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Solution of Fractional Order Optimal Control Problems Using SVD-based Rational Approximations

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***Abstract*— This paper introduces a new direction to approximately solving fractional order optimal control problems (FOCPs). A general methodology is described that can potentially solve any type of FOCPs (linear/nonlinear, time-invariant/time-variant, SISO/MIMO, state/input constrained, free terminal conditions etc.). The method uses a rational approximation of the fractional derivative operator obtained from the singular value decomposition of the Hankel data matrix of the impulse response. The FOCP is then reformulated to be solved by RIOTS_95, a general-purpose optimal control problem (OCP) solver in the form of a MATLAB toolbox. Illustrative examples from the literature are reproduced to demonstrate the effectiveness of the propose methodology and a free final time OCP is also demonstrated.**

***Index Terms*— Fractional calculus, fractional order optimal control problems, numerical methods.**

I. INTRODUCTION

The idea of fractional derivative dates back to a conversation between two mathematicians: Leibniz and L'Hopital. In 1695, they exchanged about the meaning of a derivative of order $1/2$. Their correspondence has been well documented and is stated as the foundation of fractional calculus [1].

Many real-world physical systems display fractional order dynamics, that is their behavior is governed by fractional-order differential equations [2]. For example, it has been illustrated that materials with memory and hereditary effects, and dynamical processes, including gas diffusion and heat conduction, in fractal porous media can be more adequately modeled by fractional-order models than integer-order models [3].

The general definition of an optimal control problems requires minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions. The system is subject to constrained dynamics and control variables. Additional constraints such as final time constraints can be considered. This paper introduces an original formulation and a general numerical scheme for a potentially almost unlimited class of FOCPs. An FOCP is an optimal control problem in which the criterion and/or the differential equations governing the dynamics of the system contain at least one fractional derivative operator.

Integer order optimal controls (IOOCs) have been discussed for a long time and a large collection of numerical techniques have been developed to solve IOOC problems [4]. However, the number of publications on FOCPs is limited. A general formulation and a solution scheme for FOCPs were first introduced in [5] where fractional derivatives were introduced in the RiemannLiouville sense, and FOCP formulation was expressed using the fractional variational principle and the Lagrange multiplier technique. The state and the control variables were given as a linear combination of test functions, and a virtual work type approach was used to obtain solutions. In [4], [6], the FOCPs are formulated using the definition of fractional derivatives in the sense of Caputo, the FDEs are substituted into Volterra-type integral equations and a direct linear solver helps calculating the solution of the obtained algebraic equations. In [7], the fractional dynamics of the FOCPs are defined in terms of the RiemannLiouville fractional derivatives. The Grunwald and Letnikov formula is used as an approximation and the resulting equations are solved using a direct scheme. Frederico and Torres [8], [9], [10], using similar definitions of the FOCPs, formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo and studied fractional conservation laws in FOCPs. Using a rational approximation of the fractional derivative to solve FOCP was first introduced in [11]. The approximation method used was the "Oustaloup Recursive Approximation" [12], a frequency-domain based method.

In this paper, we present a formulation and a numerical scheme for FOCP based on IOOC problem formulation. Therefore, the class of FOCP solvable by the proposed methodology is closely related to the considered IOOC solver RIOTS_95 [13], [14]. The fractional derivative operator is approximated by a state-space realization by using the singular value decomposition (SVD) of a Hankel matrix derived from the analytical impulse response of the fractional integrator. The fractional differential equation governing the dynamics of the system is expressed as an integer order state-space realization. The FOCP can then be reformulated into an IOOC problem, solvable by a wide variety of algorithms from the literature. Three examples are solved to demonstrate the performance of the method.

II. FRACTIONAL OPTIMAL CONTROL PROBLEM FORMULATION

In this section, we briefly give some definitions regarding fractional derivatives allowing us to formulate a general definition of an FOCP.

