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Geometries of Single-Input Locally Accessible Control Systems

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Abstract— **The description of a nonlinear control system as an exterior differential system suggests an interesting connection between locally accessible control systems and the existence of a natural geometry associated to the control system. The classification of single-input locally accessible control systems leads to a canonical construction of a (pseudo-) Riemannian metric on the state space. Conditions for the existence of such a metric are derived and the construction is illustrated in a simple example.**

I. INTRODUCTION

In this paper, we explore different equivalence classes for local accessible systems using exterior differential calculus. The tools used are taken from geometric control theory, which is a well-established field. One area of geometric control is the study of control systems whose state manifolds are equipped with additional structure which describes inherent properties of these classes of systems, e.g., mechanical systems [1] or port-Hamiltonian systems [16]. The additional structures are used for efficient controller synthesis for nonlinear systems and have found a wide range of applications. Another area of geometric control research focuses on determining if a nonlinear system is equivalent to a linear system, and if so, how to obtain a linear controllable description through a change of coordinates and feedback [9]. This approach has been extended to include a state-dependent time scaling such that the system described in the new time scale has a linear controllable representation, this is known as orbital feedback linearization [15], [7]. Orbital feedback linearization has been applied successfully in control problems where a state of the system can be identified with the natural time of the system. Examples of this include batch cooling crystallization processes [17] and spray dryer plants [8], and it has been used as a design methodology for tracking controller in car parking problems [11].

The present work is motivated by the following question: In what cases can we canonically construct a geometric structure on the state manifold determined by the control system? We answer this question for the special case of single-input control-affine systems under some integrability assumption. We develop a normal form in the original time scale for the class of orbital feedback linearizable systems. This normal form consists of a two-dimensional control system and a chain of $n-2$ integrators. The classification of the system can be reduced to the classification of the two dimensional subsystem. The classification of single-input planar control systems by some normal forms allows us to give conditions for the existence of a (pseudo-) Riemannian metric on the state space. We conjecture that it is possible to extend this result to the case of m controls since the tools used in the development rely on a generalized version of the Brunovsky normal form. In an example we consider one possible use of the Riemannian metric to design a path following controller for a simple car model. We consider single-input control-affine systems

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and f and g are assumed to be sufficiently smooth.

The paper is structured as follows. In Section II, we introduce some basic notation for geometric control theory and state some basic results. In Section III we develop the contribution of this paper in three stages. First, we find a normal form for orbital feedback linearizable systems, then we analyze a two-dimensional subsystem of the control system, and finally show that the classification found for the subsystem gives a classification of the original system. Next, we consider the case in which a (pseudo-) Riemannian metric can be canonically constructed. We compute this metric for a simple example in Section IV and design a path following controller based on this metric. In the last section we give a short conclusion and future directions for development.

II. MATHEMATICAL PRELIMINARIES

In this section, we introduce notation and state some basic results to develop the necessary setup for the remainder of the paper. We give a short introduction to exterior differential systems, Pfaffian systems, the method of equivalence and their applications in control theory. References will be given for each subject.

A. Exterior Differential Systems

Let M be a smooth manifold. Denote by $\Omega^k(M)$ the space of differential k-forms on M, where $\Omega^0(M) = \mathcal{C}^\infty(M)$ is the space of smooth functions on M. Then $\Omega(M) = \bigoplus_{k=1}^n \Omega^k(M)$ denotes the space of all differential forms on M. There are two operations defined on $\Omega^k(M)$ for all k, an associative but non-commutative product called the *wedge product* of two differential forms $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ and the map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k called the *exterior derivative*.

