact on the original cost. We also show that stabilizing gains K are associated to finite LRAC with $\varepsilon > 0$. These contributions are presented in a quite simple and comprehensive manner, and are fundamental for the approximation of the LRAC via the FHC and, hence, for our algorithm to provide a solution to the LRAC. Apart from these theoretical results, we employ some new strategies in the genetic algorithm for the FHC problem: (i) use of solutions given by coupled Riccati equations from the LSMJP theory to initialize the population and (ii) representation of each element of the gain in a polynomial form to increase the number of genes of each element of the population, as detailed in Section III.

The paper is organized as follows. In Section II we formalize the LRAC and FHC problems for LSMJP. The algorithm for the FHC is detailed in Section III, and in Section IV we obtain some results that allow us to use this algorithm to approximate the LRAC. Section V presents an illustrative example. We finish with some concluding remarks.

II. Problem Formulation and Preliminary Results

One simple way to describe the discrete-time LSMJP considered in this paper is to take into account, initially,

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a standard time-varying linear system in the form

$$x_{k+1} = S_k x_k + F_k u_k + w_k, \quad k \geq 0,$$

the initial condition $x_0 \in \mathscr{R}^n$, where $x \in \mathscr{R}^n$ is the state, $u \in \mathscr{R}^r$ is the input and w_k is a zero-mean independent Gaussian random variable with covariance matrix E. Consider the cost function

$$W^T = \sum_{k=0}^{T} x_k' Q x_k + u_k' R u_k,$$

with $Q = Q' \geq 0$ and $R = R' > 0$. However, instead of the standard hypothesis that the values of S_k and F_k are known a-priori for all time instants $k \geq 0$, we consider that they assume values from the collections $A = \{A_1, \ldots, A_S\}$ $B = \{B_1, \ldots, B_S\}$ respectively, accordingly to a Markov chain $\{\theta_0, \theta_1, \ldots\}$:

$$S_k = A_{\theta_k},\ F_k = B_{\theta_k}, \quad k \geq 0.$$

The transition probabilities are

$$P(\theta_{k+1} = j | \theta_k = i) = p_{ij}$$

with $i, j \in \mathcal{S} = \{1, \ldots, S\}$, and the initial distribution is $\pi = [P(\theta_0 = 1), \ldots, P(\theta_0 = S)] = [\pi_1, \ldots, \pi_S]$. We assume that the matrix $\mathbb{P} = [p_{ij}]$ and the vector π are known. In this situation, the system is inherently stochastic and x_k forms a stochastic process, in such a manner that W as defined above is a random variable, and we consider its expected value for optimization purposes,

$$Y^T = \mathscr{E}\{W^T\}.$$

It is a well known fact in the LSMJP literature that Y^T can be represented in terms of certain linear operators (denoted here by $\mathcal{T}$) involving the conditional second moment matrices of the system, as we present in the sequel. Let $\mathcal{M}^{r,s}$ denote the linear space formed by a number S of $r \times s$-dimensional matrices, $\mathcal{M}^{r,s} = \{U = (U_1, \ldots, U_S)\}$. Also, $\mathcal{M}^r \equiv \mathcal{M}^{r,r}$. For $U, V \in \mathcal{M}^r$, $U \geq V$ signifies that $U_i - V_i \in \mathcal{R}^{r0}$ for each $i \in \mathscr{N}$, and similarly for other mathematical relations. Consider $tr\{\cdot\}$ as the trace operator. It is known that $\mathcal{M}^{r,s}$ equipped with the inner product

$$\langle U, V \rangle = \sum_{i=1}^{S} tr\{U_i' V_i\}$$

forms a Hilbert space. Following the notation of [3] we define, for $U \in \mathcal{M}^n$, $V \in \mathcal{M}^n$, the operator $\mathscr{T}_U : \mathcal{M}^n \to \mathcal{M}^n$ by

$$\mathcal{T}_{U,i}(V) := \sum_{j=1}^{S} p_{ji} U_j V_j U_j', \quad \forall i \in \mathscr{N}, \tag{1}$$

and we define for convenience $\mathcal{T}^0(V) = V$, and for $t \geq 1$, $\mathcal{T}^t(V) = \mathcal{T}(\mathcal{T}^{t-1}(V))$. It is simple to check that $\mathcal{T}$ is linear.

