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Stabilization of Markovian Jump Linear Systems with Limited Information – A Convex Approach

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Abstract— **In this paper, we study the second order stabilization problem of Markovian jump linear systems (MJLSs) with logarithmically quantized state feedbacks. We give explicit constructions of the stabilizing logarithmic quantizer and controller. We also present a semi-convex way to determine the coarsest stabilizing quantization density. In addition, we show that the problem of stabilizing a linear time-invariant (LTI) system over a lossy channel can be viewed as a special example of the framework developed here. A contribution of the work is a simultaneous treatment of finite bandwidth constraints (logarithmic quantization) and latency in feedback channels.**

I. INTRODUCTION

Control over communication networks becomes more important as more complex and aggressive controls are implemented over networks. To design control systems under these conditions, we have to incorporate issues such as finite bandwidth, packet loss, and delays in a systematic way. Multi-modal systems provide a handy tool to model some of these issues.

MJLSs are convenient models for mathematically representing multi-modal stochastic systems, where the structure of the plant is subject to random changes. This is a commonly used abstraction for the hybrid automata, where the continuous state of the plant changes according to an underlying discrete-time stochastic process.

In this paper, we consider the quantized control problem of MJLSs; more specifically, we investigate the second order *mean square*[1] stabilization problem of a discrete-time MJLS subject to logarithmically quantized state feedbacks. We develop an explicit stabilizing *mode-dependent* logarithmic quantizer together with the associated controller. Necessary and sufficient conditions on the stabilizing quantization densities are given along with the semi-convex programming method to approach the coarsest stabilizing quantization density. In addition, it can also be shown (see [2]) that, in the special case where the system mode is independently identically distributed (i.i.d.), mean square stabilization is equivalent to *stochastic quadratic stabilization* studied in [3].

Second order stabilization of MJLSs have been intensively studied in [1], [4], and [5] and references therein, where Lyapunov type convex tests are developed.

Control over communication networks has received much attention in the control literature recently. Most research focuses either on the finite bandwidth issue, for example, [6], [7], [8], [9], and [10]; or on the unreliable transmission problem, for example, [11], [12], [13], [14], and [15]. Only a few very recent works start to address the more complicated combined problem (see [3], where the packet loss is restricted to being i.i.d. at each step). Among these works, the quadratic stabilization problem of an LTI system with logarithmically quantized state feedbacks over a *reliable* channel is investigated in [10]. In [11] and [16], estimation and control problems over an unreliable channel with *infinite* bandwidth are considered. Real numbers can be transmitted over the channel but are subject to random packet loss.

It is a formidable task to model the communication channel precisely. Beyond the single issues investigated previously in the literature, we consider both finite bandwidth[2] and packet loss constraints in this paper. Intuition says if packets get dropped more frequently, more information should be packed in each packet, and vice versa. Mathematically, we set up this example as a stabilization problem of a discrete-time LTI system with logarithmically quantized state feedbacks over a *lossy* channel, where the packet gets lost according to a Bernoulli process. This models the situation where measurements of the system are distorted not only by quantization due to the finite bandwidth but also by packet loss due to the unreliability of the communication channel.

We cast this problem into the general framework of stabilizing Markovian jump linear systems over bandwidth-limited communication channels developed later in this paper; and use methods developed there to illustrate the trade-off between the packet loss probability and the quantization density, which supports our above intuition. We are able to recover both results in [10] and [11] as extremal cases. This example also coincides with results in [3], where stochastic quadratic stabilization is examined via a different approach.

Main contributions of this work are that it (1) develops a general framework to stabilize MJLSs with quantized feedbacks; (2) provides a semi-convex algorithm to compute the coarsest quantization density; (3) incorporates the problem of quadratic stabilization of LTI systems and the problem of feedback control over unreliable channels into the MJLS framework; and (4) demonstrates the trade-off between packet loss and quantization; verifies the information needed to achieve stability has a lower bound.

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[1] Second-order stabilities are equivalent to each other for MJLSs, see [1].

[2] A logarithmically quantized signal still has *countably infinitely* many quantization levels (thus requires infinite bandwidth), compare to *infinitely* many levels of a real number. Proper truncation can then be introduced.

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II. PROBLEM SETUP

The sets of integers, non-negative integers, natural numbers, and real numbers are denoted by $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{R}$, respectively. Variables are represented by lower case letters, such as x. Matrices are denoted in upper case letters such as A, B, etc.

A. Markovian jump linear systems

Given a number $N \in \mathbb{N}$, let the matrix $\mathbf{P} \in \mathbb{R}^{N \times N}$ be a *row stochastic matrix*; that is, $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for $i, j \in \{1, \cdots, N\}$. Let the vector p be a *row stochastic vector* such that $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$.

Denote Θ as a random process with finite state space $\{1, \cdots, N\}$, the transition probability matrix $\mathbf{P}$, and the initial distribution p. Let $\theta(t), t \geq 0$ be a realization of Θ. Denote Ω as the sample space of all infinite sequences $\{\theta(t), t \geq 0\}$ and define $\mathbb{P}$ as the unique consistent measure on it that satisfies

$$\begin{aligned} &\mathbb{P}\{\theta(t+1) = j | \theta(t) = i\} = p_{ij} \\ &\mathbb{P}\{\theta(0) = i\} = p_i \end{aligned} \quad (1)$$

for all $t \geq 0$, and $i, j \in \{1, \cdots, N\}$. Then the process Θ is called a discrete-time Markovian process.