The space $\Omega(M)$ also forms a graded algebra under the wedge product. A subspace $\mathcal{I} \subset \Omega(M)$ is an *algebraic ideal* if it is a direct sum of homogeneous subspaces $\mathcal{I}^k \subset \Omega^k(M)$ and is closed under wedge product with arbitrary

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differential forms. An algebraic ideal is a *differential ideal* if it is closed under exterior differentiation, $d\mathcal{I} \subseteq \mathcal{I}$. A Pfaffian system I is a submodule of one-forms over $\mathcal{C}^\infty(M)$ generated by a set of one-forms $\{\omega_1, \ldots, \omega_k\}$, *i.e.* , $I = \{\sum f_k \omega_k \mid f_k \in \mathcal{C}^\infty(M)\}$. A Pfaffian system I generates a differential ideal $\mathcal{I} = \langle\{dI, I\}\rangle$ of differential one-forms on M, where $\langle\cdot\rangle$ denotes the $\mathbb{R}$-linear span of elements in $\Omega^1(M)$. Furthermore, a differential form dt determined up to multiplication by an arbitrary function is called an independence condition for $\mathcal{I}$, if dt is nowhere vanishing on any integral manifold of $\mathcal{I}$.

A solution of an exterior differential system is a smooth curve $c : [a, b] \subset \mathbb{R} \to M \times \mathbb{R}$ such that $\omega_{c(s)}(c'(s)) = 0$ for all $\omega \in \mathcal{I}$ and all $s \in [a, b]$.

For of exterior differential systems, the well-known Frobenius theorem takes the following form.

Proposition 2.1 ([10]): A Pfaffian system I is completely integrable if and only if $d\alpha_i \equiv 0 \mod \mathcal{I}, \quad \forall \alpha_i \in \mathcal{I}$.

Let I be a Pfaffian system on M. A submanifold N of M is called an integral manifold of $\mathcal{I}$ if $T_x N \subset \langle \mathcal{I}(x)\rangle^\perp$ for every $x \in N$. A curve smooth $c : [a, b] \subset \mathbb{R} \to M$ is called an integral curve of $\mathcal{I}$ if $c'(t) \in \langle \mathcal{I}(c(t))\rangle^\perp$ for every $t \in [a, b]$.

Given a Pfaffian system I, one might ask: What is the smallest integrable subsystem contained in $\mathcal{I}$? This leads to the definition of the derived flag. Set $\mathcal{I}^{(0)} = \mathcal{I}$ and

$$\mathcal{I}^{(i)} = \{\omega \in \mathcal{I}^{(i-1)} \mid d\omega \equiv 0 \mod \mathcal{I}^{(i-1)}\}$$

This definition gives rise to a filtration

$$\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots \supset \mathcal{I}^{(i)} \supset \cdots$$

called the *derived flag*. The filtration stabilizes for some integer N, *i.e.* , $\mathcal{I}^{(N)} = \mathcal{I}^{(N+1)}$ called the *derived length*, the system $\mathcal{I}^{(N)}$ is then called the *bottom derived system*. A derived system is called trivial if it only contains the null element. The derived length is an integer invariant associated to the derived flag. Another set of integer invariants are the Kronecker indices $\kappa_1, \ldots, \kappa_n$ defined by $\kappa_i = \dim(\mathcal{I}^{(i)}/\mathcal{I}^{(i+1)})$.

For further information on the subject of exterior differential systems the reader is refered to [3].

B. Local Accessibility

We turn our attention to control systems. For ease of notation, we consider a control system in the form $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Consider the Pfaffian system

$$\begin{aligned} I = \{&dx_1 - f^1(x, u)dt,\\ &dx_2 - f^2(x, u)dt,\\ &\vdots\\ &dx_n - f^n(x, u)dt\} \end{aligned}$$

on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with independence condition dt where $f = [f^1, f^2, \ldots, f^n]^T$. Then it is easy to show that (c, u), with $c : [0, T] \subset \mathbb{R} \to M$ and $u \in \mathbb{R}$, is a solution to the control system if and only if it is a solution to the Pfaffian system I. From now on we will revert back to control systems in control affine form (1).

We can formulate accessibility [16] in terms of the derived flag.

Proposition 2.2 ([12]): A control-affine system is:

- Strongly locally accessible if and only if $\mathcal{I}^{(N)} = \{0\}$, and
- Locally accessible if and only if $\operatorname{rank}(\mathcal{I}^{(N)}) \le 1$ and the integral manifolds of the bottom derived system are time-dependent.