There are different definitions of the fractional derivative operator [5]. The Left Riemann-Liouville Fractional Derivative (LRLFD) of a function $f(t)$ is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \quad (1)$$

where the order of the derivative α satisfies $n-1 \leq \alpha < n$. The Right Riemann-Liouville Fractional Derivative (RRLFD) is defined as

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dt})^n \int_t^b (t-\tau)^{n-\alpha-1} f(\tau)d\tau. \quad (2)$$

Another fractional derivative is the left Caputo fractional derivative $LCFD$ defined as

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} (\frac{d}{dt})^n f(\tau)d\tau. \quad (3)$$

The right Caputo fractional derivative $RCFD$ defined as

$$ {}_t^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (t-\tau)^{n-\alpha-1} (\frac{d}{dt})^n f(\tau)d\tau. \quad (4)$$

From any of these definitions, we can specify a general FOCP: Find the optimal control $u(t)$ for a fractional dynamical system that minimizes the following performance criterion

$$ J(u) = G(x(a), x(b)) + \int_a^b L(x,u,t)dt \quad (5)$$

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subject to the following system dynamics

$$ {}_aD_t^{\alpha}x(t) = H(x,u,t) \tag{6}$$

with initial condition

$$x(a) = x_a \tag{7}$$

and with the following constraints

$$u_{min}(t) \le u(t) \le u_{max}(t), \tag{8}$$

$$x_{min}(a) \le x(a) \le x_{max}(a), \tag{9}$$

$$L_{ti}^{\nu}(t, x(t), u(t)) \le 0, \tag{10}$$

$$G_{ei}^{\nu}(x(a), x(b)) \le 0, \tag{11}$$

$$G_{ee}^{\nu}(x(a), x(b)) = 0. \tag{12}$$

where x is the state variable, $t \in [a,b]$ stands for the time, and F, G and H are arbitrary given nonlinear functions. The subscripts o, ti, ei, and ee on the functions $G(.,.)$ and $L(.,.,.)$ stand for, respectively, objective function, trajectory constraint, endpoint inequality constraint and endpoint equality constraint.

III. LINEAR APPROXIMATION OF FRACTIONAL TRANSFER FUNCTIONS

A. Approximation Method

This methodology was derived from [15]. Consider the analytical impulse response $h(t)$ of a given fractional system. The approximation problem consists in obtaining a linear system of order n whose impulse response $h_a(t)$ coincides with $h(t)$ well. The linear system is modeled by the following state-space realization:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + bu(t) \tag{13}$$

$$y(t) = c\mathbf{x}(t) \tag{14}$$

where the state $\mathbf{x}(t)$ is of size n and the system matrix A is n by n. The impulse response $h_a(t)$ can be expressed in terms of A, b, and c by [16]

$$h(t) = ce^{At}b \tag{15}$$

where the state-transition matrix e^{At} denotes the exponential of the matrix At. Let us describe the methodology for solving the approximation problem. We consider a set of sampled data $h(kT)$ from the analytical impulse response $h(t)$, with T standing for the sampling period. An approximate linear system would have the following property

$$h(kT) \approx ce^{AkT}b = c(e^{AT})^k b \tag{16}$$

which can be reformulating in the following way

$$h(kT) \approx c(e^{A_d})^k b \tag{17}$$

with

$$A_d = e^{AT} \tag{18}$$

We then take $2p$ data points from the sampled impulse response to form a Hankel data matrix H defined as

$$H = \begin{pmatrix} h(0) & h(1) & \dots & h(p-1) \\ h(1) & h(2) & \dots & h(p) \\ \vdots & \vdots & \dots & \vdots \\ h(p) & h(p-1) & \dots & h(2p-1) \end{pmatrix}_{p+1,p} \tag{19}$$

that is

$$H = \begin{pmatrix} cb & cA_db & \dots & cA_d^{p-1}b \\ cA_db & cA_d^2b & \dots & cA_d^pb \\ \vdots & \vdots & \dots & \vdots \\ cA_d^pb & cA_d^{p+1}b & \dots & cA_d^{2p-1}b \end{pmatrix} \tag{20}$$