Define the following finite set:

$$\mathscr{S} = \{(A_1, B_1, C_1), \cdots, (A_N, B_N, C_N)\}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$, and $C_i = \mathbb{R}^{n_y \times n}$ for all $i \in \{1, \cdots, N\}$.

The discrete-time Markovian jump linear system defined by the tuple $(\mathscr{S}, \mathbf{P}, p)$ has the following state-space representation:

$$\begin{aligned} x(t+1) &= A_{\theta(t)} x(t) + B_{\theta(t)} u(t), \quad x(0) = x_0 \\ y(t) &= C_{\theta(t)} x(t) \end{aligned} \quad (2)$$

When $\theta(t) = i$, the system is said to be in *mode* i at time t.

We assume the initial state x_0 is a second-order random variable and is independent of $\theta(t)$ for all $t \geq 0$. We also assume all random variables are on the same probability space.

When $B_i \neq 0$ for some $i \in \{1, \cdots, N\}$, the MJLS $(\mathscr{S}, \mathbf{P}, p)$ is called a controlled Markovian jump linear system (C-MJLS).

Remark 1: In this paper, we restrict ourselves to the scalar-input system with state feedbacks; that is, $B_i \in \mathbb{R}^{n \times 1}$, and $C_i = I_n$, the n-dimensional identity matrix, for all $i = 1, \cdots, N$.

Given the following state feedback controller $\mathscr{H}$,

$$u(t) = H(t) x(t)$$

the closed-loop system $\Sigma(\mathscr{S}, \mathscr{H}, \mathbf{P}, p)$ is given by

$$x(t+1) = \left(A_{\theta(t)} + B_{\theta(t)} H(t)\right) x(t), \quad x(0) = x_0 \quad (3)$$

We define an *autonomous Markovian jump linear system* (A-MJLS) to be of the following form:

$$x(t+1) = A_{\theta(t)} x(t), \quad x(0) = x_0 \quad (4)$$

where the matrix A_i and the transition sequence θ are defined previously.

Definition 1: A transition sequence θ of an MJLS $(\mathscr{S}, \mathbf{P}, p)$ is *admissible* if it satisfies

1) $p_{\theta(0)} > 0$
2) $p_{\theta(t)\theta(t+1)} > 0$ for any $t \geq 0$

Any finite transition sequence $(\theta(t_0), \cdots, \theta(t))$ is admissible if θ is.

B. Logarithmic quantizers

Definition 2: A logarithmic quantizer with density $\rho \in (0, 1)$ is a function $\mathcal{Q} : \mathbb{R} \to \mathbb{Z} \times \{-1, 0, 1\}$, given by

1) If $x = 0$, $\mathcal{Q}(0) = (0, 0)$.
2) If $x \neq 0$, $\mathcal{Q}(x) = (n, \text{sgn}(x))$, when $\rho^{n+1} < |x| \leq \rho^n$, and $n \in \mathbb{Z}$.

When $\rho = 1$ or $\rho = 0$ we define $\mathcal{Q}(x) = x$ or $\mathcal{Q}(x) = 0$, respectively, to extend the above definition. The logarithmic quantizer defined here does not saturate. The second element of the output identifies the sign of x.

Definition 2 defines a one-dimensional logarithmic quantizer on the real line. We can extend this concept to the n-dimensional space $\mathbb{R}^n$ as well by defining a one-dimensional logarithmic quantizer in the following way: first choose a vector $H_q \in \mathbb{R}^n$; then, for all $x \in \mathbb{R}^n$, define $\mathcal{Q}(x) = \mathcal{Q}(H_q^T x)$. This is equivalent to quantizing the projection of $x \in \mathbb{R}^n$ on the one-dimensional subspace generated by the vector H_q, which is called the *quantization direction*.

It is also possible to design higher-order quantizers which quantize the projection of $x \in \mathbb{R}^n$ on a higher-order subspace. However, we are not interested in such quantizers in this paper since for a scalar-input system, a one-dimensional quantizer is sufficient and optimal, [10].

We now define the notion of an exponential decoder.

Definition 3: An exponential decoder with density $\rho \in (0, 1)$ is a function $\hat{x} : \mathbb{Z} \times \{-1, 0, 1\} \to \mathbb{R}$, which satisfies

1) $\hat{x}(0, 0) = 0$
2) $\hat{x}(n, \pm 1) = \mp \rho^n$, for $n \in \mathbb{Z}$

By combining a logarithmic quantizer $\mathcal{Q}$ with density ρ, an exponential decoder with the same ρ, and a constant gain β_u, we define the *log-controller* as follows:

Definition 4: A static log-controller with density $\rho \in (0, 1)$ and constant gain $\beta_u > 0$ is a function $u : \mathbb{R} \to \mathbb{R}$, such that

$$u(x) = -\text{sgn}(x) \beta_u \rho^{\lfloor \log_\rho |x| \rfloor}$$

where $\lfloor \cdot \rfloor$ is the standard floor function, rounding down to the closest smaller integer.