A control-affine system is called *weakly locally accessible* if it is locally accessible but not strongly locally accessible, in which case the bottom derived system has rank one and depends on time. We will assume that all systems are locally accessible.

C. Method of Equivalence

In Section III we use results on equivalence of control systems based on the method of equivalence developed by Élie Cartan [2]. We give a short overview and refer the reader to the treatment of the subject in [5], [14], [10], [3]. In connection to control theory the equivalence method has found application as a tool for studying equivalence to linear systems [6].

The method of equivalence is the study of equivalence between mathematical objects described in a basis, called coframe, of the cotangent bundle of some manifold. Given two control systems $\dot{x} = f(x, u)$ on M and $\dot{X} = F(X, U)$ on N, we say that the systems are *feedback equivalent* if there exists a local diffeomorphism $\Phi : M \to N$ such that $\Phi(x, u) = (\phi(x), \psi(x, u))$, where ϕ is a coordinate transformation on the state space and ψ is a static state feedback. The primary object of study in this framework is the pullback of the diffeomorphism Φ as a linear map that is pulling a coframe on N back to a coframe on M up to the action of some group. The data of the specific problem, in our case the feedback equivalence, is completely contained in the coframe and the group action.

III. MAIN RESULT

The development in this section can be divided into three parts. First, we divide a n-dimensional control system into a two-dimensional system and a chain of integrators. Next, we summarize the complete classification of two-dimensional control systems from [4]. Finally, we show how this classification can be used to classify the original system.

A. Reduction

In the case of a locally accessible control system with a single input it is easy to see that $\kappa_1 = \kappa_2 = \cdots = \kappa_{n-1} = 1$,

- $\kappa_n = 1$ and $\mathcal{I}^{(n)} = \mathcal{I}^{(N)}$, depends on time if the system is weakly locally accessible or
- $\kappa_n = 0$, in which case the system is strongly locally accessible.

Since local accessibility only determines the structure of the derived flag but not the integrability of the derived systems, we require additional assumptions on $\mathcal{I}^{(n-2)}$. Furthermore, integrability of $\mathcal{I}^{(n-2)}$ implies integrability of the remaining derived systems, this property allows us to reduce the analysis of the system to $\mathcal{I}^{(n-2)}$. We make the following observation: The derived flag of a locally accessible system takes the form (2) if the $\mathcal{I}^{(n-2)}$ derived system is integrable.

$$\begin{aligned}
&d\omega_n \equiv dt \wedge du \bmod \mathcal{I} \\
&d\omega_i \equiv dt \wedge \omega_{i+1} \bmod \mathcal{I}^{(n-i)}, \quad 3 \leq i \leq n-1 \\
&\left.\begin{aligned} d\omega_2 &\equiv dt \wedge \alpha_2 \\ d\omega_1 &\equiv dt \wedge \alpha_1 \end{aligned}\right. \bmod \mathcal{I}^{(n-2)},
\end{aligned} \tag{2}$$

where α_1 can vanish in which case

$$d\omega_1 \equiv 0 \bmod \omega_1$$

defines the bottom derived system for which the integral manifold depends on time. It is easy to see that for $\alpha_1 = 0$ the system is weakly locally accessible. Note that this is also the derived flag of an orbital feedback linearizable [15] system. Next, we find a normal form for this type of systems. Consider the system

$$\begin{aligned}
\dot{x}_1 &= f^1(x_1, x_2, v) \\
\dot{x}_2 &= f^2(x_1, x_2, v) \\
\dot{v} &= \dot{y}_3 = y_4 \\
&\vdots \\
\dot{y}_n &= u,
\end{aligned} \tag{3}$$

then we can make the following statement.

Lemma 3.1: The system (1) fulfills the congruence (2) if and only if it is feedback equivalent to (3).