H is further reformulated by the factorization

$$H = \begin{pmatrix} c \\ cA_d \\ \vdots \\ cA_d^{p-1} \end{pmatrix}_{p+1,n} \begin{pmatrix} b & A_db & \dots & A_d^{p-1}b \end{pmatrix} = OC \tag{21}$$

where n is the approximated numerical rank of the Hankel data matrix H and is determined by its singular values (square roots of eigenvalues of HH^T). By examining singular values of H, we are able to choose a proper integer n to be the dimension of the approximating linear system. In other words, n is the number of state variables of the linear system which are 'adequate' in describing the distributed system specified by $h(t)$. Since the matrix H is given, factorization of H into a product of two matrices is always possible using the singular value decomposition. After O and C are generated from the Hankel data matrix, matrices A, b and c can be obtained as follows:

$$c = \text{1st row of O} \tag{22}$$

$$b = \text{1st column of C} \tag{23}$$

Define

$$O_1 = \text{O without the last row} \tag{24}$$

$$O_2 = \text{O without the first row} \tag{25}$$

then

$$O_2 = O_1A_d \tag{26}$$

Solving the above equation yields

$$A_d = (O_1^TO_1)^{-1}O_1^TO_2 \tag{27}$$

Finally, we recall the relationship A, = e^{AT} and obtain A from A_d by

$$A = ln(A_d)/T \tag{28}$$

where ln denotes the natural log of a matrix

B. Sub-Optimal Approximation of the Fractional Integrator

We try to approximate the following fractional transfer function

$$H(s) = \frac{1}{s^{\alpha}} \tag{29}$$

with $\alpha \in [0,1]$. The analytical impulse response of such a system is given by

$$h(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \tag{30}$$

where $\Gamma(\cdot)$ represents the Gamma function. For a given transfer function, an infinite number of approximation can be performed. Therefore, for a given order n of the state-space realization of the approximation, we wish to find the values of T and p that give the best approximation. In addition, the impulse response of a fractional integrator displays a singularity at the origin ($t = 0$) as observed in (30). Therefore, to avoid this infinite term, $h(0)$ has to be approximated by a finite value. This finite initial value giving the best approximation is also sought. The best approximation is obtained via an exhaustive search. The performance criteria used to assess the quality of an approximation is the ITSE of the step response because of the absence of singularity and improved results. The analytical step response of the system described by (29) is

$$s(t) = \frac{t^{-\alpha}}{\Gamma(-\alpha+1)} \tag{31}$$

The search is performed for approximation orders n ranging from 1 to 10. Table I summarizes the different values used in the search for the best parameters set. These values were upper-bounded by the computer's memory.

ITSE	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$
$n=1$	$1.05e-2$	$2.62e-3$	$6.26e-2$	$1.77e-2$	$3.12e-3$	$2.32e-3$	$8.98e-4$	$4.72e-4$	$1.85e-4$
$n=2$	$3.61e-4$	$1.39e-3$	$1.56e-3$	$6.47e-4$	$8.50e-4$	$7.07e-4$	$3.96e-4$	$2.10e-4$	$6.01e-5$
$n=3$	$2.81e-4$	$1.34e-3$	$1.40e-4$	$5.40e-5$	$1.49e-4$	$9.39e-5$	$4.47e-5$	$1.56e-5$	$3.42e-6$
$n=4$	$3.45e-5$	$1.30e-3$	$1.01e-4^*$	$2.01e-6$	$4.22e-5$	$2.89e-5$	$1.26e-5$	$4.12e-6$	$1.02e-6$
$n=5$	$3.25e-6^*$	$1.25e-3$	$2.27e-4$	$1.49e-6$	$1.86e-5$	$1.51e-5$	$6.93e-6$	$2.40e-6$	$6.77e-7$
$n=6$	$8.40e-6$	$1.25e-3$	$3.13e-4$	$1.51e-7^*$	$1.15e-5$	$1.01e-5$	$5.35e-6$	$1.93e-6$	$4.00e-7$
$n=7$	$2.80e-5$	$1.31e-3$	$3.61e-4$	$1.41e-6$	$6.09e-6$	$4.50e-6$	$2.31e-6$	$8.51e-7$	$2.38e-7^*$
$n=8$	$2.00e-4$	$1.14e-3$	$3.92e-4$	$3.26e-6$	$2.94e-6$	$3.85e-6^*$	$2.06e-6^*$	$7.80e-7^*$	$2.53e-7$
$n=9$	$4.33e-4$	$8.96e-4$	$4.14e-4$	$5.10e-6$	$2.53e-6$	$4.17e-6$	$2.28e-6$	$8.71e-7$	$2.78e-7$
$n=10$	$6.80e-4$	$7.55e-4^*$	$4.32e-4$	$6.84e-6$	$2.46e-6^*$	$4.39e-6$	$2.44e-6$	$9.29e-7$	$2.89e-7$

