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Control of many robots moving in the same direction with different speeds: a decoupling approach

David DeVon and Timothy Bretl

Abstract— **This paper considers the problem of controlling a group of microrobots that can move at different speeds but that must all move in the same direction. To simplify this problem, the movement direction is made a periodic function of time. Although the resulting control policy is suboptimal for an infinite-horizon quadratic cost, a bound is provided on how suboptimal it is. This bound is extended to show that, in theory, the design compromise making all robots move in the same direction only increases the expected cost by a factor of at most $\sqrt{2}$. Results are shown in simulation.**

I. INTRODUCTION

The past two decades have seen rapid progress in the development and deployment of microscale and nanoscale robotic systems [1]. These systems are intended for a wide range of applications that include microfabrication, minimally-invasive medical diagnosis and treatment, adaptive optics, regenerative electronics, and biosensing for environmental monitoring and toxin detection [2]. Some of these systems are mechanical, consisting for example of nickel nanowires that can be used to assemble scaffolding for opto-electronic devices [3]–[7]. Other systems are biological, consisting for example of magnetotactic bacteria that can be used as carriers for targeted therapy [8]–[12]. In either case, two aspects of microscale and nanoscale robotic systems present key control challenges. First, these systems involve hundreds or millions of robots, entire *ensembles* that have to be steered from one configuration to another. Second, these systems involve actuation mechanisms that apply global inputs to all robots at once, *programmable force fields* that are created by electromagnetic or acoustic fields, optical or chemical gradients, or fluid flow. Recent work has begun to address the resulting control challenges, both for ensembles that are homogenous [13], [14] and for those that are inhomogenous [15]–[20].

As a case study, in this paper we consider a particular microscale robotic system, the "Magmite," that was developed recently by researchers at the Institute of Robotics and Intelligent Systems (ETH-Zurich) [21], [22]. It is controlled by an external magnetic field: rotating the field rotates the robot, while oscillating the field drives the robot to resonance and propels it forward. Many robots can be driven at the same time, as long as each one has a different resonant frequency. *However, although it is possible for each robot to move at a different speed, all robots must move in the same direction.*

Our goal is to design a control policy for a group of these microrobots that achieves closed-loop stability and that minimizes an infinite-horizon quadratic cost function (Section III). It is clearly possible to move the robots from any initial configuration to any final configuration, for example, by moving them one at a time. But, because all robots must move in the same direction, the dynamic system is both nonlinear and coupled, so the cost function is not easily minimized.

To simplify this problem, we make the direction of movement a periodic function of time (Section IV). This choice both decouples and linearizes the dynamic system, allowing us to apply a standard linear quadratic control approach to minimize the expected cost and to prove stability for the closed-loop system (Section V). The resulting control policy is easy to implement and, although our choice of movement direction makes this policy suboptimal, we can give a bound on how suboptimal it is. We show the expected results of this approach in simulation (Section VI).

One interesting aspect of our approach is that it allows us to comment on the microrobot design. In particular, it was a design compromise to restrict the movement of all robots to the same direction while allowing different speeds [21], [22]. We can say exactly what the expected cost would be if this constraint were lifted. So by providing an upper bound on the cost function when the movement direction rotates at a fixed rate ω, we are in fact providing an upper bound on the cost of coupling in general. We show in Section V-D that, in theory, the expected cost for a group of these coupled microrobots is no higher than $\sqrt{2}$ times what the expected cost would be if they were not coupled. This exact bound holds only in the limit as $\omega \to \infty$, but convergence is rapid and it is possible to achieve a bound arbitrarily close to $\sqrt{2}$ even with small ω.