Proof: Straight forward computations show sufficiency. For the necessary part, assume that (1) fulfills the congruence (2), we can write each generator $\omega_i = \eta_i - b_i dt$, $i = 1, \ldots, n$, where η_i are one-forms on $\mathbb{R}^n$ and b_i are functions on $\mathbb{R}^n$ independent of time. Considering the $\{\omega_1, \omega_2\}$ subsystem we see that the integrability condition

$$\begin{aligned}
d\omega_1 \wedge \omega_1 \wedge \omega_2 \wedge dt &= 0 \\
d\omega_2 \wedge \omega_2 \wedge \omega_1 \wedge dt &= 0
\end{aligned}$$

are satisfied, which implies the following integrability condition on η_1, η_2

$$\begin{aligned}
d\eta_1 \wedge \eta_1 \wedge \eta_2 &= 0 \\
d\eta_2 \wedge \eta_2 \wedge \eta_1 &= 0.
\end{aligned}$$

Hence, we can find functions y_1, y_2 such that $\langle dy_1, dy_2 \rangle = \langle \eta_1, \eta_2 \rangle$ and determine a new set of generators for $I^{(n-2)}$ given by $\bar{\omega}_i = dy_i - \bar{b}_i dt$, $i = 1, 2$. Next, we want to find a set of exact generators for $\mathcal{I}^{(n-3)}$ since ω_1 and ω_2 are integrable we can generate a integrable form ω_3 by exterior differentiation, repeating this integration yields a set of coordinates and a feedback such that (1) takes the form of (3). ∎

Hence, the classification of systems given by (1) can be reduced to the classification of a two dimensional control system with a single input. The drawback of this reduction is that the obtained two dimensional system is not necessarily in control-affine form.

From a control perspective, we separated the system into two subsystems, the first one being a feedback linearizable part, which was transformed into a Brunovsky normal form, resulting in a simple integrator chain. The second part is a two dimensional subsystem for which the linearizing output of the first system served as a "virtual input".

B. Local equivalence of systems with two states and one control

The complete problem of equivalence for control systems with two states and one input has been solved by Gardner and Shadwick [4]. We summarize this result. Given a general control system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$ and an equivalence relation defined by state and feedback transformation, there are two sets of equivalence classes. The first equivalence classes can be summarized as the class of linear systems, since they can be represented in the form $\dot{x} = Ax + bu$. They are describe by their normal forms:

(I) $\dot{x}_1 = 1$, $\dot{x}_2 = 0$
(II) $\dot{x}_1 = u$, $\dot{x}_2 = 0$
(III) $\dot{x}_1 = u$, $\dot{x}_2 = x_2$
(IV) $\dot{x}_1 = u$, $\dot{x}_2 = x_1$.

It should be noted that a sufficient condition for a system to be within one of these classes is given by

$$\frac{\partial f_2}{\partial u}\frac{\partial^2 f_1}{\partial u^2} = \frac{\partial f_1}{\partial u}\frac{\partial^2 f_2}{\partial u^2}. \tag{4}$$

The second class is described by the structure equations

$$\begin{aligned}
d\omega_1 &= \varepsilon \omega_1 \wedge \omega_2 \\
d\omega_2 &= \omega_3 \wedge \omega_1 + I\omega_3 \wedge \omega_2 \\
d\omega_3 &= J\omega_3 \wedge \omega_2 - K\omega_1 \wedge \omega_2
\end{aligned} \tag{5}$$

where $\varepsilon = \pm 1$ and I, J, K are the torsion coefficients of the e-structure [5] and $\{\omega_1, \omega_2\}$ are part of the adapted coframe and only depend on $f(x, u)$. These quantities are invariants of the system. Note that these are also the structure equations for Lagrangian particle in the plane developed by Cartan [2]. Here all remaining non-zero invariance of the lifted coframe $\{\omega_1, \omega_2, \omega_3\}$ depend only on $f(x, u)$ and its partial derivatives. Define the generalized derivatives of h with respect to $\{\omega_1, \ldots, \omega_i\}$ by $dh = \sum h_{\omega_i}\omega_i$. With this notation, the following integrability conditions on the torsion coefficients are obtained:

$$J = -I_{\omega_1}, \qquad J_{\omega_1} = -K_{\omega_3} - KI$$

which allow, following [5], to decide on two different case based only on the invariance I, assuming that the torsion coefficients are constant. Here we want to focus on the case where $I = 0$, implying $J = 0$ this structure equations are the same as the one of (pseudo)-Riemannian metric

$$ds^2 = (\omega_1)^2 + \varepsilon(\omega_2)^2. \tag{6}$$

on the state space with Gauss curvature K. Hence the curvature of the system and ε are the only remaining non zero invariants of the system.

