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On the Transient Control of Linear Time Invariant Systems

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***Abstract*— Practical problems require the synthesis of a set of stabilizing controllers that guarantee transient performance specifications such as a bound on the overshoot of its closed loop step response. A majority of these specifications for Linear Time Invariant (LTI) systems can be converted to the requirement of synthesizing a set of stabilizing controllers guaranteeing the non-negative impulse response of an appropriate transfer function whose coefficients are functions of the controller parameters. The main topic of investigation of this paper is to find a bound for the set of control parameters, K, so that a rational, proper transfer function, $\frac{N(z,K)}{D(z,K)}$ has a decaying, non-negative impulse response. For Single Input Single Output (SISO) LTI systems, one may assume that the coefficients of the polynomials $N(z,K)$ and $D(z,K)$ are affine in K. An earlier result by the authors provides an approximation of the set of stabilizing controller parameters in terms of unions of polyhedral sets. In this paper, we provide necessary and sufficient conditions for a rational proper stable transfer function to have a non-negative impulse response. For the synthesis problem, we show that these conditions translate into a sequence of polynomial matrix inequalities in K using Markov-Lucaks' theorem. We propose an outer approximation of the feasible set of matrix inequalities using Lasserre's moment method.**

I. MOTIVATION

The problem of controlling the transient response is important in the design of controllers for practical applications. Despite its importance, very little is known in terms of a systematic solution technique, even for Single Input Single Output (SISO) Linear Time Invariant (LTI) systems. A large class of problems involving transient response can be posed in this form:

Consider the problem of synthesizing a (stabilizing) first order controller, $C(z) = \frac{az+b}{z+c}$, for a plant whose transfer function is

$$H_p(z) = \frac{N_p(z)}{D_p(z)}.$$

The feedback system corresponding to this controller is shown in Fig. 1.

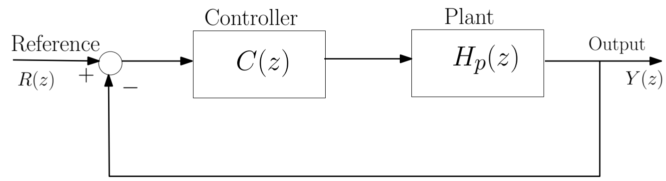

Fig. 1. Feedback control system.

Suppose $s_l(k)$ and $s_u(k)$ be the lower and upper bounds for the step response of the closed loop system and let $\frac{N_l}{D_l}(z)$ and $\frac{N_u}{D_u}(z)$ be the Z transforms of $s_l(k)$ and $s_u(k)$. Then this problem may be posed as requiring the impulse response of the following two transfer function

$$H_1(z) = \frac{z}{z-1}\frac{(az+b)N_p(z)}{(az+b)N_p(z)+(z+c)D_p(z)} - \frac{N_l}{D_l}(z),$$

and

$$H_2(s) = \frac{N_u}{D_u}(z) - \frac{z}{z-1}\frac{(az+b)N_p(z)}{(az+b)N_p(z)+(z+c)D_p(z)}$$

to be non-negative.

The problem of synthesizing a system with a desired undershoot and overshoot can be solved by specifying $s_l(k) = -\epsilon_u$ and $s_u(k) = (1+\epsilon_o)$, where ϵ_u and ϵ_o is the tolerable undershoot and overshoot respectively for the step response of the closed loop system. Similarly, one may specify the speed of response by setting $s_l(k) = 1-\lambda^k$, where $\lambda > 0$ is chosen appropriately to reflect the desired speed of the closed loop response.

A major difficulty with designing controllers guaranteeing transient specification for LTI system is that the currently available techniques only deal with techniques that alter the frequency response of the closed loop system and hence are indirect. A part of the problem lies in the unavailability of a characterization of the non-negativity of impulse response of a transfer function that can be used for synthesis. The work of S. N. Bernstein and the generalization of this result by Widder [1] provide us with a characterization of non-negative impulse response of a rational, proper transfer function through the complete monotonicity of the transfer function. A similar characterization for discrete-time LTI systems is lacking and will be provided in this paper.

The problem of synthesizing a controller which renders the step response of an LTI system was considered earlier in [2], [3], [4]. These results point to the existence of a 2-parameter stabilizing compensator of a *sufficiently high order* that can achieve a monotonically increasing step response. Non-overshooting step response is possible only if the LTI plant has no real, non-minimum phase zeros. Other approaches that have been adopted for this problem especially for discrete-time LTI systems is to place the poles at zero and use additional freedom in parameters to ensure that the numerator polynomial is non-negative [5]. Fixed order and PID controller synthesis for satisfying transient specifications in continuous time LTI systems was considered by the authors in an earlier conference paper [6].