II. RELATED WORK

Recent work in areas as diverse as integrated circuit design, micro-electro-mechanical systems, structural dynamics [23], autonomous digital agent modeling for computer graphics [24], and microscale robotics, have led to a considerable interest in the analysis and synthesis of coupled dynamical systems. However, there are few general techniques, either analytical or numerical, for addressing these problems. Direct methods of numerical integration have been proposed, but convergence of these methods is not guaranteed and the time for deriving the numerical solution may be unacceptable [23]. An alternative approach uses feedback linearization to transform the coupled differential equations into sets of decoupled linear differential equations, for which a stabilizing time-varying control scheme can

D. DeVon and T. Bretl are with the Department of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL. devon@illinois.edu, tbretl@illinois.edu

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be implemented [25]. We take a similar approach in this paper, transforming a coupled nonlinear system into a linear periodic one.

The analysis and control of linear periodic systems, and of more general time-varying systems, has a rich history. In particular, Floquet theory is often used to analyze linear periodic systems [26]–[30]. Unlike time-invariant systems, the time-varying eigenvalues of the closed-loop system do not determine stability [26], [27]. Moreover, with time-varying systems, the use of a Lyapunov stability argument becomes more difficult as LaSalle's theorem for ordinary differential equations is not applicable [31]. So, one approach for controlling linear periodic systems is to convert them to a time-invariant form, either with a Lyapunov-Floquet transformation [26] or with feedback linearization [25], [32]. A linear periodic state feedback approach has been proposed for the placement of the closed-loop poles of an equivalent time-invariant system [33], [34]. More recently, integral quadratic constraints have been used for not only the analysis but also the control synthesis of linear time-varying systems [28], [35]. In addition, robust H_∞ methods have been proposed for the analysis and control of linear time-varying systems [36] and for state estimation of linear periodic systems [37]. We use an approach based on optimal linear quadratic control [38], [39], with closed-loop stability shown by a Lyapunov argument.

III. PROBLEM STATEMENT

Consider a group of n robots moving in a plane with the following dynamics:

$$\begin{aligned}\dot{x}_1(t) &= u(t)v_1(t)\\ &\vdots \\ \dot{x}_n(t) &= u(t)v_n(t).\end{aligned} \tag{1}$$

The position of each robot is $x_i(t) \in \mathbb{R}^2$ for $i = 1, \dots, n$. All robots must move in the same direction $u(t) \in \mathbb{R}^2$ where $\|u(t)\|_2 = 1$, but each can move at a different speed $v_i(t) \in \mathbb{R}$ for $i = 1, \dots, n$. We are given an initial configuration $x_1(t_0), \dots, x_n(t_0)$ and assume without loss of generality that the desired final configuration is $x_1^{\text{final}} = \cdots = x_n^{\text{final}} = 0$. Then, our goal is to select the inputs $u(t)$ and $v_1(t), \dots, v_n(t)$ that minimize the infinite-horizon quadratic cost function

$$J_\infty = \int_{t_0}^{\infty} \sum_{i=1}^{n} \left(x_i(t)^T Q x_i(t) + r v_i^2(t)\right) dt \tag{2}$$

where $Q > 0$ penalizes errors in position and $r > 0$ penalizes speed (or control effort).

IV. SOLUTION APPROACH

Because all robots must move in the same direction $u(t)$, the system (1) is both nonlinear and coupled, and so the quadratic cost (2) is not easily minimized. To simplify the problem, we make the movement direction $u(t)$ rotate at a fixed rate, defining

$$u(t) = \begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix}$$

for some $\omega > 0$. The advantage of this choice is that it both decouples and linearizes the dynamic system (1), which we can now write as

$$\begin{aligned}\dot{x}_1(t) &= \begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix} v_1(t)\\ &\vdots \\ \dot{x}_n(t) &= \begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix} v_n(t).\end{aligned}$$

In fact, we can now consider each robot separately, and our problem becomes minimizing the quadratic cost

$$J_\infty = \int_{t_0}^{\infty} \left[x(t)^T Q x(t) + r v^2(t)\right] dt \tag{3}$$

for the single linear-periodic dynamic system

$$\dot{x}(t) = \begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix} v(t). \tag{4}$$

As we will show, the resulting control policy $v(t)$ is simple and easy to implement. The only disadvantage of this approach is that it is not optimal—there are choices of $u(t)$ and $v_1(t), \dots, v_n(t)$ that will result in a smaller total cost (2). In particular, our approach may perform much worse relative to optimal as the frequency ω decreases, and in practice it is a good idea to keep ω as small as possible. Nonetheless, we will prove an explicit bound on the cost of our control policy in the following section.