For completeness, we note that in the case $I = \text{constant} \neq 0$, implying $J = 0$ and $K = 0$ the equations (5) are the structure equations of a generalized geometry, which we will not discuss at this point.

Another interpretation in terms of centro-affine geometry can be found in [18], where the invariants are connected to invariance of curves in the plane under actions of $GL(2, \mathbb{R})$ giving a more geometric interpretation. Also some normal forms have been developed for the different cases contained in the second class.

Another interesting case is considered in [18], where the torsion coefficient I is constant on each fiber, meaning that I varies smoothly only with the states of the reduced system but is independent of the control. In this case, one can show that J gives the rate of change under which the curvature K changes while the system moves along the direction of the input on the state space.

In summary we have the following two cases

(a) The system is feedback linearizable corresponding to one of the cases $(I) - (IV)$

(b) The final structure equation are given by (5).

We refer to [4] for complete discussion on this subject.

C. Classification of a control system with n states and one control

So , we have seen in the last two sections that we can write a control affine system in the normal form (3) and classify any two dimensional system by either being linear represented by the class (a) above or not linear represented by the class (b). The next step is to combine these two results.

A system (1) is said to belong to the class (A), written in normal form (3) if the f^1, f^2 subsystem with control input v belongs to one of the classes $(III) - (IV)$.

A system (1) is said to belong to the class (B), written in normal form (3) if the f^1, f^2 subsystem with control input v belongs to the equivalence class (b). Furthermore, there exists a (pseudo-) Riemannian metric if the system is in the equivalence class (B) and the torsion is constant with $I = 0$ as given in (5) then a normal form is given by

$$
\begin{aligned}
\dot{y}_1 &= \frac{\cos y_3}{1 + K(y_1^2 + y_2^2)} \\
\dot{y}_2 &= \frac{\sin y_3}{1 + K(y_1^2 + y_2^2)} \\
\dot{y}_3 &= y_4 \\
&\vdots \\
\dot{y}_n &= u.
\end{aligned}
$$

The main proposition of this note is the following.

Proposition 3.2: A locally accessible control affine system with n states and one control satisfying the congruence (2) is equivalent to either of the following:

(A) The system is feedback linearizable.

(B) The final structure equation are given by (5) and a chain of integrators.

Proof: As noted above, the generators of the complement of $I^{(n-2)}$ in $I^{(0)}$ are canonically constructed by differentiation with respect to time. Furthermore, we can choose a set of exact generators $\{dy_3, dy_4, \ldots, dy_n\}$ for this complement. Define $v = y_3$ as control for the implicit control system defined by the $I^{(n-2)}$ derived system. Let $\{\omega_1, \omega_2\}$ be a set of generators for $I^{(n-2)}$ and let dv the generator of $I^{(n-3)}$ in the complement of $I^{(n-2)}$ in $I^{(n-3)}$. Following [4], we get the two equivalence classes (a) and (b) for this system. Then a feedback transformation to a normal form amounts to change of coordinates such $v' = u(x_1, x_2) - v$ is the fiber coordinate on the (x_1, x_2)-plane. ■

Remark 3.3: The equation (6) is defined in terms of the lifted coframe and therefore defines the (pseudo)-Riemannian metric uniquely on the reduced space. In order to have a metric defined on the complete state space, we can always construct a Riemannian metric on the state space such that its restriction agrees with the (pseudo)-Riemannian metric on the reduced space [13].