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Controllers of fixed structure such as PID controllers and first order controllers have been employed successfully in many industrial control applications. This paper builds on an earlier paper of the authors [7], where a technique has been presented to approximate the set of fixed order/structure controllers for SISO systems through polyhedral sets in the space of controller parameters. The specific problem we address is as follows: Let $N(z,K) := N_0(z) + \sum_{i=1}^{l} K_i N_i(z)$ and similarly let $D(z,K) := D_0(z) + \sum_{i=1}^{l} K_i D_i(z)$. Here $N_i(z), D_i(z),\ i = 0,1,2,\ldots,l$ are given polynomials (part of the data) and the degree of $N(z,K)$ is always at most the degree of the polynomial $D(z,K)$. Given a *bounded* polyhedral set $\mathcal{F} = \{K : FK \leq g\}$ where F, g are appropriate matrices of finite dimension and K is the vector of controller parameters that render $D(z,K)$ Schur, find a *non-trivial* approximate set $\mathcal{F}_a \subset \mathcal{F}$ that contains $\mathcal{F}_b := \{K \in \mathcal{F} : \frac{N(z,K)}{D(z,K)}$*has a non-negative impulse response*$\}$.

The primary motivation to approximate the sets of controllers is that it provides flexibility to the control engineer to incorporate additional specifications without having to start from scratch. An immediate application of the characterization result we develop along the lines of the results of Bernstein and Widder [1] leads to a sequence of numerator polynomials (whose coefficients depend polynomially on the controller parameters) that must have no real, positive non-minimum phase zeros. The Markov-Lucaks theorem [8] guarantees a sum-of-squares representation for univariate polynomials that are non-negative on any interval of the real axis. Using such a representation, one can obtain a sequence of polynomial matrix inequalities, the first of which is a Linear Matrix Inequality (LMI), the second one being a Quadratic Matrix Inequality(QMI) and so on. If one considers the feasible set of the LMI and intersects it with the polyhedral set $\mathcal{F}$, one readily obtains an outer approximation $\mathcal{F}_a$. If $\mathcal{F}_a$ is non-empty, one may consider the QMI and use it to generate a cut if possible to obtain a refinement of $\mathcal{F}_a$; in such a case, one may pick a $K_0 \in \mathcal{F}_a$ and if such a K_0 does not satisfy the QMI. Say the QMI is $Q(K) \geq 0$, while for some v, $v^T Q(K_0) v < 0$. The vector v may just be the eigen-vector corresponding to a negative eigenvalue of $Q(K_0)$. To eliminate K_0 from the outer approximation, one may enforce the cut constraint $v^T Q(K) v \geq 0$, which is a quadratic inequality constraint. Since $\mathcal{F}_a$ is bounded by virtue of $\mathcal{F}$ being bounded, for some large enough M, one has $M^2 - \|K\|^2 \geq 0$, another quadratic inequality. From the results of Lasserre, the convex hull of the feasible set of two quadratic inequalities can be representable by the feasible set of a Linear Matrix Inequality. This provides a semi-definite constraint. It will be a proper cut if K_0 is eliminated. Nevertheless, the addition of this semi-definite constraint provides a refinement of the outer approximation. Further advances in approximating semi-algebraic sets can be utilized for the refinement of the outer approximation $\mathcal{F}_a$.

This paper is organized as follows: In section II, we provide the main results. In section III, we provide some corroborating numerical results.

II. Main Results

A. Discrete-time LTI systems with a non-negative response

We will begin with a characterization of a discrete-time LTI system with a non-negative impulse response. Let $H(z)$ be the z-transformation of the impulse response, $h(k)$, of a discrete-time causal, LTI system. Since $H(z)$ is analytic in $|z| > R$ for sufficiently large R, the series

$$\sum_{l=0}^{\infty} h(l) z^{-l}$$

converges to $H(z)$. Let us define inductively the following sequence of transfer functions $\{H_k(z)\}_{k=0}^{\infty}$ associated with $H(z)$ as follows:

$$\begin{aligned} H_0(z) &:= H(z), \\ H_{k+1}(z) &:= -z\frac{dH_k(z)}{dz}, \ \ \forall k \geq 0. \end{aligned}$$

The following well-known result is useful in characterization:

Lemma 1. *Let $G(z)$ be a rational, proper transfer function with a decaying impulse response, $g(k)$. If $G(z_0) = 0$ for some $z_0 \geq 1$, then $g(k)$ changes sign at least once.*