V. CONTROL POLICY

A. Design of the controller

For the linear-periodic system (4), we will derive an optimal state feedback linear quadratic regulator that minimizes the cost function (3) and that is closed-loop stable. Linear quadratic optimal control is a standard technique that can be derived, for example, from a variational approach or dynamic programming [38], [39].

It is easy to show that the input $v(t)$ minimizing the quadratic cost (3) for the linear periodic system (4) has the form

$$v(t) = -\frac{1}{r}\begin{bmatrix}\cos(\omega t) & \sin(\omega t)\end{bmatrix} S(t)x(t) \tag{5}$$

where $S(t) \geq 0$ is a limiting, periodic solution of the matrix differential Riccati equation

$$\dot{S}(t) = -Q + S(t)\begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix}\frac{1}{r}\begin{bmatrix}\cos(\omega t) & \sin(\omega t)\end{bmatrix} S(t). \tag{6}$$

Let

$$Q = \begin{bmatrix} q & 0\\ 0 & q\end{bmatrix}$$

where $q > 0$ so position errors in any direction are penalized equally. Then we can show

$$S(t) = \begin{bmatrix} S_1(t) & S_2(t) \end{bmatrix}$$

is a solution to (6) where

$$S_1(t) = \begin{bmatrix} c + a\sin(2\omega t) + b\cos(2\omega t) \\ d + b\sin(2\omega t) - a\cos(2\omega t) \end{bmatrix}$$

and

$$S_2(t) = \begin{bmatrix} d + b\sin(2\omega t) - a\cos(2\omega t) \\ c - a\sin(2\omega t) - b\cos(2\omega t) \end{bmatrix}$$

by solving directly for the coefficients

$$\begin{aligned} a &= -\omega r + \sqrt{\omega^2 r^2 + qr} \\ b &= \frac{-a}{2\omega r}\sqrt{2qr - a^2} \\ c &= -b + \sqrt{2qr - a^2} \\ d &= 0. \end{aligned}$$

We can also show that

$$2qr - a^2 = 2rwa + qr$$

so, by inspection, we have $a > 0$, $b < 0$, and $c > 0$. The eigenvalues of $S(t)$ are given by

$$\lambda(S) = c \pm \sqrt{b^2 + a^2} > 0, \tag{7}$$

which are strictly positive and also invariant over time. Hence, the periodic Riccati solution $S(t)$ is positive definite, or in other words $x^T S(t) x > 0$ for all $x \in \mathbb{R}^2$ such that $x \neq 0$. We will use this property in the following section to show that the control policy is closed-loop stable.

Note that our control policy is easy to implement and that its complexity is constant as the number of microrobots increases, although the problem of measuring the state of each microrobot may get more difficult.

B. Closed-loop stability

Under the optimal control policy $v(t)$ given by (5), the closed-loop system is

$$\dot{x}(t) = A(t)x(t), \tag{8}$$

where

$$A(t) = -\frac{1}{r}\begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \end{bmatrix} S(t).$$

The stability of the closed-loop system can be demonstrated using a variety of different approaches, for example Floquet theory [26]–[30] or the method of integral quadratic constraints [35], [40]. For time-varying systems, the state transistion matrix characterizes the uniform asymptotic stability of the system, but requires solving the differential equation (8). For linear systems, uniform asymptotic stability is equivalent to exponential stability [27]. However, the time-varying eigenvalues of $A(t)$ do not determine stability of the system [26], [27]. Therefore, we use a Lyapunov approach to characterize the stability of the closed-loop system.