Figure 1 provides a schematic view of a log-controller.

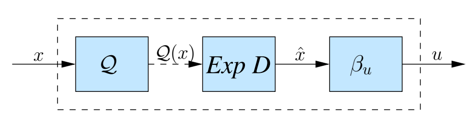

Fig. 1. A Log-controller

Remark 2: In later sections, we make the constant gain β_u a function of the quantization density ρ, in which case the parameter ρ completely specifies the log-controller.

III. QUADRATIC STABILIZATION OF SINGLE-MODE LINEAR SYSTEMS

The quadratic stabilization problem of a scalar-input LTI system with logarithmically quantized state feedbacks is studied in [10]. The coarsest quantization density is obtained by solving an optimization problem associated with an *algebraic Riccati equation* (ARE). In this section, we show that this can also be achieved by solving a *linear matrix inequality* (LMI) powered optimization problem. This is the cornerstone of the semi-convex optimization in later sections.

Consider the following discrete-time LTI system

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{5}$$

where $x \in \mathbb{R}^n$ is the state variable, and $u \in \mathbb{R}$ is the scalar control. Matrices A, B are of compatible dimensions and (A, B) is controllable. State feedback is assumed. Initial state $x_0 \in \mathbb{R}^n$ is assumed unknown but deterministic. This is a special C-MJLS where there is only a single mode, and the initial state is deterministic.

Definition 5: A dynamical system $x(t+1) = f(x(t))$ with the origin as the equilibrium point is quadratically stable if there exists a positive definite matrix P such that the quadratic function $V(x) = x^T Px$ is a valid Lyapunov function for the system; that is, for all $x \neq 0$, $V(x) > 0$, and

$$(\Delta V)(x) = V(f(x)) - V(x) < 0$$

Definition 6: Given a feedback control system $x(t+1) = f(x(t), u(t))$ with the origin as the equilibrium point, where u is the feedback control; a function $V(x)$ is a *control Lyapunov function* (CLF) for this system if and only if for all $x \neq 0$, $V(x) > 0$, and

$$\inf_u (\Delta V)(x, u) < 0$$

where $(\Delta V)(x, u) = V(f(x, u)) - V(x)$, and u is an admissible control.

A CLF is a Lyapunov function for the closed-loop system.

For a linear system, it is well known that stabilizability is equivalent to quadratic stabilizability; thus we have the following lemma on stabilizing system (5),

Lemma 1: Suppose system (5) is stabilizable. There exists a log-controller with density $\rho \in (0, 1)$ that quadratically stabilizes system (5) if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P - A^T PA + \lambda A^T PB(B^T PB)^{-1} B^T PA > 0 \tag{6}$$

where $0 < \lambda = \frac{4\rho}{(1+\rho)^2} < 1$.

Proof: ($\Longleftarrow$) Suppose there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that Riccati inequality (6) is satisfied. Then consider the following log-controller with density ρ

$$u = \begin{cases} -\beta_u \rho^\ell & \rho^{\ell+1} < Hx \le \rho^\ell \\ 0 & Hx = 0 \\ \beta_u \rho^\ell & -\rho^\ell \le Hx < -\rho^{\ell+1} \end{cases} \tag{7}$$

where $\beta_u = \frac{2\rho}{1+\rho}$, and $H = \frac{B^T PA}{B^T PB}$.

It is sufficient to show that the function $V(x) = x^T Px$ is a CLF for system (5) with controls defined in Equation (7). Clearly for all $x \neq 0$ and all $t \ge 0$, we have $V(x(t)) > 0$; therefore, we only need to show $(\Delta V)(x(t), u(t)) = V(x(t+1)) - V(x(t)) < 0$.

The difference $(\Delta V)(x(t), u(t))$ can be computed as

$$\begin{aligned} &(\Delta V)(x(t), u(t)) \\ &= x(t)^T (A^T PA - P)x(t) + 2x(t)^T A^T PBu(t) \\ &\quad + u(t)^T B^T PBu(t) \qquad (8) \\ &< \lambda x(t)^T A^T PB(B^T PB)^{-1} B^T PAx(t) \\ &\quad + 2x(t)^T A^T PBu(t) + u(t)^T B^T PBu(t) \qquad (9) \end{aligned}$$

where the last inequality comes from by adding and subtracting the term $\lambda x(t)^T A^T PB(B^T PB)^{-1} B^T PAx(t)$ to the right-hand side (RHS) of Equation (8) and then applying Riccati inequality (6).

Divide the RHS of Equation (9) by $B^T PB$ (which is a scalar due to the scalar input assumption) and let $q(t) = Hx(t) = \frac{B^T PA}{B^T PB} x(t)$; we now only need to show:

$$\lambda q(t)^2 + 2q(t)u(t) + u(t)^2 \le 0 \tag{10}$$

The left-hand side (LHS) of inequality (10) defines an upwards parabola and control values defined in Equation (7) make it non-positive for any $q(t)$, $t \ge 0$. This makes $V(x)$ a legitimate CLF with the log-controller (7). Therefore, system (5) is quadratically stabilizable.