Proof. Since $z_0 > 1$ and $g(k)$ is decaying, the series

$$\sum_{k=0}^{\infty} g(k) z_0^{-k}$$

converges and from the definition of Z-transform, it equals $G(z_0)$. If $g(k)$ does not change sign, then $G(z_0) \neq 0$ since z_0^k is always positive. But $G(z_0) = 0$ by the hypothesis. Hence, $g(k)$ must change sign at least once. □

Remark 1. It is sufficient for the impulse response to change sign if a real, positive zero of $G(z)$, namely z_0, lies outside the disk containing all its poles. To see this, one can see that $g(k) z_0^{-k}$ will be decaying exponentially and hence the series $\sum_{k=0}^{\infty} g(k) z_0^{-k}$ converges to $G(z_0)$. The rest of the proof carries through.

Theorem 1. *Suppose $H(z)$ is analytic in $|z| \geq 1$. The impulse response $h(k)$ is non-negative if and only if the sequence of transfer function $H_k(z)$ do not have a real positive zero outside the unit disk $|z| \geq 1$.*

Proof. (Necessity) We note by induction that for all $|z| \geq 1$

$$H_k(z) = \sum_{l=0}^{\infty} l^k h(l) z^{-l}.$$

Clearly, for $k = 0$, the above holds. Suppose it holds for $k = 0,1,2,\ldots,m$. Consider

$$\begin{aligned} -z\frac{dH_m(z)}{dz} &= -z\frac{d}{dz}[\sum_{l=0}^{\infty} l^m h(l) z^{-l}] \\ &= \sum_{l=0}^{\infty} l^{m+1} h(l) z^{-l} = H_{m+1}(z). \end{aligned}$$

We further note that $H_m(z)$ is also analytic in $|z| \geq 1$ and its impulse response $\{l^m h(l)\}_{l=0}^{\infty}$ decays to zero asymptotically. From the earlier lemma, if $H_m(z)$ has at least one real, positive zero outside the unit disk for any m, its impulse response $l^m h(l)$ will change sign implying that the impulse response, $h(l)$, of $H(z)$ will also change sign.
(Sufficiency) Associated with every natural number t, one can define for every k,

$$\mathcal{D}_{k,t}(H(z)) := (\frac{e}{t})^k H_k(e^{\frac{k}{t}}),$$

and further define

$$\mathcal{D}_t(H(z)) := \lim_{k\to\infty} \mathcal{D}_{k,t}(H(z)).$$

Clearly,

$$\begin{aligned}\mathcal{D}_{k,t}((H(z)) &= \sum_{l=0}^{\infty} h(l) l^k e^{-\frac{lk}{t}} (\frac{e}{t})^k \\ &= \sum_{l=0}^{\infty} h(l)(\frac{le}{t} e^{-\frac{l}{t}})^k.\end{aligned}$$

Let $y = \frac{l}{t}$ and consider the sequence of functions $\phi_k(y) := (ye^{-(y-1)})^k,\ k = 0, 1, \ldots$ It is clear that $\phi_k(y) = \phi_0(y)^k$. It is easy to notice that $\phi_0(y)$ is monotonically increasing in $[0, 1]$ and monotonically decreasing in $[1, \infty)$ and has exactly one maximum at $y = 1$. The corresponding maximum value of $\phi_0(y)$ is 1. Hence, as $k \to \infty$, the function $\phi_k(y)$ approaches the Kronecker delta function $\delta(y - 1)$ (which equals one if $y = 1$ and equals 0 otherwise).

With this observation, it is easy to see that $\mathcal{D}_t(H(z))$ approaches $\sum_{l=0}^{\infty} h(l)\delta(\frac{l}{t} - 1) = \sum_{l=0}^{\infty} h(l)\delta(l - t) = h(t)$ for every natural number t. Suppose there is a sign change in the impulse response; then there must exist a t_1 and $t_2 > t_1$ such that $h(t_1)h(t_2) < 0$. Clearly, for a sufficiently large k, it must be the case that $H_k(e^{\frac{k}{t_1}})H_k(e^{\frac{k}{t_2}}) < 0$; otherwise, the limit will not hold. Hence, for all sufficiently large k, there will be a change in sign of $H_k(z)$ for some real positive z lying between $e^{\frac{k}{t_2}}$ and $e^{\frac{k}{t_1}}$. □

B. Results concerning controlling the overshoot of a discrete-time LTI system

The problem of achieving non-overshooting step response is important in control systems and several results have been reported [9], [2], [10], [11], [12]. The question of whether overshoot can be eliminated or not was first was answered by Deodhare and Vidyasagar [13]. They established that, for discrete-time LTI systems, there is a deadbeat closed loop system that can be synthesized which has a non-overshooting step response. Techniques from l_1 optimal control were applied to arrive at this result, since their focus was primarily on minimizing the l_1 norm of the error response to a step input. The continuous-time counterpart of their results for discrete-time LTI systems was established by the authors in [14] directly. This approach is used in this paper to establish directly the discrete-time counterpart of the result.