TABLE II

ITSE OF THE BEST MODEL FOR DIFFERENT APPROXIMATION ORDERS AND FRACTIONAL ORDERS. * POINTS THE BEST APPROXIMATE

T	10^{-3}	$5\cdot10^{-4}$	10^{-4}	$5\cdot10^{-5}$	10^{-5}
	$5\cdot10^{-6}$	10^{-6}	$5\cdot10^{-7}$	10^{-7}	
p	25	50	75	100	250
	500	750	1000		
$h(0)$	$10\cdot h(1)$	$10^2\cdot h(1)$	$10^3\cdot h(1)$	$10^4\cdot h(1)$	

TABLE I

PARAMETER VALUES USED FOR THE EXHAUSTIVE SEARCH OF THE BEST APPROXIMATION

IV. FRACTIONAL OPTIMAL CONTROL PROBLEM REFORMULATION

With our state-space approximation of the fractional derivative operator, it is now possible to reformulate the FOCP described in equations (5)-(12). Find the optimal control $u(t)$ for a dynamical system that minimizes the performance criterion

$$\begin{aligned} J(u) &= G(cz(a), cz(b)) \\ &+ \int_a^b L(cz, u, t)dt \end{aligned} \tag{32}$$

subject to the following dynamics

$$\dot{z}(t) = Az + b(H(cz, u, t)) \tag{33}$$

with initial condition

$$z(a) = x_a w/(cw) \tag{34}$$

Equation (34) ensures the the initial condition $cz(a) = x_a$ is maintained. and with the following constraints

$$u_{min}(t) \le u(t) \le u_{max}(t) \tag{35}$$

$$x_{min}(a) \le Ccz(a) \le x_{max}(a) \tag{36}$$

$$L_{ti}^{\nu}(t, cz(t), u(t)) \le 0 \tag{37}$$

$$G_{ei}^{\nu}(cz(a), cz(b)) \le 0 \tag{38}$$

$$G_{ee}^{\nu}(cz(a)), cz(b)) = 0 \tag{39}$$

where z is the state vector, w is a vector of size N, $t \in [a, b]$ stands for again the time, and F, G and H are arbitrary nonlinear functions. The subscripts o, ti, ei, and ee on the functions $G(.,.)$ and $L(.,.,.)$ stand for, respectively, objective function, trajectory constraint, endpoint inequality constraint and endpoint equality constraint.

The choice for the vector w is indeed important as it can improve the convergence of the optimization. To make computation faster, our experiments have shown that choosing w as

$$w = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \tag{40}$$

represents the best choice. The state $x(t)$ of the initial FOCP can be retrieved from

$$x(t) = cz(t). \tag{41}$$

This framework allows us to approximately solve a large variety of FOCPs thanks to the link we created with the traditional optimal control problems. In fact, the proposed conversion allows us to apply any readily available IOOC solver to find an approximate solution of almost any given FOCP problem. For this paper, we decide to use the RIOTS_95 Matlab Toolbox to be briefly introduced in the next section.

V. RIOTS_95 MATLAB TOOLBOX: A BRIEF INTRODUCTION

The acronym RIOTS means "recursive integration optimal trajectory solver." It is a Matlab toolbox developed to solve a large class of optimal control problems. For details, refer to [13] and the references therein.