($\Longrightarrow$) If system (5) is quadratically stabilizable, then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that $V(x) = x^T Px$ is a CLF when linear state feedback is allowed. From [10], we know that if we restrict the admissible controls to those generated by static log-controllers with quantization density $\rho > \frac{\gamma-1}{\gamma+1}$ with $\gamma = \sqrt{\frac{B^T PAQ^{-1} A^T PB}{B^T PB}}$ and $Q := P - A^T PA + A^T PB(B^T PB)^{-1} B^T PA > 0$, then the function $V(x) = x^T Px$ is still a valid CLF. Thus quadratic stability of the closed-loop system is preserved. The constraints on ρ is equivalent to Riccati inequality (6) by some arithmetic manipulations. See [10] for details. ■

Remark 3: Notice that ρ is strictly greater than $\frac{\gamma-1}{\gamma+1}$ since we need the function $V(x)$ to strictly decrease. The infimum itself is not achievable.

Lemma 2: Riccati inequality (6) is solvable for some positive definite matrix $P \in \mathbb{R}^{n\times n}$ with $0 < \lambda < 1$ if and only if the following linear matrix inequality is feasible for some positive definite matrix $Y \in \mathbb{R}^{n\times n}$ and some matrix $Z \in \mathbb{R}^{n_u \times n}$

$$\begin{bmatrix} Y & \sqrt{1-\lambda} YA^T & \sqrt{\lambda}(YA^T + Z^T B^T) \\ (\star) & Y & 0 \\ (\star) & 0 & Y \end{bmatrix} > 0 \tag{11}$$

where $(\star)$ denotes the conjugate transpose of the corresponding term.

This follows directly from [11, Theorem 5].

Now, by combining the previous two lemmas, we state the first result of this paper. It states that the existence problem of quadratically stabilizing log-controllers for the LTI system (5) is equivalent to the feasibility problem of the LMI (11).

Theorem 1: Suppose system (5) is stabilizable. There exists a log-controller with density $\rho \in (0, 1)$, that quadratically stabilizes system (5) if and only if there exist a positive definite matrix $Y \in \mathbb{R}^{n\times n}$ and a matrix $Z \in \mathbb{R}^{n_u\times n}$ such that the LMI (11) is feasible with $0 < \lambda = \frac{4\rho}{(1+\rho)^2} < 1$.

In order to find the coarsest quantization density, we can solve the following semi-convex optimization problem:

$$\begin{array}{c}\texttt{inf} \quad \rho \in (0,1) \\ \texttt{subject to} \quad \texttt{LMI (11)}\end{array}$$

The optimal value of ρ can then be approached via bi-section to any degree of desired precision. From [10], we know the closed-form solution of the infimum of the quadratically stabilizing quantization density is given by

$$\rho_{inf} = \frac{\prod \mathbf{eig}^u(A) - 1}{\prod \mathbf{eig}^u(A) + 1} \tag{12}$$

where $\mathbf{eig}^u(A)$ denotes all unstable eigenvalues of A.

Example 1: Figure 2 shows the infimum of the quadratically stabilizing quantization density for a scalar system

$$x(t+1) = Ax(t) + u(t)$$

where $1.1 \leq A \leq 5$. The circles represent the infimum solved via the LMI approach developed in this section (using SeDuMi LMI solver). The 45 degree line is given by the closed-form solution. Clearly, they coincide with each other.

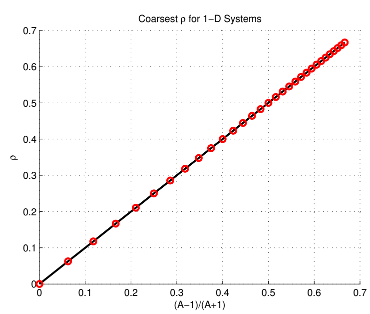

Fig. 2. Coarsest Quantization Density for 1-D Systems

IV. SECOND-ORDER STABILIZATION OF MARKOVIAN JUMP LINEAR SYSTEMS

In this section, we consider the mean square stabilization problem of MJLSs with a mode-dependent static log-controller (defined later). Necessary and sufficient conditions on the stabilizing quantization densities are given along with the semi-convex algorithm to approach the optimal value.

Definition 7: An A-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is *mean square stable* (MSS) if for any $\theta(0) \in \{1, \cdots, N\}$ and vector $x(0) \in \mathbb{R}^n$ we have

$$\mathbb{E}[||x(t)||^2 | x(0)] \to 0 \text{ as } t \to \infty$$

Definition 8: Given a set of matrices $H_i \in \mathbb{R}^{n_y\times n_u}$, where $i = 1, \cdots, N$, the zeroth-order time varying controller $\mathscr{H}$ defined by

$$H(t) = H_{\theta(t)}$$

is called a *mode-dependent static gain controller*.