The next result is an adaptation of the results in [6, ch 3], see also [19].

Proposition 1. *Let $X \in \mathcal{M}^n$, $\mathcal{Q} \in \mathcal{M}^n$ and $\Sigma \in \mathcal{M}^n$ be defined by $X_i = x_0 x_0' \pi_i$, $\mathcal{Q}_i = Q$ and $\Sigma_i = E$, $\forall i \in \mathcal{S}$. Then,*

$$Y^T = \sum_{k=0}^{T} \Big\langle \mathcal{T}_A^k(X) + \sum_{l=0}^{k-1} \mathcal{T}_A^l(\Sigma)\,,\, \mathcal{Q} \Big\rangle. \tag{2}$$

The collection of matrices X in Proposition 1 represents the conditional second moment of the initial condition, $X_i = \mathscr{E}\{x_0 x_0' | \theta_0 = i\} P(\theta_0 = i)$. We write $Y(X)$ to emphasize the dependence of Y on X.

Observation and Control Structures

We assume that only the quantity x_k is available to the controller at each time instant k. θ_k is not observed. In connection, we consider a linear state feedback control in the form

$$u_k = K x_k. \tag{3}$$

K is referred to as a static gain (in opposition to time-dependent gains of the form K_k). Static gains are among the simplest to implement controls, thus being of interest for many applications.

For the complete observation case (x_k and θ_k observed), with controls in the form

$$u_k = K_{\theta_k} x_k, \tag{4}$$

there are strong results in LSMJP literature, parallelling the deterministic linear systems theory. One result that shall be useful to us is the solution of finite horizon quadratic cost problems via coupled algebraic Riccati equations, allowing to compute optimal gains K_i in a simple way, as presented in the Appendix.

For a given gain K (or a collection of gains $K = \{K_i, i \in \mathcal{S}\}$ in the complete observation case), we have a closed loop form $x_{k+1} = (S_k + F_k K)$ (respectively, $x_{k+1} = (S_k + F_k K_k)$), giving rise to collections of "closed loop" matrices $A_K = (A_1 + B_1 K, \ldots, A_S + B_S K)$ (respectively, $A_K = (A_1 + B_1 K_1, \ldots, A_S + B_S K_S)$), operators $\mathcal{T}_{A_K}$ and costs given by (2). We denote Y_K^T and $Y_K^T(X)$ to emphasize the dependence on K and on X,

$$Y_K^T(X) = \sum_{k=0}^{T} \Big\langle \mathcal{T}_{A_K}^k(X) + \sum_{l=0}^{k-1} \mathcal{T}_{A_K}^l(\Sigma), \mathcal{Q} \Big\rangle. \tag{5}$$

Problem Formulation

We are interested in the optimization problems

$$P_1: \quad min_G Y_G^T(X),$$

in the scenario with $T < \infty$, and

$$P_2: \quad min_G Z_G,$$

when $T = \infty$, where

$$Z_G = \lim_{T\to\infty} \sup_{T} Y_G^T(X)/T.$$

P_2 is referred to as the LRAC problem. The scenario with $T = \infty$ is more complex in the sense that finite Z does not ensure that the controlled system is stable, as we shall see later.