In general, we can only guarantee that the solution $S(t)$ of the periodic Riccati differential equation (6) is semi-definite and so could not use it as a Lyapunov function [41]. But in this case, as we showed in the previous section, the periodic Riccati solution $S(t)$ is indeed positive definite. This fact can be used to prove the following proposition:

Proposition 5.1: Under the control policy (5), the closed-loop system (8) is exponentially stable about the origin.

Proof: Consider the candidate Lyapunov function

$$V(t, x) = x^T S(t) x$$

which satisfies $V(t, x) > 0$ for all t and $x \neq 0$. The derivative along the trajectories is given by

$$\begin{aligned} \dot{V}(x,t) &= \dot{x}^T S(t) x + x^T \dot{S}(t) x + x^T S(t) \dot{x} \\ &= -x^T Q x - \frac{1}{r} x^T M x, \end{aligned}$$

where

$$M = S(t) \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \end{bmatrix} S(t).$$

Hence, since $S(t)$ is symmetric, we can write $M = N^T N$, where

$$N = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \end{bmatrix} S(t)$$

as in [42]. Therefore, we have

$$\dot{V}(x,t) = -x^T Q x - \|Nx\|_2^2 < 0$$

for all $x \neq 0$. Since $V(t, x)$ is radially unbounded, the system is globally exponentially stable about the origin [27], [30]. ■

C. Expected cost as the parameter ω varies

Under the optimal control policy $v(t)$, the expected cost is given by

$$J_\infty^*(\omega, x(t_0)) = x(t_0)^T S(t_0) x(t_0), \tag{9}$$

where $S(t_0)$ is an explicit function of ω as well as the control parameters q and r. Hence, we have both

$$\frac{\|J_\infty^*(\omega, x(t_0))\|_2}{\|x(t_0)\|_2} = \frac{\|x(t_0)^T S(t_0) x(t_0)\|_2}{\|x(t_0)\|_2}$$

and

$$\lambda_{\text{min}}(S) \leq \frac{\|x(t_0)^T S(t_0) x(t_0)\|_2}{\|x\|_2} \leq \lambda_{\text{max}}(S).$$

Since (7) tells us that

$$\lambda_{\text{min}}(S) = c - \sqrt{b^2 + a^2}$$

and

$$\lambda_{\text{max}}(S) = c + \sqrt{b^2 + a^2},$$

then we can bound the expected cost for any initial condition $x(t_0)$ as a function of ω by

$$\|x(t_0)\|_2 \left(c - \sqrt{b^2 + a^2}\right) \leq J_\infty^*(\omega, x(t_0))$$
$$J_\infty^*(\omega, x(t_0)) \leq \|x(t_0)\|_2 \left(c + \sqrt{b^2 + a^2}\right)$$

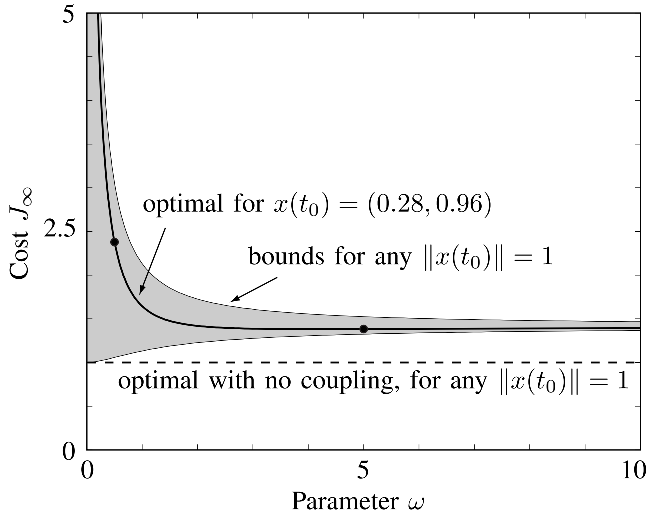

Fig. 1. The optimal cost as a function of ω. The cases $\omega = 0.5$ and $\omega = 5.0$, corresponding to Figs. 2-3, are also indicated.