The corresponding mean square stabilizability of a C-MJLS is defined as follows,

Definition 9: A C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is *mean square stabilizable* if there exists a mode-dependent controller $\mathscr{H}$ such that the closed-loop A-MJLS $\Sigma(\mathscr{S}, \mathscr{H}, \mathbf{P}, p)$ is mean square stable.

From [1] and [5], we have the following lemma on the mean square stabilizability of a C-MJLS,

Lemma 3: A C-MJLS is mean square stabilizable if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n_y}$, $i = 1, \cdots, N$, such that for all $i = 1, \cdots, N$, we have

$$(A_i - B_i H_i)^T \left(\sum_{j=1}^{N} p_{ij} P_j \right) (A_i - B_i H_i) - P_i < 0 \tag{13}$$

Clearly, if $B_i = 0$ for all $i = 1, \cdots, N$, Lemma 3 provides a Lyapunov test on the mean square stability of an A-MJLS.

We define the following stochastic version Lyapunov function for a switched dynamical system:

Definition 10: For a switched dynamical system $x(t+1) = f(x(t), \theta(t))$ with the origin as the equilibrium point, where $\theta \in \{1, \cdots, N\}$ is the mode of the system; A process $V(x, \theta)$ is a *stochastic Lyapunov function* (SLF) if it is a positive super-Martingale; that is, for all $x \neq 0$ and all $\theta \in \{1, \cdots, N\}$, we have $V(x, \theta) > 0$, and

$$(\Delta V)(x, \theta) = \mathbb{E}[V(f(x, \theta), \theta^+) - V(x, \theta) | x, \theta] < 0$$

where θ^+ is the system mode at the next step.

Notice that since the system is Markovian, instead of conditioning on the *filtration*,[3] we only need to condition on random variables of the current time.

Definition 11: For a switched feedback control system

$$x(t+1) = f(x(t), \theta(t), u(t))$$

with the origin as the equilibrium point, where $\theta \in \{1, \cdots, N\}$ is the mode of the system, and u is the feedback control; A process $V(x, \theta)$ is a *stochastic control Lyapunov function* (SCLF) if for all $x \neq 0$ and $\theta \in \{1, \cdots, N\}$, we have $V(x, \theta) > 0$, and

$$\inf_u (\Delta V)(x, \theta, u) < 0$$

where $(\Delta V)(x, \theta, u) = \mathbb{E}[V(f(x, \theta, u), \theta^+) - V(x, \theta) | x, \theta]$, θ^+ is as defined previously; and u is an admissible control.

In short, it is an SLF for the closed-loop system.

The following lemma connects the mean square stabilizability of a C-MJLS to the existence of an SCLF:

Lemma 4: A C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is mean square stabilizable if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1, \cdots, N$ such that the follwoing quadratic process (14) is an SCLF for this C-MJLS

$$V(x(t), \theta(t)) = x(t)^T P_{\theta(t)} x(t) \tag{14}$$

where $P_{\theta(t)} = P_i$ when $\theta(t) = i$.

Proof: ($\Longrightarrow$) From Lemma 3, we know that if the system is mean square stabilizable, then there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n_y}$, $i = 1, \cdots, N$, such that LMIs (13) are satisfied. For all $x \neq 0, \theta \in \{1, \cdots, N\}$, and $t \geq 0$, the process $V(x(t), \theta(t)) > 0$. We only need to show it is a super-Martingale; that is,

$$\mathbb{E}[V(x(t+1), \theta(t+1)) - V(x(t), \theta(t)) | x(t), \theta(t)] < 0 \tag{15}$$

[3] Simply put, the union of algebras generated by random variables up to the current time.

By using P_i and H_i given in Lemma 3, we can check the point-wise property,

$$\begin{aligned}&\mathbb{E}[V(x(t+1),\theta(t+1)) - V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i]\\&= x_t^T\left\{(A_i - B_iH_i)^T\mathbb{E}[P_{\theta(t+1)}|\theta(t)=i](A_i - B_iH_i) - P_i\right\}x_t\\&= x_t^T(\text{LHS of (13)})x_t < 0\end{aligned}$$

Therefore, Equation (15) is valid, which means the process (14) is a legitimate SCLF.

($\Longleftarrow$) If there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$ such that the process (14) is an SCLF, then from the definition of SCLF, we know

$$\begin{aligned}&\mathbb{E}\left[(A_{\theta(t)}x(t) + B_{\theta(t)}u(t))^TP_{\theta(t+1)}(A_{\theta(t)}x(t) + B_{\theta(t)}u(t))\right.\\&\left.- x(t)^TP_{\theta(t)}x(t)|x(t),\theta(t)\right] < 0 \qquad (16)\end{aligned}$$

Given $x(t) = x_t$ and $\theta(t) = i$, the control that makes the difference $\Delta V(x,\theta)$ most negative is $u_i(t) = -H_ix_t$ with $H_i = \frac{B_i^T\tilde{P}_iA_i}{B_i^T\tilde{P}_iB_i}$ and $\tilde{P}_i = \sum_{j=1}^N p_{ij}P_j$.