TABLE I
GENETIC ALGORITHM FOR THE FHC PROBLEM P1

1)	Set the GA and system parameters
2)	Initialize the population K^ℓ as in (6) and (7)
3)	Perform genetic and evolutionary operators to obtain a new population and the gains K^ℓ (7)
4)	For each K^ℓ, calculate $Y^\ell(X)$ via (5)
5)	If the stop criterion is not satisfied, return to (3)

III. A GENETIC ALGORITHM FOR THE PROBLEM P1

One important issue that arises when dealing with GAs is how to create an adequate initial population. In principle, one could simply take randomly generated gains $K^1,\ldots$. However, we have observed in many numerical examples that the cost associated to many of these gains are extremely high, leading to a high probability of being excluded in the selection process. As a result, the "genetic variety" of the the second generation of the population decays, and the algorithm performance is poor, see e.g. [18], [15], [17]. In order to overcome this difficulty, we initially solve the optimization problem within the class of controls in the form (4) via the recursive coupled algebraic Riccati equations described in the Appendix, leading to a collection of gains $L_1,\ldots,L_S$ that are optimal for the complete state observation problem, and then we use these gains to obtain the initial population as follows:

$$K^\ell = \alpha_1 L_1 + \ldots + \alpha_S L_S, \quad \ell = 1,\ldots,q \tag{6}$$

where $L_1,\ldots,L_S$ are given by (14) (or by (15) in the context of the LRAC problem P2), q is the size of the initial population and α_i, $i \in S$, are zero-mean independent Gaussian variables with some arbitrary covariance matrix P.

The static gain K (or, in the complete observation case, each element the collection of static gains K) may be low dimensional, as in Example 1, where $K = [K(1,1)\, K(1,2)\, K(1,3)]$. In these situations, if we simply identify each element of K as one gene of each element of the population, we will have a few genes, and we have observed a poor performance of the GA. We employ the polynomial representation for each element of $K = [K(i,j)]$,

$$K(i,j) = \beta_1(i,j)^1 + \beta_2(i,j)^2 + \ldots + \beta_p(i,j)^p. \tag{7}$$

For the initialization of each element of the population, K^ℓ, we employ randomly generated $\beta_1^\ell(i,j),\ldots,\beta_{p-1}^\ell(i,j)$ and define $\beta_p^\ell(i,j) = K^\ell(i,j) - \beta_1^\ell(i,j)^1 + \ldots + \beta_{p-1}^\ell(i,j)^{p-1}$; the GA will determine the β's of the following generations. The Algorithm is presented in Table I.

IV. AN ALGORITHM FOR THE PROBLEM P2 (LRAC)

The problem P2 (LRAC problem for LSMJP) is far more complex than the finite horizon problem P1 because the infinite horizon scenario involves the question of stability and related issues of sensitivity to the initial condition. For instance, we may have $\lim_{T\to\infty}(1/T)Y_G^T(X) < \infty$ and $\lim_{T\to\infty}(1/T)Y_G^T(X) = \infty$, $W \neq X$. Moreover, lack of positiveness of Σ and Q may lead to finite cost controls that are not stabilizable, provided certain conditions of detectability and stabilizability are not satisfied. In order to overcome these difficulties, we consider the standard hypothesis that the Markov chain is ergodic, in such a manner that the limiting distribution $\mathbb{P}^\infty\pi$ is unique [1]. We also consider the modified cost

$$Y_K^\varepsilon(X) = \sum_{k=0}^{T} \left\langle \mathcal{T}_{A_K}^k(X) + \sum_{l=0}^{k-1} \mathcal{T}_{A_K}^l(\Sigma + \varepsilon\mathbb{I}), \mathcal{Q} + \varepsilon I \right\rangle,$$

where $\mathbb{I} = (I,\ldots,I) \in \mathcal{M}^n$, and the related LRAC Z_G^ε, with the property that Z_K^ε is stabilizing if and only if the gain K is stabilizing, and we evaluate the impact on the original costs. Let us start formalizing the stabilizability notion.

Definition 1. *We say that a static feedback gain K (respectively, a collection of gains $K \in \mathcal{M}^{r,n}$ in the complete observation scenario) is mean square (MS) stabilizing when, for each $\Sigma \in \mathcal{M}^n$ satisfying $\Sigma = \Sigma' \geq 0$, there exists $\Gamma \in \mathcal{M}^n$, $\Gamma = \Gamma' \geq 0$, such that*

$$\mathcal{T}_{A_K}^k(X) + \sum_{l=0}^{k-1} \mathcal{T}_{A_K}^l(\Sigma) < \Gamma \tag{8}$$

for each $X \in \mathcal{M}^n$, $X = X' \geq 0$, provided $k \geq \bar{k}$ for some $\bar{k}$ (possibly dependent on X).