VI. ILLUSTRATIVE EXAMPLES

In this section, we demonstrate the capability of the introduced approach. First we solve two widely used examples from the literature and then we introduce a new problem that none of the previously introduced methodologies attempted to solve. For each problem, we examine the solution for different values of α. For this purpose, α was taken between 0.1 and 1. Problems are first stated in the traditional FOCP framework and then reformulated via our introduced methodology. Results of these studies are given at the end of each subsection.

A. Linear Time-Invariant Problem

Our first example can be found in [5], [7], [4], [11]. It is a linear time invariant (LTI) fractional order optimal control problem stated as follows. Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{42}$$

subject to the following dynamics

$${}_0D_t^{\alpha} x = -x + u \tag{43}$$

with free terminal condition and the initial condition

$$x(0) = 1. \tag{44}$$

According to [17], the analytical solution of the problem defined above for $\alpha = 1$ is

$$x(t) = \cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t) \tag{45}$$

$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t) \tag{46}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2}t)}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2}t)} \approx -0.98.$$

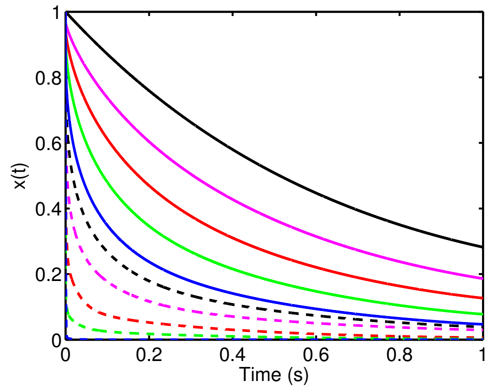


Fig. 1. State $x(t)$ as a function of t for the LTI problem for different values of α (dashed-blue: $\alpha = 0.1$, dashed-green: $\alpha = 0.2$, dashed-red: $\alpha = 0.3$, dashed-magenta: $\alpha = 0.4$, dashed-black: $\alpha = 0.5$,solid-blue: $\alpha = 0.6$, solid-green: $\alpha = 0.7$, solid-red: $\alpha = 0.8$, solid-magenta: $\alpha = 0.9$, solid-black: $\alpha = 1$)

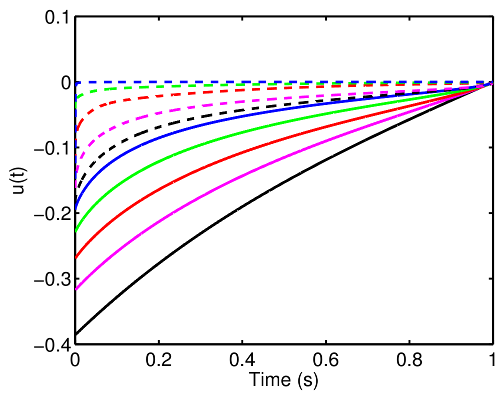


Fig. 2. Control $u(t)$ as a function of t for the LTI problem for different values of α (dashed-blue: $\alpha = 0.1$, dashed-green: $\alpha = 0.2$, dashed-red: $\alpha = 0.3$, dashed-magenta: $\alpha = 0.4$, dashed-black: $\alpha = 0.5$,solid-blue: $\alpha = 0.6$, solid-green: $\alpha = 0.7$, solid-red: $\alpha = 0.8$, solid-magenta: $\alpha = 0.9$, solid-black: $\alpha = 1$)

Using the proposed methodology, we reformulate the problem defined by Eqn.(42)-(44). Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 (cz(t))^2 + u^2(t)dt \tag{47}$$

subject to the following dynamics

$$\dot{z} = Az + B(-(cz) + u) \tag{48}$$

and the initial condition

$$z(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T. \tag{49}$$

Figures 1 and 2 show the state $x(t)$ and the control input $u(t)$ as functions of time t for different values of α. For $\alpha = 1$, the results match those of the analytical solution. Results are comparable to those obtained in [5], [7], [11].