It is now straightforward to verify that state feedbacks H_i together with matrices P_i satisfy Equation (13). Thus the system is mean square stabilizable. ■

We denote $H_i = \frac{B_i^T\tilde{P}_iA_i}{B_i^T\tilde{P}_iB_i}$ with $\tilde{P}_i = \sum_{j=1}^N p_{ij}P_j$ as the *average gradient descendent direction* for mode i.

Similar to the mode-dependent controller, we define the *mode-dependent* logarithmic quantizer as follows,

Definition 12: Let $\mathcal{Q}_1,\cdots,\mathcal{Q}_N$ be a set of logarithmic quantizers; then the time varying quantizer $\mathscr{Q}$ defined by

$$\mathcal{Q}(t) = \mathcal{Q}_{\theta(t)}$$

is called a *mode-dependent logarithmic quantizer*.

Notice that for a mode-dependent quantizer, both the quantization density and the quantization direction can change from mode to mode.

Similarly, we define the *mode-dependent static log-controller (with density ρ_i and gain β_i)* as the combination of the mode-dependent quantizer with density ρ_i, the obviously defined exponential decoder, and the constant gain β_i.

Now consider the stabilization problem of a C-MJLS with a mode-dependent static log-controller. Suppose the system is mean square stabilizable with a linear feedback mode-dependent static gain controller; then we have the following lemma on stabilizing the C-MJLS subject to logarithmically quantized state feedbacks.

Lemma 5: Suppose an N-mode C-MJLS $(\mathscr{S},\mathbf{P},p)$ is mean square stabilizable. There exists a mode-dependent log-controller with density $\rho_i \in (0,1)$, $i = 1,\cdots,N$, that mean square stabilizes the C-MJLS if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$ such that

$$P_i - A_i^T\tilde{P}_iA_i + \lambda_iA_i^T\tilde{P}_iB_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i > 0 \quad (17)$$

where $0 < \lambda_i = \frac{4\rho_i}{(1+\rho_i)^2} < 1$ and $\tilde{P}_i = \sum_{j=1}^N p_{ij}P_j$ for all $i = 1,\cdots,N$.

Proof: The proof is an extension of the single-mode case. The SCLF introduced in Definition 11 plays a significant role.

($\Longleftarrow$) Suppose Riccati inequalities (17) are solvable. Define the following mode-dependent log-controller with density $\rho_i \in (0,1)$ and constant gain $\beta_i = \frac{2\rho_i}{1+\rho_i}$

$$u_i(x) = \begin{cases} -\beta_i\rho_i^\ell & \rho_i^{\ell+1} < H_ix \le \rho_i^\ell \\ 0 & H_ix = 0 \\ \beta_i\rho_i^\ell & -\rho_i^\ell \le H_ix < -\rho_i^{\ell+1}\end{cases} \qquad (18)$$

where $H_i = \frac{B_i^T\tilde{P}_iA_i}{B_i^T\tilde{P}_iB_i}$ is the average gradient descendant direction for mode i.

Consider the stochastic process defined by Equation (14). It is obvious that $V(x(t),\theta(t)) > 0$ for all $x(t) \ne 0$, $\theta(t) \in \{1,\cdots,N\}$, and $t \ge 0$. We only need to show

$$\mathbb{E}[V(x(t+1),\theta(t+1)) - V(x(t),\theta(t))|x(t),\theta(t)] < 0$$

By using same techniques as in the proof of Lemma 1, it is easy to verify that

$$\mathbb{E}[V(x(t+1),\theta(t+1)) - V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i] < 0$$

Thus, $V(x(t),\theta(t))$ is a valid SCLF; and therefore the C-MJLS is mean square stabilizable with log-controller (18).

($\Longrightarrow$) Now assume the C-MJLS is stabilizable in the mean square sense by a mode-dependent linear feedback controller $\mathscr{H}$, then by Lemma 4 there exists a legitimate SCLF $V(x(t),\theta(t)) = x(t)^TP_{\theta(t)}x(t)$ such that

$$\begin{aligned}&\mathbb{E}[V(x(t+1),\theta(t+1)) - V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i]\\&= x_t^T(A_i^T\tilde{P}_iA_i - P_i)x_t + 2x_t^TA_i^T\tilde{P}_iB_iu(t) + u(t)^TB_i^T\tilde{P}_iB_iu(t)\\&< 0 \qquad (19)\end{aligned}$$

The only difference here from the single-mode case is instead of $\tilde{P}_i$, we have P_i in the first term of Equation (19). However, this does not affect the choice of the optimal control u_i. We can again follow the proof of Lemma 1 to reach Riccati inequalities (17). See [2] for details. ■

We now show that the feasibility of the coupled Riccati inequalities (17) can be converted to an equivalent convex problem.