Remark 1. *MS-stabilizability as defined above and the related notion of MS-stability are equivalent to the standard MS notions in the LSMJP literature, e.g. K is MS-stabilizing if and only if*

$$\sum_{l=0}^{k} \mathcal{T}_{A_K}^l(X) < \infty, \quad \forall X = X' \geq 0. \tag{9}$$

For the equivalences, we refer to [6].

Proposition 2. *If K is not MS-stabilizing, then for each $M \in \mathcal{M}^n$, $M = M' \geq 0$, there exists a sufficiently large integer $\bar{T}$ such that $\sum_{k=0}^{\bar{T}} \mathcal{T}_{A_K}^k(I) \geq M$.*

Proof: From (9), if K is not stabilizable we have that there exists X such that, for each Γ, $\sum_{l=0}^{k} \mathcal{T}_{A_K}^k(X) \geq \Gamma$ for some k. The linearity of $\mathcal{T}$ and the fact that $\upsilon\mathbb{I} \geq X$ for some $\upsilon \geq 0$, allow to write $\sum_{l=0}^{k} \mathcal{T}_{A_K}^l(\mathbb{I}) \geq \upsilon^{-1}\sum_{l=0}^{k} \mathcal{T}_{A_K}^l(X) \geq \upsilon^{-1}\Gamma$.

Lemma 1. *K is MS-stabilizing if and only if $Z_K^\varepsilon < \infty$.*

Proof: Necessity. Replacing Σ by $\Sigma + \varepsilon\mathbb{I}$ in Definition

1 and employing Proposition 1 we write

$$\begin{aligned}\frac{Y_K^{\varepsilon,T}(X)}{T} &= \frac{1}{T}\sum_{k=0}^{T-1}\left\langle \mathcal{T}_{A_K}^k(X)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\Sigma+\varepsilon\mathbb{I}), \mathcal{Q}+\varepsilon I\right\rangle \\ &= \frac{1}{T}\sum_{k=0}^{\bar{k}-1}\left\langle \mathcal{T}_{A_K}^k(X)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\Sigma+\varepsilon\mathbb{I}), \mathcal{Q}+\varepsilon I\right\rangle \\ &+ \frac{1}{T}\sum_{k=\bar{k}}^{T-1}\left\langle \mathcal{T}_{A_K}^k(X)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\Sigma+\varepsilon\mathbb{I}), \mathcal{Q}+\varepsilon I\right\rangle \\ &\le \frac{\Delta}{T}+\frac{1}{T}\sum_{k=\bar{k}}^{T-1}\left\langle \Gamma, \mathcal{Q}\varepsilon I\right\rangle = \frac{\Delta}{T}+\frac{T-\bar{k}}{T}\left\langle \Gamma, \mathcal{Q}+\varepsilon I\right\rangle,\end{aligned} \tag{10}$$

where we set $\Delta = \sum_{k=0}^{\bar{k}-1}\left\langle \mathcal{T}_{A_K}^k(X)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\Sigma+\varepsilon\mathbb{I}), \mathcal{Q}+\varepsilon I\right\rangle$. This leads to

$$\limsup_{T\to\infty}\frac{Y_K^{\varepsilon,T}(X)}{T} \le \left\langle \Gamma, \mathcal{Q}+\varepsilon I\right\rangle$$