Note that for fixed parameters q and r, this bound converges as ω increases, as shown in Fig. 1. In particular, as $\omega \to \infty$, the solution $S(t)$ converges to the constant matrix

$$S(t) \to \begin{bmatrix} \sqrt{2qr} & 0 \\ 0 & \sqrt{2qr} \end{bmatrix},$$

which can be determined directly from the coefficient equations. Similarly, the eigenvalues converge to $\lambda(S) = \sqrt{2qr}$ (with multiplicity 2). So in theory, increasing ω lowers the worst-case expected cost, but may cause problems in practice due to high-frequency changes in the movement direction (and consequent high-frequency changes in the commanded speed). For real systems, the choice of ω will be a balance between minimizing the expected cost and avoiding high-frequency noise. Fortunately, as shown in Fig. 1, the value of ω does not have to be large—choosing $\omega = 5$ (in other words, rotating the movement direction $u(t)$ at a frequency of slightly less than 1 Hz) already significantly lowers the worst-case cost. Furthermore, for the robotic system of interest that we described in Section I [21], [22], $\omega = 5$ is several orders of magnitude smaller than the resonant frequencies used to drive the microrobots.

D. Expected cost for uncoupled robots

We can also compute exactly what the expected cost would be if the group of microrobots could move in different directions as well as at different speeds. In this case, the dynamics of each robot would be given by

$$\dot{x}(t) = w(t)$$

where $w(t) \in \mathbb{R}^2$, resulting in the optimal control policy

$$w(t) = -\sqrt{\frac{q}{r}} x(t)$$

and the total cost

$$J_\infty^*(x(t_0)) = \sqrt{qr} \left(x(t_0)^T x(t_0) \right).$$

Fig. 1 compares this cost to the one we computed in the previous section, for any $x(t_0)$ such that $\|x(t_0)\| = 1$. In particular, as $\omega \to \infty$, we see that the expected cost for a group of coupled microrobots is no higher than $\sqrt{2}$ times what the expected cost would be if they were not coupled. So in theory, the design compromise that restricts the movement of all robots to the same direction does not significantly impact system performance.

VI. Results in Simulation

In this section we evaluate our control policy from the previous section in simulation. First, we compare the trajectory of a single robot moving to the origin for two different values of ω. Then, we show the trajectories of five robots moving simultaneously from distinct initial configurations. We use the parameters $q = r = 1$ in each case.

A. Robot trajectories for two different values of ω

First, consider a single robot starting with an arbitrary initial condition

$$x(0) = \begin{bmatrix} 0.28 \\ 0.96 \end{bmatrix}$$

at the initial time $t_0 = 0$, which satisfies $\|x(t_0)\| = 1$. We define the movement direction by

$$u(t) = \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}$$

where $\omega = 0.5$. The simulation is executed for 4π seconds, with the results shown in Fig. 2. The dashed circle signifies all initial conditions of unit distance from the origin (the desired final position). We show the robot's trajectory $x(t)$, along with its speed $v(t)$ (given by our control policy (5)) and movement direction $u(t)$. Since ω is small, the movement direction changes slowly, and the robot's trajectory deviates significantly from a straight line to the origin. (It is clear that if the system were uncoupled, the optimal trajectory would be this straight line.) The robot's speed converges to zero as it approaches the goal position. The time the robot arrives at the goal is slightly more than 2π seconds. The total cost, as plotted in Fig. 1, is slightly less than $J_\infty = 2.5$.

Similarly, Fig. 3 shows the results of the same simulation with a higher value of $\omega = 5.0$. Since the movement direction changes more rapidly, the resultant trajectory deviates less from the straight line to the origin. As shown in Fig. 1, the total cost in this case is approximately $J_\infty = \sqrt{2}$. By inspection of Figures 2 and 3, the increase in ω results in a lower cost (and shorter path), but requires an increase in speed changes corresponding to faster changes in movement direction. Therefore, we expect large ω may cause problems if used in practice. Fortunately, as stated in the previous section, $\omega = 5.0$ (slightly less than 1 Hz) is high enough to significantly reduce the worst-case total cost.