Given an MJLS $(\mathscr{S},\mathbf{P},p)$ and positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$ with the following relationship:

$$\mathbb{P}[P_{\theta(t+1)} = P_j|\theta(t)=i] = p_{ij} \qquad (20)$$

$$\mathbb{P}[P_{\theta(0)} = P_i] = p_i \qquad (21)$$

We introduce the following auxiliary functions for convenience:

$$g_i(\mathbf{P},P_1,\cdots,P_N) = A_i^T\tilde{P}_iA_i - \lambda_iA_i^T\tilde{P}B_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i$$

$$\phi_i(\mathbf{P},H_i,P_1,\cdots,P_N) = (1-\lambda_i)A_i^T\tilde{P}_iA_i + \lambda_iF_i^T\tilde{P}_iF_i$$

where $\lambda_i \in (0,1)$ are constants, matrices $F_i = A_i - B_iH_i$ for some $H_i \in \mathbb{R}^{n_u\times n}$, and $\tilde{P}_i := \sum_{j=1}^N p_{ij}P_j$ for $i = 1,\cdots,N$.

Lemma 6: There exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, such that $P_i > g_i$ is feasible for all $i = 1,\cdots,N$ if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n}$, such that $P_i > \phi_i$ is feasible for all $i = 1,\cdots,N$.

The proof is an extension of that of [11, Theorem 1]. Interested readers can refer to [2] for details.

Using this result, we are able to convert Riccati inequalities (17) into a set of computable LMIs.

Lemma 7: There exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1, \cdots, N$, such that the coupled Riccati inequalities (17) are satisfied if and only if the following coupled LMIs are feasible for some positive definite matrices $Y_i \in \mathbb{R}^{n\times n}$, and matrices $Z_i \in \mathbb{R}^{n_u\times n}$, $i = 1, \cdots, N$,

$$\begin{bmatrix} Y_i & \mathscr{L}_i^1 & \cdots & \mathscr{L}_i^N & \mathscr{M}_i^1 & \cdots & \mathscr{M}_i^N \\ (\star) & Y_1 & & & & & \\ \vdots & & \ddots & & & & \\ (\star) & & & Y_N & & & \\ (\star) & & & & Y_1 & & \\ \vdots & & & & & \ddots & \\ (\star) & & & & & & Y_N \end{bmatrix} > 0 \qquad (22)$$

where for $i, j = 1, \cdots, N$,

$$\mathscr{L}_i^j = \sqrt{(1-\lambda_i)p_{ij}}Y_i A_i^T$$
$$\mathscr{M}_i^j = \sqrt{\lambda_i p_{ij}}(Y_i A_i^T + Z_i^T B_i^T)$$

Notation $(\star)$ denotes the conjugate transpose of the corresponding term.

Proof: The coupled Riccati inequalities (17) are satisfied for some positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1, \cdots, N$ is equivalent to $P_i > g_i(\mathbf{P}, P_1, \cdots, P_N)$ with g_i defined in Lemma 6. From the same lemma, we know this is equivalent to the existence of positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n}$, $i = 1, \cdots, N$ such that

$$\begin{aligned} P_i &> \phi_i(\mathbf{P}, H_i, P_1, \cdots, P_N) &(23)\\ &= (1-\lambda_i)A_i^T \tilde{P}_i A_i + \lambda_i F_i^T \tilde{P}_i F_i \\ &= \sum_{j=1}^N \left\{(1-\lambda_i)p_{ij}A_i^T P_j A_i + \lambda_i F_i^T P_j F_i\right\} &(24) \end{aligned}$$

Apply Schur complement to the last inequality; then pre- and post-multiply $\begin{bmatrix} P_i^{-1} & \\ & I \end{bmatrix}^T$ and its conjugate transpose. Then define $Y_i = P_i^{-1}$ and $Z_i = Y_i H_i^T$, the last inequality is feasible if and only if LMIs (22) are feasible. ■

Lemmas 5 and 7 can now be combined to give a complete convex solution to determine the mean square stabilizability of a C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ with logarithmically quantized state feedbacks.

Theorem 2: Suppose an N-mode C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is mean square stabilizable. There exists a mode-dependent log-controller with density $\rho_i \in (0, 1)$, $i = 1, \cdots, N$, that mean square stabilizes the C-MJLS if and only if there exist positive definite matrices $Y_i \in \mathbb{R}^{n\times n}$, and matrices $Z_i \in \mathbb{R}^{n_u\times n}$, $i = 1, \cdots, N$, such that the coupled LMIs (22) are feasible with $0 < \lambda_i = \frac{4\rho_i}{(1+\rho_i)^2} < 1$.

In general, the coarsest quantizer may not exist for the MJLS due to the lack of a proper definition of *total ordering* for sets $(\rho_1, \cdots, \rho_N)$. However, one reasonable approach is to solve the following *min-max* problem, which gives the infimum of the upper bound of the quantization density of stabilizing mode-dependent quantizers.

$$\texttt{inf} \quad \rho \in (0, 1)$$
$$\texttt{subject to} \quad \rho_i \le \rho \quad \texttt{and} \quad \texttt{LMI (22)}$$

This is a semi-convex problem. The optimal value can be found to any desired degree of accuracy via bi-section.

The stabilizing mode-dependent log-controller can be designed as follows:

1) Choose any $\rho \in (\rho_{inf}, 1)$.
2) Solve the coupled LMIs (22) for Y_i. Let $P_i = Y_i^{-1}$.
3) The quantization direction is given by $H_i = \frac{B_i^T \tilde{P}_i A_i}{B_i^T \tilde{P}_i B_i}$.
4) The log-controller can be then designed according to the definition in Section II-B.