Sufficiency. Let $\Gamma = Z_K^\varepsilon < \infty$. Let us deny the statement by assuming that K is not stabilizing. In Proposition 2, set $M = \varepsilon^{-2}(\Gamma\mathbb{I}+1)$ and consider the corresponding $\bar{T}$. We can write

$$\begin{aligned}\Gamma &= \lim_{T\to\infty}\sup_X \frac{1}{T}\sum_{\ell=0}^{T-1}\left\langle \mathcal{T}_{A_K}^\ell(X)+\sum_{l=0}^{\ell-1}\mathcal{T}_{A_K}^l(\Sigma+\varepsilon\mathbb{I}), \mathcal{Q}+\varepsilon\mathbb{I}\right\rangle \\ &\ge \varepsilon^2\lim_{T\to\infty}\sup_X \frac{1}{T}\sum_{\ell=0}^{T-1}\left\langle \sum_{l=0}^{\ell-1}\mathcal{T}_{A_K}^l(\mathbb{I}), \mathbb{I}\right\rangle \\ &= \varepsilon^2\lim_{T\to\infty}\frac{1}{T}\sum_{\ell=0}^{T-1}\left\langle \sum_{l=0}^{\ell-1}\mathcal{T}_{A_K}^l(\mathbb{I}), \mathbb{I}\right\rangle \\ &\ge \varepsilon^2\lim_{T\to\infty}\frac{1}{T}\sum_{\ell=\bar{T}}^{T-1}\left\langle \sum_{l=0}^{\bar{T}}\mathcal{T}_{A_K}^l(\mathbb{I}), \mathbb{I}\right\rangle \\ &\ge \varepsilon^2\lim_{T\to\infty}\frac{1}{T}\sum_{\ell=\bar{T}}^{T-1}\left\langle M, \mathbb{I}\right\rangle \ge \varepsilon^2\lim_{T\to\infty}\frac{T-\bar{T}}{T}\left\langle M, \mathbb{I}\right\rangle \\ &\ge \varepsilon^2\varepsilon^{-2}\left\langle (\Gamma\mathbb{I}+1), \mathbb{I}\right\rangle \ge \Gamma+1,\end{aligned}$$

which is an absurd.

The algorithm we propose in this section seeks for the LRAC using the approximation via $Y^{\varepsilon,T}(X)/T$. Thus, it is important to show that this quantity converges to Z^ε as $T\to\infty$. This issue was studied and solved in [19, Corollary 2]; for convenience, we present an adaptation to the present context, next.

Proposition 3. *Assume the Markov chain θ is ergodic and that K is MS-stabilizing. Then, there exist scalars $\alpha,\beta \ge 0$ such that*

$$\left|Z_K - \frac{Y^{\varepsilon,T}(X)}{T}\right| \le \frac{\alpha\|X(\ell)\|+\beta}{T} \tag{11}$$

Next, the impact of the scalar ε in Z_K^ε is evaluated in terms of the original LRAC Z_K, showing that Z_K^ε converges to Z_K as ε tends to zero, provided the gain K is stabilizing.

Lemma 2. *If K is MS-stabilizing, then there exists $\xi \ge 0$ such that $Z_K \le Z_K^\varepsilon \le Z_K+\varepsilon\xi$.*

Proof: The statement that $Z_K \le Z_K^\varepsilon$ is trivial. Let us denote the quantities Γ and $\bar{k}$ in Definition 1 by Γ_Σ and $\bar{k}_X$ to emphasize the dependence on Σ and X respectively. Let $\tilde{k} = max(\bar{k}_X,\bar{k}_0)$. It is straightforward to check from (5) and the linearity of $\mathcal{T}$ that

$$\mathcal{T}_{A_K}^k(0)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\varepsilon\mathbb{I}) = \varepsilon\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\mathbb{I}) \le \varepsilon\Gamma_{\mathbb{I}} \tag{12}$$

for $k\ge\bar{k}_0$. One can check that

$$\begin{aligned}Y_K^{\varepsilon,T}(X)-Y_K^T(X) &= \sum_{k=0}^{T-1}\left\langle \sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\varepsilon\mathbb{I}), \mathcal{Q}\right\rangle \\ &+ \sum_{k=0}^{T-1}\left\langle \mathcal{T}_{A_K}^k(X)+\sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\Sigma), \varepsilon\mathbb{I}\right\rangle + \sum_{k=0}^{T-1}\left\langle \sum_{l=0}^{k-1}\mathcal{T}_{A_K}^l(\varepsilon\mathbb{I}), \varepsilon\mathbb{I}\right\rangle\end{aligned} \tag{13}$$