Given any plant $P(z) = \frac{N_p(z)}{D_p(z)}$; consider a 2-parameter compensator as shown in Fig. 2. To avoid trivially unsolvable

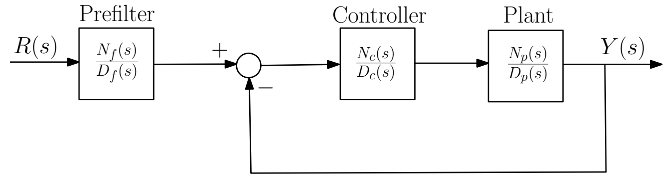

Fig. 2. Block diagram of the compensation scheme

problems, we only consider plants that satisfy the following conditions:

1) Are stabilizable by feedback controllers, and
2) Do not have a zero at unity.

Plants satisfying the above conditions will be called admissible plants.

Theorem 2. *For every admissible plant, there is a two-parameter compensator, as shown in Fig. 2, that renders the closed loop step response non-overshooting.*

Proof. The proof is by construction. Let $\frac{N_c}{D_c}(z)$ be a feedback controller so that $N_p(z)N_c(z) + D_p(z)D_c(z)$ is Schur, and write $N_p(z)N_c(z)$ as $n_0 + n_1 z + \ldots + n_q z^q$. Choose

$$N_f(z) = N_p(z)N_c(z) + D_p(z)D_c(z),$$

and $D_f(z) = (z - \alpha)^p$, where $1 > \alpha > 0$ can be chosen and the degree, p, of $D_f(z)$ is greater than or equal to that of $N_f(z)$.

This choice results in the following transfer function from the reference to the output:

$$P_{cl}(z) = \frac{N_p(z)N_c(z)}{N_p(z)N_c(z) + D_p(z)D_c(z)} \frac{N_f(z)}{D_f(z)},$$

and is stable.

To track a step input, it is necessary to have $N_c(1) \neq 0$, and this always can be achieved by a small perturbation of the constant numerator coefficient of the controller, since the plant satisfies condition 2 above. In essence, one may assume that $\bar{n}_0 := n_0 + n_1 + \ldots + n_q \neq 0$.

To determine if the closed loop system has a non-overshooting step response, it is sufficient to verify that the response to a step input of magnitude $\frac{(1-\alpha)^p}{\bar{n}_0}$ does not exceed unity at any instant of time. Therefore, it is sufficient to check if the following error transfer function has a non-negative impulse response:

$$E(z) = \frac{z}{z-1} - \frac{n_0 + n_1 z + \ldots + n_q z^q}{(z-\alpha)^p} \frac{(1-\alpha)^p}{\bar{n}_0} \frac{z}{z-1}.$$

Upon simplification,

$$\begin{aligned}E(z) &= \frac{z}{z-1} \frac{1}{(n_0 + \ldots + n_q)(z - 1 + \alpha)^p}[\\ &\quad (n_0 + n_1 + \ldots + n_q)(z-\alpha)^p \\ &\quad -\alpha^p(n_0 + n_1 z + \ldots + n_q z^q)].\end{aligned}$$

Therefore,

$$E(\alpha z) = \frac{\alpha z}{\alpha z - 1} \frac{X(z)}{(n_0 + n_1 + \ldots + n_q)(z-1)^p \alpha^p},$$

where

$$\begin{aligned} X(z) &:= (n_0+\ldots+n_q)(z-1)^p\alpha^p \\ &\quad -(1-\alpha)^p(n_0+n_1\alpha z+\ldots+n_q\alpha^q z^q). \end{aligned}$$

The polynomial $X(z)$ may be simplified as

$$\begin{aligned} X(z) &= (n_0+\ldots+n_q)[(\alpha z-\alpha)^p-(1-\alpha)^p] \\ &\quad +(1-\alpha)^p[(n_0+\ldots+n_q) \\ &\quad -(n_0+n_1\alpha z+\ldots+n_q\alpha^q z^q)] \\ &= (\alpha z-1)[(n_0+\ldots+n_q)[\alpha^{p-1}(z-1)^{p-1}+ \\ &\quad \alpha^{p-2}(z-1)^{p-2}(1-\alpha)+\ldots+(1-\alpha)^{p-1}] \\ &\quad -(1-\alpha)^p F(\alpha,z), \ \textit{where} \\ F(\alpha,z) &= [n_1+n_2(\alpha z+1)+\ldots \\ &\quad +n_q(\alpha^{q-1}z^{q-1}+\ldots+1)]. \end{aligned}$$