B. Linear Time-Variant Problem

The second example studied here is also studied in [5], [7], [4], [11]. It is a linear time variant (LTV) problem stated as follows. Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{50}$$

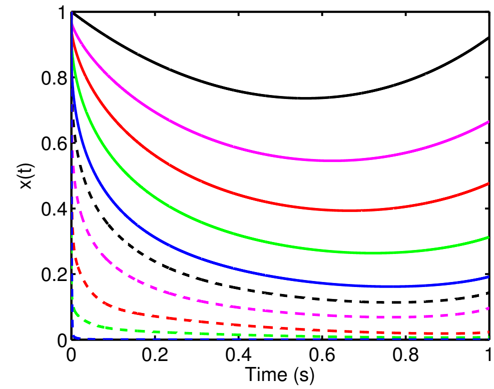


Fig. 3. State $x(t)$ as a function of t for the LTV problem for different values of α ((dashed-blue: $\alpha = 0.1$, dashed-green: $\alpha = 0.2$, dashed-red: $\alpha = 0.3$, dashed-magenta: $\alpha = 0.4$, dashed-black: $\alpha = 0.5$,solid-blue: $\alpha = 0.6$, solid-green: $\alpha = 0.7$, solid-red: $\alpha = 0.8$, solid-magenta: $\alpha = 0.9$, solid-black: $\alpha = 1$)

subject to the following dynamics

$$_0D_t^\alpha x = tx + u \tag{51}$$

with free terminal condition and the initial condition

$$x(0) = 1 \tag{52}$$

Using the proposed methodology, we reformulate the problem defined by Eqn.(50)-(52). Find the control $u(t)$, which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 (cz(t))^2 + u^2(t)dt \tag{53}$$

subjected to the following dynamics

$$\dot{z} = Az + b((cz)t + u) \tag{54}$$

and the initial condition

$$z(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T. \tag{55}$$

Figures 3 and 4 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of α $(0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1)$. For $\alpha = 1$, the optimal control problem has been solved in [17]. In that paper, the author uses a scheme specific to integer order optimal control problems. The numerical solution obtained with the proposed methodology for $\alpha = 1$ is accurate and results for fractional orders of α matches those found in the literature [5], [7], [11].

C. Fractional Order Bang-Bang Optimal Control

The third example studied here is called a free final time problem. It is a linear time invariant problem stated as follows. Find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{56}$$

subject to the following dynamics

$$_0D_t^\alpha x = u, 1 < \alpha \le 2 \tag{57}$$

and the initial condition

$$\begin{array}{rcl} x(0) & = & 0 \\ \dot{x}(0) & = & 0 \end{array} \tag{58}$$

final state constraints are

$$\begin{array}{rcl} x(T) & & 300 \\ \dot{x}(T) & = & 0 \end{array} \tag{59}$$

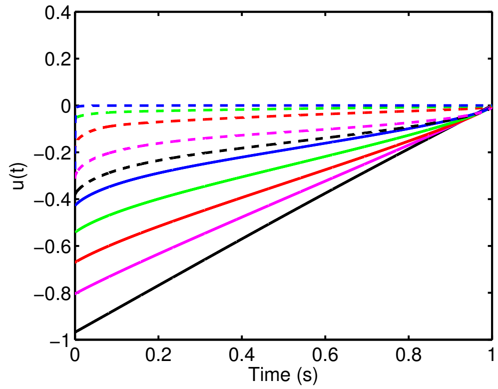


Fig. 4. Control $u(t)$ as a function of t for the LTV problem for different values of α (dashed-blue: $\alpha = 0.1$, dashed-green: $\alpha = 0.2$, dashed-red: $\alpha = 0.3$, dashed-magenta: $\alpha = 0.4$, dashed-black: $\alpha = 0.5$,solid-blue: $\alpha = 0.6$, solid-green: $\alpha = 0.7$, solid-red: $\alpha = 0.8$, solid-magenta: $\alpha = 0.9$, solid-black: $\alpha = 1$)