Now, evaluations similar to the one in (10) for the terms on the right-hand side of (13) provides

$$\begin{aligned}&\frac{Y_K^{\varepsilon,T}(X)-Y_K^T(X)}{T} \\ &\le \frac{\Delta}{T}+\frac{T-\tilde{k}}{T}\left(\varepsilon\left\langle\Gamma_\Sigma, \mathbb{I}\right\rangle+\varepsilon\left\langle\Gamma_{\mathbb{I}}, \mathcal{Q}\right\rangle+\varepsilon^2\left\langle\Gamma_{\mathbb{I}}, \mathbb{I}\right\rangle\right),\end{aligned}$$

which leads to

$$\limsup_{T\to\infty}\frac{Y_K^{\varepsilon,T}(X)-Y_K^T(X)}{T} \le \varepsilon\xi$$

where we set $\xi = \left\langle\Gamma_\Sigma, \mathbb{I}\right\rangle+\left\langle\Gamma_{\mathbb{I}}, \mathcal{Q}\right\rangle+\varepsilon\left\langle\Gamma_{\mathbb{I}}, \mathbb{I}\right\rangle$.

In the algorithm for the LRAC, we use the coupled algebraic Riccati equation in (15) (in the Appendix) to initialize the population instead of the coupled recursive Riccati equations in (14). The solution of (15) with positive weighting matrices is connected to stabilizing solutions of the LRAC problem, see e.g. [12], thus providing stabilizing initial gains L_i for (6). We include in the Appendix a simple method for solving the coupled algebraic Riccati equation, for ease of reference.

V. Numerical Example

Example 1 (Magnetic suspension system)**.** *We take into account the model of a magnetic suspension system, as presented in [5]. For ease of reference, next we present the data of the system,*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1750 & 0 & -34.07 \\ 0 & 0 & -0.0383 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1.9231 \end{bmatrix}.$$

Here we consider failures in the control action u. Let the transition probability matrix $\mathbb{P}$ be given by

$$\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix},$$

and assume that whenever $\theta_k = 2$ the system is not affected by the controller, and we set $A_2 = A_1$ and $B_2 = 0$ accordingly. We also consider zero-mean additive noise with covariance matrix GG', with

$$G = 1e^{-6}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

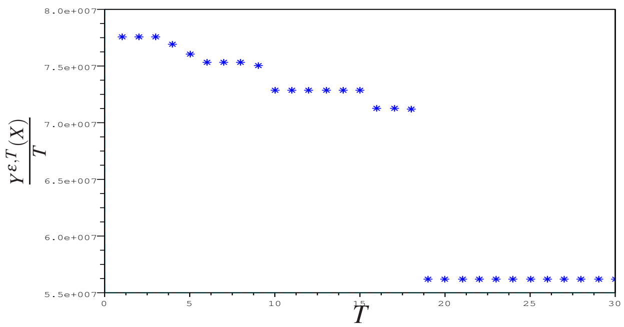

Fig. 1. Cost $Y^{\varepsilon,T}(X)/T$ with $T = 10$ versus the number of iterations t of the GA.

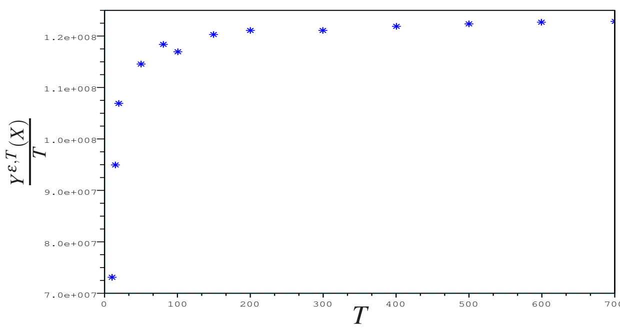

Fig. 2. $Y^{\varepsilon,T}(X)/T$ converging to the LRAC as the horizon T increases.