The classes $(I) - (II)$ correspond to cases in which the original system is not locally accessible and the case (III) corresponds to the case where the system is weakly locally accessible, hence the only linear case in which the system is strongly accessible is the case (IV), which represents the feedback linearizable case. Also, note that the conditions in Proposition 3.2 are the same as for orbital feedback linearizable and within this class we have the case for which we can find a Riemannian metric and the remaining cases also define a more general geometry. Orbital feedback linearization is used to construct a state-dependent time transformation to obtain a linear system representation in the new time scale. Instead of changing time to get a linear system, we can use the existence of some geometric structures to apply other control techniques. We consider one possible application of this in the next section.

IV. Example

We study a simple car model presented in [15] to show some implications of the results. The system dynamics are given by

$$
\begin{aligned}
\dot{x} &= \cos\theta \\
\dot{y} &= \sin\theta \\
\dot{\theta} &= u
\end{aligned}
$$

where $(x, y) \in \mathbb{R}^2/\{0\}$ and $\theta \in \mathbb{S}^1$. We will only consider an open neighborhood $U \subset \mathbb{R}^2/\{0\} \times \mathbb{S}^1$ around an arbitrary point x_0. The derived flag is $I^{(0)} = \{dx - \cos\theta dt, dy - \sin\theta dt, d\theta - u dt\}$, $I^{(1)} = \{dx - \cos\theta dt, dy - \sin\theta dt\}$, hence has the required form (2). We also see that $\theta = v$ is our control input for the two dimensional subsystem and that the system is not feedback linearizable since $f_v^2 f_{vv}^1 - f_v^1 f_{vv}^2 = -1 \neq 0$, where $f^1 = \cos\theta$ and $f^2 = \sin\theta$. Hence, the system falls into the second equivalence class (b). We can readily calculate the torsion coefficient and obtain the following structure equations

$$\begin{aligned} d\omega_1 &= -\omega_3 \wedge \omega_2 \\ d\omega_2 &= \omega_3 \wedge \omega_1 \\ d\omega_3 &= 0. \end{aligned}$$

where $\omega_1 = dx\cos\theta + dy\sin\theta$ and $\omega_2 = -dy\sin\theta + dx\cos\theta$. This guarantees the existence of a Riemannian metric on the (ω_1, ω_2) state space with Gauss curvature $K = 0$. Also, we see that the system is already in the normal form suggested by [5]. In the given coordinates (x, y, u), the Riemannian metric g takes the form

$$\begin{aligned} g &= (\omega_1)^2 + (\omega_2)^2 \\ &= (\cos\theta dx + \sin\theta dy)^2 + (\cos\theta dy - \sin\theta dx)^2 \\ &= (dx)^2 + (dy)^2 \end{aligned}$$

where $(\cdot)^2$ denotes the tensor product of a one-form with itself. This is just the euclidian metric given in the adapted coframe $\{\omega_1, \omega_2\}$. Note that for any constant heading angle θ the trajectories of the reduced system agree with the geodesics of the metric g. Hence, we know that the car will travel along a straight line in the direction of the heading angle.

Finally, we consider the path following problem to illustrate how the metric can be used for control purposes. We want to find a feedback that steers the car on the unit circle

$$S = \{(x, y, \theta) \in U \mid x^2 + y^2 - 1 = 0\}.$$

First, we determine a coordinate transformation of the complete state space such that the Jacobian of this transformation agrees with the change of frames on the reduced system, *i.e.* we want to find a coordinate coframe such that the coordinate forms agree with ω_1 and ω_3 on the reduced space. We propose the following transformation:

$$\begin{aligned} z_1 &= x\cos\theta + y\sin\theta \\ z_2 &= -x\sin\theta + y\cos\theta \\ z_3 &= x^2 + y^2 - 1. \end{aligned}$$

It is easy to see that $dz_1 = \omega_1$ and $dz_2 = \omega_2$ on the reduced space. This turns out to be the coordinate transformation that transforms the system in partial feedback linearization normal form:

$$\begin{aligned} \dot{z}_1 &= 1 + z_2 u \\ \dot{z}_2 &= -z_1 u \\ \dot{z}_3 &= z_1. \end{aligned}$$

The unit circle in the new coordinates is given by $S = \{z_3 = 0\}$ and we can consider the path following problem as an output regulation problem and we get the following feedback that steers the system to the unit circle $u = -1/z_2(-1+z_1+z_3)$, under the assumption that $z_2 \neq 0$.