The analytical solution for this system for $\alpha = 2$, is given in [18] by $T^* = 30$ as

$$u(t) = \begin{cases} 1 & \text{for} \quad 0 \le t < 20 \\ -2 & \text{for} \quad 20 \le t \le 30 \end{cases} \tag{60}$$

$$x(t) = \begin{cases} t^2/2 & \text{for} \quad 0 \le t < 20 \\ -t^2 + 60t - 600 & \text{for} \quad 20 \le t \le 30 \end{cases} \tag{61}$$

$$\dot{x}(t) = \begin{cases} t & \text{for} \quad 0 \le t < 20 \\ 60 - 2t & \text{for} \quad 20 \le t \le 30 \end{cases} \tag{62}$$

Free final time problems can be transcribed into fixed final time problems by augmenting the system dynamics with additional states (one additional state for autonomous problems). The idea is to specify a nominal time interval, $[a, b]$, for the problem and to use a scaling factor, adjustable by the optimization procedure, to scale the system dynamics and hence, in effect, scale the duration of the time interval. This scale factor, and the scaled time, are represented by the extra states. Then RIOTS_95 can minimize over the initial value of the extra states to adjust the scaling.

The problem defined by Eqn.(56)-(59) can accordingly be reformulated as: find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{63}$$

subject to the following dynamics

$$\begin{array}{rcl} \dot{x}_1 & = & Tx_2 \\ {}_0D_t^{\beta} x_2 & = & Tu \\ \dot{(T)} & = & 0 \end{array} \tag{64}$$

where $\beta = \alpha - 1$ and the initial conditions are

$$\begin{array}{rcl} x_1(0) & = & 0 \\ x_2(0) & = & 0 \\ T(0) & = & 10 \end{array} \tag{65}$$

where $T(0)$ is the initial value chosen by the user. Final state constraints are

$$\begin{array}{rcl} x_1(T) & = & 300 \\ x_2(T) & = & 0. \end{array} \tag{66}$$

To ensure the applicability of our method, we need to define a new state vector $y(t)$ such that

$$y(t) = \begin{bmatrix} x_1(t) \\ z(t) \\ T \end{bmatrix} \tag{67}$$

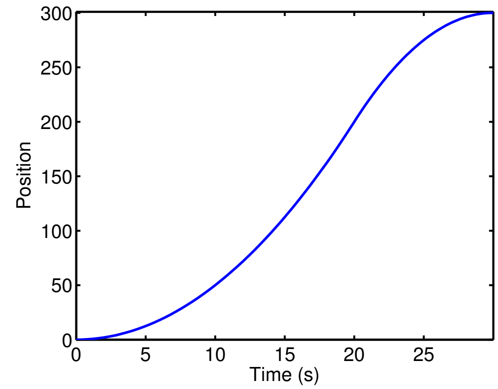


Fig. 5. States $x(t)$ and $\dot{x}(t)$ as functions of time t for the Bang-Bang control problem for $\alpha = 2$

TABLE III

OPTIMAL DURATIONS OF THE SOLUTION OF THE BANG-BANG FOCP

α	2	2	1.9	1.8	1.7	1.6
T^*(s)	30	30.56	32.7	36	40.2	46.7
α	1.5	1.4	1.3	1.2	1.1	
T^*(s)	55.5	69.9	91.3	129.8	188.6	

where $z(t)$ is the state vector of the ORA for the fractional order system described by ${}_0D_t^{\beta} x_2 = u$

Using the proposed methodology, we reformulate the problem defined by Eqn.(56)-(59). Find the control $u(t)$ (satisfying $-2 \le u(t) \le 1$), which minimizes the quadratic performance index

$$J(u) = T \tag{68}$$

subjected to the following dynamics

$$\dot{y} = \begin{bmatrix} c[y_2(t) \cdots y_{N+1}(t)]^T \\ A[y_2(t) \cdots y_{N+1}(t)]^T + bu(t) \\ 0 \end{bmatrix} \tag{69}$$

and the initial condition

$$y(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & T \end{bmatrix}^T \tag{70}$$