The weighting matrices are $R = 1$ and $Q = I$. We consider initial condition $X = I$, and we set $\varepsilon = 10^{-3}$. For the initialization of the algorithm, we employ the coupled Riccati equations (15) *(solved using the method presented in the Appendix), which yields*

$$L_1 \approx [2797\ \ 66.87\ \ -53.71];\ L_2 = [0\ \ 0\ \ 0].$$

We start with horizon length $T = 10$. Figure 1 shows the behaviour of the best average cost $Y^{\varepsilon,T}(X)/T$ as a function of the number of iterations (number of population) of the method of Table 1. Figure 2 shows how the average cost evolves as T increases, illustrating the convergence to the LRAC, $Z^{\varepsilon}(X)$. The obtained static gain K and the attained LRAC are given by

$$K \approx [2324\ \ 57.91\ \ -44.99];\ Z^{\varepsilon}(X) \approx 117569345.$$

VI. Conclusions

In this paper we have presented a genetic algorithm for the problems of finite horizon cost and of long run average cost, with static feedback controls, for discrete-time linear systems with Markov jump parameters and partial state observation. This is a relevant problem for applications, taking into account the simplicity of the controller (in the form $u_k = Kx_k$) and its implementation, and the fact that perfect observation of the jump variable θ may be difficult in many practical situations.

In order to overcome some intrinsic difficulties of the problem, we have proposed a modification on the original cost, parameterized by a scalar ε, and we show that the modified cost Z_K^{ε} converges to the original one when ε tends to zero, provided K is MS-stabilizing. We also shown that finite Z_K^{ε} is strongly connected with MS-stabilizing K, in such a manner that, when the method converges, the obtained solution is stabilizing.

References

[1] E. Cinlar. *Introduction to Stochastic Processes*. Prentice-Hall, 1975.

[2] E. F. Costa and J. B. R. do Val. Weak detectability and the linear quadratic control problem of discrete-time Markov jump linear systems. *International Journal of Control. Special issue on Switched and Polytopic Linear Systems*, 75:1282–1292, 2002.

[3] E. F. Costa, J. B. R. do Val, and M. D. Fragoso. A new approach to detectability of discrete-time infinite Markov jump linear systems. *SIAM Journal on Control and Optimization*, 43:2132–2156, 2005.

[4] E. F. Costa, A. L. P. Manfrim, and J. B. R. do Val. Weak controllability and weak stabilizability concepts for linear systems with Markov jump parameters. In *Proc. ACC'06 American Control Conference*, Minneapolis,USA, 2006.

[5] E. F. Costa, V. A. Oliveira, and J. B. Vargas. Digital implementation of a magnetic suspension control system for laboratory experiments. *IEEE Transactions on Automatic Control*, 42:315 – 322, 1999.

[6] O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markovian Jump Linear Systems*. Springer-Verlag, New York, 2005.

[7] O. L. V. Costa and M.D. Fragoso. Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems. *IEEE Transactions on Automatic Control*, AC-40:2076–2088, 1995.

[8] O. L. V. Costa and E. F. Tuesta. Finite horizon quadratic optimal control and a separation principle for Markovian jump linear systems. *IEEE Trans. Automat. Control*, 48, 2003.

[9] C. E. de Souza and D. F. Coutinho. Robust stability of a class of uncertain Markov jump nonlinear systems. *IEEE Transactions on Automatic Control*, 51(11):1825–1831, 2006.

[10] C. E. de Souza and M. D. Fragoso. H_∞ filtering for discrete-time linear systems with Markovian jumping parameters. *International Journal of Robust and Nonlinear Control*, 13:1299–1316, 2003.

[11] J. do Val and T. Basar. Receding horizon control of jump linear systems and a macroeconomic policy problem. *Journal of Economic Dynamics & Control*, 23:1099–1131, 1999.

[12] J. B. R. do Val and E. F. Costa. Stabilizability and positiveness of solutions of the jump linear quadratic problem and the coupled algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 50:691–695, 2005.