B. Five robots moving at the same time

Finally, we consider the case of $n = 5$ microrobots, where we have chosen $\omega = 1.0$. Recall that the system is decoupled by fixing the movement direction $u(t)$, so each robot was

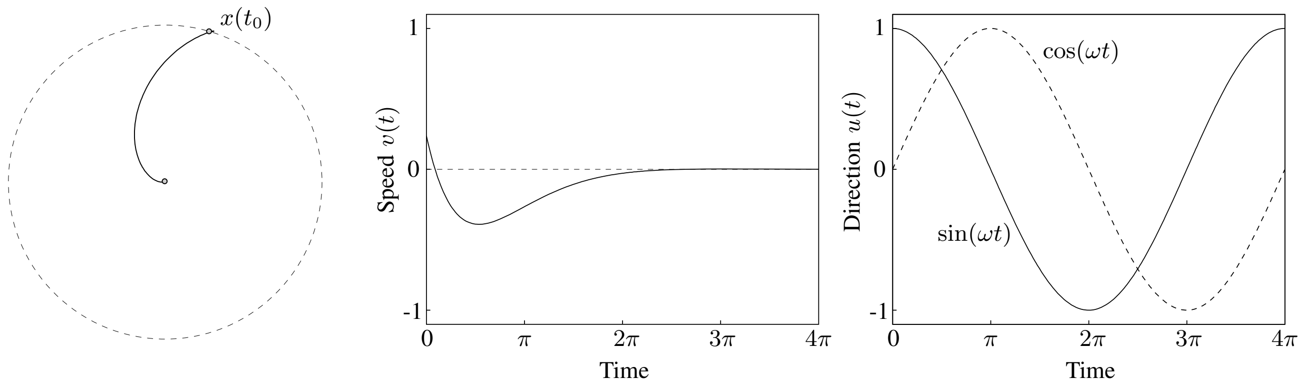

Fig. 2. The optimal path $x(t)$ and speed $v(t)$ given the direction $u(t)$ to reach the origin when $t_0 = 0$ and $q = r = 1$ for $\omega = 0.5$.

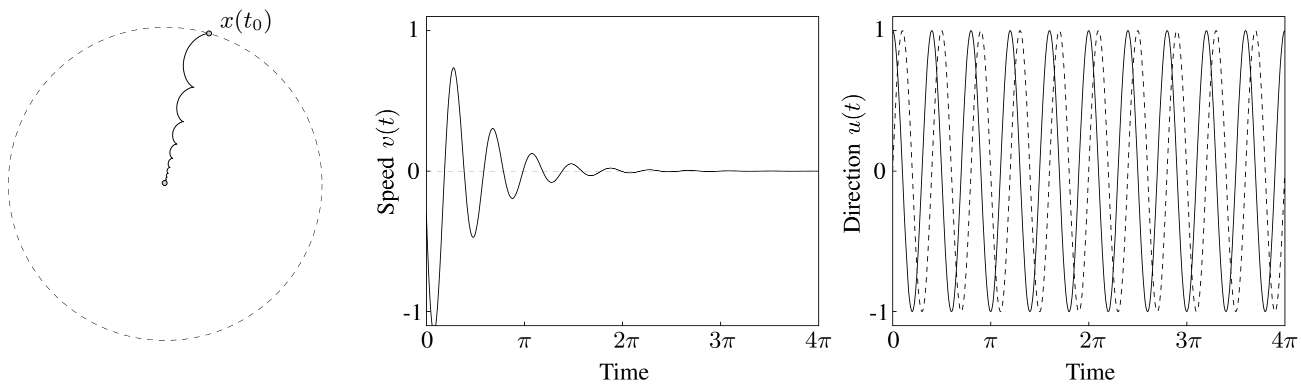

Fig. 3. The optimal path $x(t)$ and speed $v(t)$ given the direction $u(t)$ to reach the origin when $t_0 = 0$ and $q = r = 1$ for $\omega = 5.0$.