Remark 4: Of course, we can choose other optimization objective functions; for example, we can minimize the weighted average quantization density $\sum p_i \rho_i$ if the system mode is i.i.d.; this is in some sense trying to minimize the average bandwidth requirement.

V. EXAMPLE: CONTROL OVER BANDWIDTH LIMITED UNRELIABLE CHANNELS

Now, let us go back to our motivating example: control of an LTI system with logarithmically quantized state feedbacks transmitted over an unreliable communication channel, which is depicted in Figure 3.

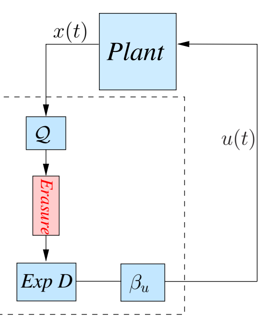

Fig. 3. An LTI System over an Unreliable Channel with Logarithmically Quantized State Feedbacks

The system is a discrete-time LTI system with scalar input; it has the following state space representation:

$$x(t+1) = Ax(t) + Bu(t) \qquad (25)$$

The state feedback is first quantized by a logarithmic quantizer with density ρ and then transmitted over a lossy channel. The packet loss is modeled as a Bernoulli process with dropping probability $0 \le \alpha \le 1$. Other types of Markovian processes can be modeled similarly.

It is clear that this problem can be modeled as a C-MJLS with logarithmically quantized state feedback, where the system mode is i.i.d. with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{bmatrix} \qquad (26)$$

Suppose system (25) is stabilizable when measurements can be transmitted to the controller with no distortion; Theorem 2 can be applied directly to solve the second order stabilization problem when logarithmically quantized feedbacks are transmitted over a lossy communication channel.

Remark 5: A special feature of this packet dropping problem is that when the packet is lost, the direction and density of the quantizer become irrelevant, since the control value is set to 0 then. Therefore, we only need to implement the logarithmic quantizer designed for the case when the packet is delivered successfully, and there is no need to switch depending on the system mode. Then this *time-invariant* static logarithmic quantizer is sufficient for stabilization purposes. In other words, the min-max optimization problem in Section IV degenerates to a minimizing problem.

In order to find the trade-off between the quantization density and the packet dropping probability, we solve the following semi-convex problem for a fixed packet dropping probability α via bi-section,

$$\begin{array}{c}\texttt{inf } \rho \in (0,1) \\ \texttt{subject to LMI } (22) \texttt{ and } \mathbf{P} \text{ in } (26)\end{array}$$

We can then grid the dropping probability α from 0 to 1.

We can still adopt the controller design method as in Section IV. However, under this special i.i.d. setup, following the same logic as in Remark 5, we can simply use the controller designed for mode 2 to handle both cases.

The following numerical examples illustrate the trade-off between the packet dropping probability α and the quantization density ρ.

Example 2: Consider the following 2-D systems with state space representation given in Equation (25)

$$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 4 \end{bmatrix}, A_3 = \begin{bmatrix} 4 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, A_4 = \begin{bmatrix} \frac{1}{4} & 1 \\ 0 & 4 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, A_6 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The region above the curve in Figure 4 is second-order stable.

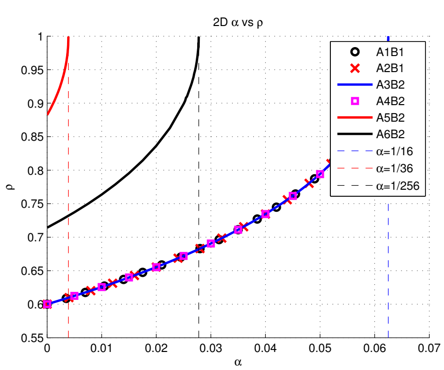

Fig. 4. Trade-off between α and ρ for 2-D Systems

From the figure, we observe that:

1) A *minimum* amount of information is required to stabilize the system; thus, if the packet dropping probability is high, one needs to include more information in each packet by quantizing finer, and vice versa. This observation supports our intuition.

2) It is clear that unstable eigenvalues are the only decisive factor on the trade-off between the packet dropping probability and the quantization density. The value of B does not matter as long as (A, B) is controllable.

3) When $\alpha = 0$, we get the result in [10]; whereas when ρ explodes (basically no quantization), results in [11] are recovered. Thus our framework solves both problems as special cases. Furthermore, these examples also coincide with results in [3]. This supports our proof that for switched systems with i.i.d. modes, mean square stability is equivalent to stochastic quadratic stability [2].

VI. Conclusions

In this paper, we investigated the quantized second order stabilization problem of MJLSs. We provided explicit constructions of stabilizing mode-dependent logarithmic quantizers and controllers. The coarsest quantization density is approached via a semi-convex algorithm. In addition, by using tools developed here, we show that the quadratic stabilization problem of an LTI system and the stabilization problem over bandwidth-limited unreliable channels can be solved as special cases. One of our main contributions is to provide a general framework to integrate several issues on communication networks of interest to control engineers.

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