[13] M. D. Fragoso and O. L. V. Costa. A unified approach for stochastic and mean square stability of continuous-time linear systems with Markovian jumping parameters and additive disturbances. *SIAM Journal on Control and Optimization*, 44(4):1165–1191, 2005.

[14] Z. Gajic and R. Losada. Solution of the state-dependent noise optimal control problem in terms of Lyapunov iterations. *Automatica*, 35:951–954, 1999.

[15] G. P. H. Chou and C.-H. Chu. Genetic algorithms for communications network design-an empirical study of the factors that influence performance. volume 5, pages 612–619. IEEE Transactions on Evolutionary Computation, June 2001.

[16] Y. Ji and H. J. Chizeck. Jump linear quadratic Gaussian control: Steady-state solution and testable conditions. *Control Theory and Advanced Technology*, 6(3):289–319, 1990.

[17] J. Liu S. P. M. Choi and S. P. Chan. A genetic agent-based negotiation system. *Computer Networks*, 37(2):195–204, 2001.

[18] A. B. Simões and E. Costa. Transposition versus crossover: An empirical study. pages 612–619, Orlando, Florida, USA, July 1999. Proceedings of the Genetic and Evolutionary Computation Conference.
[19] A. N. Vargas, E. F. Costa, and J. B. R. do Val. Bounds for the finite horizon cost of Markov jump linear systems with additive noise and convergence for the long run average cost. In *Conference on Decision and Control 2006*, 2006.

Appendix

Consider the following coupled recursive Riccati equation, with initial condition $X_i^0 = Q_i$, $i \in \mathcal{S}$,

$$X_i^{k+1} = A_i'\mathcal{E}_i^k A_i - (A_i'\mathcal{E}_i^k A_i B_i)(R + B_i'\mathcal{E}_i^k B_i)^{-1}(B_i'\mathcal{E}_i^k A_i) + \mathcal{Q}_i \tag{14}$$

where $\mathcal{E}_i^k = \sum_{j=1}^S p_{ij} X_j^k$ is the coupling term. Consider the associated coupled algebraic Riccati equation in the variables $X_i = X_i \geq 0$, $i \in \mathcal{S}$,

$$X_i = A_i'\mathcal{E}_i A_i - (A_i'\mathcal{E}_i A_i B_i)(\mathcal{R} + B_i'\mathcal{E}_i B_i)^{-1}(B_i'\mathcal{E}_i A_i) + \mathcal{Q}_i \tag{15}$$

where $\mathcal{E}_i = \sum_{j=1}^S p_{ij} X_j$. (14) can be solved recursively. We present here, for ease of reference, the following algorithm [2], [4], which converges to the solution of (15) if and only if a solution exists.

Method for solving (15)

Step 1. Set $\kappa_i \leq 1$ as the largest integer for which $\sqrt{\kappa_i p_{ii}} A_i$ is stable.

Step 2. Set $X^0 = (X_1^0, ..., X_N^0) \in \mathcal{M}^{n+}$.

Step 3. For $k = 1, 2, ..$ and $i = 1, 2, ..., N$ solve the standard algebraic Riccati equations:

$$\begin{aligned} -X_i^k + \kappa_i p_{ii} A_i' X_i^k A_i + A_i'\tilde{\mathcal{E}}_i^k A_i - (\kappa_i p_{ii} A_i' X_i^k B_i + A_i'\tilde{\mathcal{E}}_i^k B_i) \\ \times (R + \kappa_i p_{ii} B_i' X_i^k B_i + B_i'\tilde{\mathcal{E}}_i^k B_i)^{-1} \\ \times (\kappa_i p_{ii} B_i' X_i^k A_i + B_i'\tilde{\mathcal{E}}_i^k A_i) + \mathcal{Q}_i = 0 \end{aligned} \tag{16}$$

where

$$\tilde{\mathcal{E}}_i^k = \sum_{j=1}^{i} p_{ij} X_j^k + (1 - \kappa_i) p_{ii} X_i^k.$$