V. CONCLUSIONS

We have considered the classification of accessible control-affine system with single-input satisfying the congruence (2). A complete list of equivalence classes has been found by applying the results presented in [5]. The first class of feedback linearizable systems have been widely studied, the second class has received less attention. It was shown that in this case the control systems define a geometric structure on the state space. Some implications of this result have been presented in an example. By assuming integrability conditions on the derived systems we considered a specific case of locally accessible systems. A next step is to weaken this assumption by assuming integrability conditions on $I^{(n-3)}$. This leads to an implicit description of $I^{(n-2)}$ and a new equivalence problem on the reduced state space for which Cartan's method needs to be applied.

REFERENCES

[1] F. Bullo and A. D. Lewis (2005). Geometric Control of Mechanical Systems. Springer-Verlag, New York.

[2] É. Cartan (1908). Les sous-groupes des groupes continus de transformations. *Ann. Ec. Norm.*, 25: 57–194 (Oeuvres Completes II, 719–856).

[3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths (1991). Exterior Differential Systems. *Mathematical Sciences Research Institute Publications 18*, Springer-Verlag, New York.

[4] R. B. Gardner and W. F. Shadwick (1987). Feedback Equivalence of Control Systems. *Systems & Control Lett.*, 8: 463–465.

[5] R. B. Gardner (1989). The Method of Equivalence and Its Applications. Society for Industrial and Applied Mathematics. Philadelphia, PA.

[6] R. B. Gardner and W. F. Shadwick (1992). The GS Algorithm for Exact Linearization to Brunovsky Normal Form. *IEEE Transactions on Automatic Control*, 37(2):224–230.

[7] M. Guay (1999). An Algorithm for Orbital Feedback Linearization of Single-Input Control Affine Systems. *Systems & Control Lett.*, 38: 271–281.

[8] J. M. Igreja, J. M. Lemos and R. Neves-Silva, (2004). Dynamic Motion Planning for a Spray Dryer Plant. *In Proceedings of the VI Portuguese Conference on Automatic Control*, Faro, Portugal.

[9] A. Isidori (1995). Nonlinear Control Systems, 3rd ed. Springer-Verlag, New York.

[10] T. A. Ivey and J. M. Landsberg (2003). Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems. American Mathematical Society. Providence, RI.

[11] B. Kiss and E. Szádeczky-Kardoss (2003). Tracking Control of the Orbitally Flat Kinematic Car with a New Time-scaling Input. *In Proceedings of the 46th IEEE Conference on Decision and Control, 2007*. 1969–1974

[12] A. D. Lewis (1995). Aspects of Geometric Mechanics and Control of Mechanical Systems. Ph.D. Thesis. California Institute of Technology. Pasadena, CA

[13] R. Montgomery (2002). A Tour of Subriemannian Geometries, Their Geodesics, and Applications. American Mathematical Society. Providence, RI.

[14] P. J. Olver (1995). Equivalence, Invariants, and Symmetry. Cambridge University Press.

[15] W. Respondek (1998). Orbital Feedback Linerization of Single-input Nonlinear Control Systems. *In Proceedings of IFAC NOLCOS'98*. Enschede, The Netherlands, 499–504.

[16] A. van der Schaft (2000). L_2-Gain and Passivity Techniques in Nonlinear Control. Springer-Verlag. New York.

[17] U. Vollmer and J. Raisch (2003). Control of Batch Cooling Crystallization Processes Based on Orbital Flatness. *International Journal of Control*,76(16): 1635–1643

[18] G. R. Wilkens (1999). Centroaffine Geometry in the Plane and Feedback Invariants of Two-state Scalar Control Systems. *In Differential Geometry and Control: Summer Research Institute on Differential Geometry and Control*. 64: 319–333, American Mathematical Society. Providence, RI.