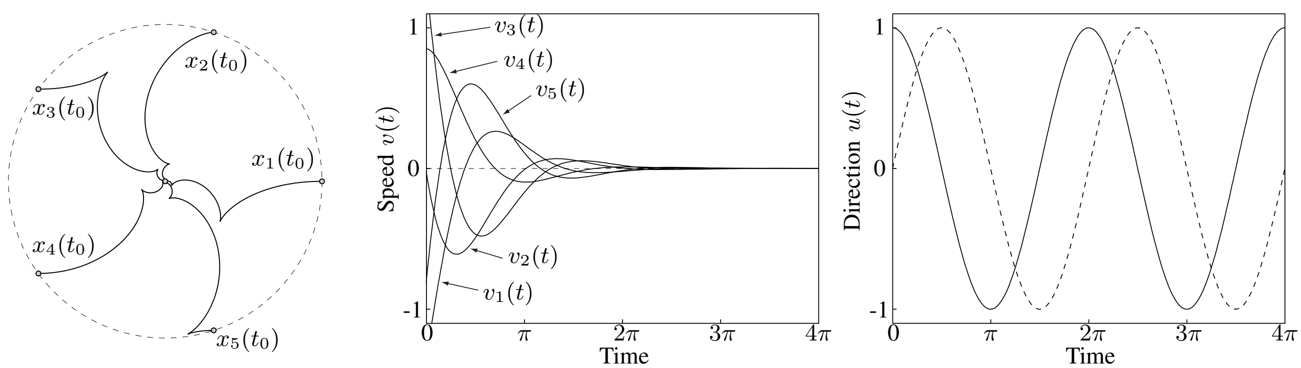

Fig. 4. The optimal path $x(t)$ and speed $v(t)$ given the direction $u(t)$ for five robots when $t_0 = 0$, $q = r = 1$, and $\omega = 1.0$.

considered completely independently. Hence, the number of robots is, in theory, irrelevant to system stability and performance. The results of the multi-robot simulation are given in Fig. 4. As with the previous simulations, each robot asymptotically moves to the corresponding final desired position (all in about the same time). All robots move in the same direction, while each robot's speed is used to control its trajectory and minimize the cost function. Similar results could be shown for any number of robots, with arbitrary initial and final goal positions.

VII. CONCLUSION

As a case study in the control of microscale and nanoscale robotic systems, this paper considered the problem of controlling a group of microrobots that can move at different speeds but that must all move in the same direction [21], [22]. Since all robots must move in the same direction, the dynamic system is both nonlinear and coupled. To simplify this problem, we chose to make the movement direction a periodic function of time. We showed that, although the resulting control policy is suboptimal for an infinite-horizon quadratic cost, we could provide a bound on how suboptimal it is. In particular, we showed that, in theory, the design compromise making all robots move in the same direction only increases the expected cost by a factor of at most $\sqrt{2}$.

Although the results in this paper were shown entirely in simulation, we are currently focusing on a hardware implementation. This implementation raises a number of issues that were not addressed here: (1) the effects of actuator and sensor noise as well as of model error; (2) the problem of state estimation; (3) collision avoidance by a higher-level motion planner; and (4) limits of performance as the number of microrobots grows large enough so that the effects of multiplexing can no longer be ignored.

Several other extensions are also of interest. For example, we assumed in this paper that the cost function was quadratic and, further, that the matrix Q penalizing state error was diagonal. If Q is not diagonal, then for a fixed ω we may not get the same bound on performance—with a time-varying rotation rate, it may be possible to recover this bound. We would also like to extend our work to time-optimal control, which may be more appropriate in practice.

VIII. ACKNOWLEDGMENTS

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