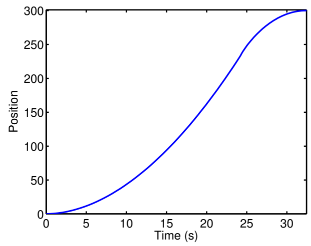
and the final state constraints given by

$$\begin{array}{c} y_1(T) = 300 \\ c[y_2(T) \cdots y_{N+1}(T)]^T = 0. \end{array} \tag{71}$$

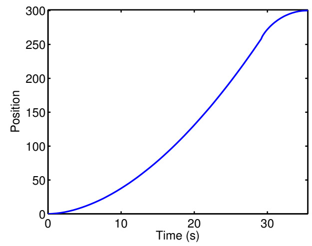
Figure 5 shows the state $x(t)$ as a function of t for $\alpha = 2$. Figure VI-C shows the state $x(t)$ as a function of t for different values of α (1.9, 1.8, 1.7, 1.6, 1.5, 1.4, 1.3, 1.2 and 1.1, respectively). As the order α approaches 2, the optimal duration nears its value for the double integrator case. Table III summarizes the optimal time durations of the different simulations under various α.

VII. CONCLUSION

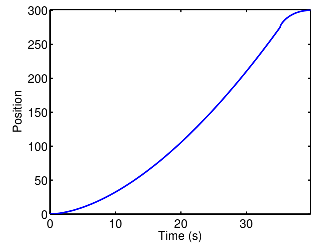
A new formulation towards solving a wide class of fractional optimal control problems has been introduced. The formulation made use of an analytical impulse response based-approximation to model the fractional dynamics of the system in terms of a state space realization. This approximation created a bridge with classical optimal control problem and a readily-available optimal control solver was used to solve the fractional optimal control problem. The methodology allowed to reproduce results from the literature as well as solving a more complex problem of a fractional free final time problem. Numerical results show that the methodology, though simple, achieves good results. For all examples, the solution for the integer order case of the problem is also obtained for comparison purpose.



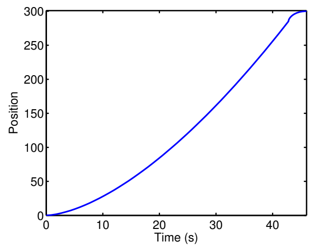
(a) $\alpha = 1.9$



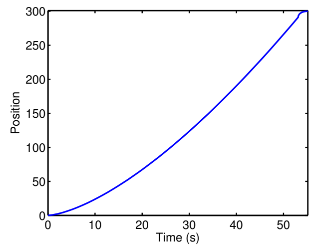
(b) $\alpha = 1.8$



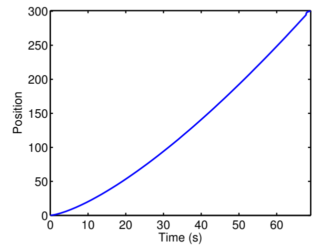
(c) $\alpha = 1.7$



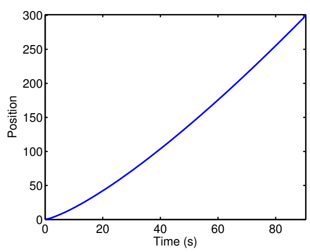
(d) $\alpha = 1.6$



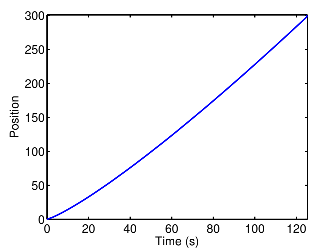
(e) $\alpha = 1.5$



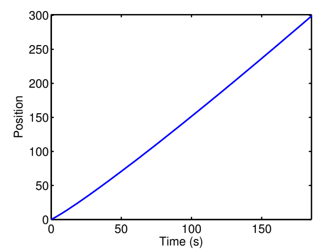
(f) $\alpha = 1.4$



(g) $\alpha = 1.3$



(h) $\alpha = 1.2$



(i) $\alpha = 1.1$

Fig. 6. State $x(t)$ as a function of time t for the Bang-Bang control problem for different orders α

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