If one writes $F(\alpha,z)=F_0(\alpha)+F_1(\alpha)(z-1)+\ldots+F_{q-1}(\alpha)(z-1)^{q-1}$, where $F_i=\frac{1}{i!}\frac{d^iF}{dz^i}|_{z=1}$, the polynomial $X(z)$ may be expressed as:

$$\begin{aligned} X(z) &= (\alpha z-1)\sum_{l=0}^{p-1}[(n_0+\ldots+n_q)\alpha^l(1-\alpha)^{p-1-l} \\ &\quad -(1-\alpha)^pF_l](z-1)^l. \end{aligned}$$

In the above expression, $F_l=0$ for every $l>q-1$.

In essence, the expression for $E(\alpha z)$ may now be simplified as:

$$\begin{aligned} E(\alpha z) &= \frac{\alpha z}{\bar{n}_0\alpha^p}\sum_{l=0}^{p-1}[(n_0+\ldots+n_q)\alpha^l(1-\alpha)^{p-1-l} \\ &\quad -(1-\alpha)^pF_l]\frac{1}{(z-1)^{p-l}} \\ &:= \sum_{l=0}^{p-1}\frac{C_l}{(z-1)^{p-1-l}}, \end{aligned}$$

where the coefficients C_l are given by:

$$C_l=\frac{\alpha}{\bar{n}_0\alpha^p}[\bar{n}_0\alpha^l(1-\alpha)^{p-1-l}-(1-\alpha)^pF_l].$$

Let $e(k)$ be the inverse Z-transformation of $E(z)$. Since the Z-transformation of $\frac{e(k)}{\alpha^k}$ is $E(\alpha z)$, it is clear that $e(k)$ will be non-negative if the coefficients $C_l\geq 0$ for all l. From the formulae for C_l, if one treats the second term on the right hand side to be a perturbation to the first term, the second term is of the order $(1-\alpha)^p$ or higher, while the first term is positive and of order less than or equal to $(1-\alpha)^{p-1}$. Clearly, by making $1-\alpha$ sufficiently small but positive, all the coefficients, C_l, can be made positive. This will ensure that the error to a step input will always be positive. This implies that the step response will not overshoot.

Therefore, every admissible continuous-time LTI plant can be controlled to have a non-overshooting step response. $\square$

Remark 2.
1) As can be seen from the proof, choosing a sufficiently small $1-\alpha>0$ is equivalent to sufficiently slowing down the speed of the response; this is the mechanism we have adopted in our constructive proof.
2) The degree of the two parameter stabilizing compensator guaranteeing a non-overshooting step response can be bounded by $3n-1$, where n is the order of the transfer function of the plant.

The synthesis of a stable, non-overshooting closed loop step response can be accomplished if one has no restrictions on slowing down the input or on the order of the compensator that can be chosen. In the following section, we will deal with this case.

C. Approximation of the set of controllers guaranteeing a non-negative impulse response

The problem statement for the synthesis of the set of stabilizing controllers guaranteeing a non-negative impulse response is as follows:

PROBLEM: *Given a proper, rational transfer function*

$$H(z,K)=\frac{N(z,K)}{D(z,K)},$$

where the coefficients of $N(z,K)$ and $D(z,K)$ are affine in the controller parameter vector K, determine the set of K's such that the impulse response, $h(k)$, of $H(z,K)$ is non-negative and decaying.

Through a bilinear transformation, the problem of rendering a polynomial Schur can be converted to some other polynomial being made Hurwitz. The results of [7] can be brought to bear to find an approximate set of stabilizing controllers; recently, using Chebychev polynomials, a direct method of approximating the set of controllers, K, that render $D(z,K)$ Schur is presented in [15]. We can approximate the set of stabilizing controllers as $\cup_{i=1}^{N}\mathcal{P}_i$, where $\mathcal{P}_\rangle$ is a polyhedral set with a description $F_iK\leq g_i$ for some appropriate vector g_i and an appropriate matrix F_i. With this as a starting point, we can focus on how to make the impulse response of $h(k)$ non-negative.

The following lemma is of relevance and is also proved in [4]:

Lemma 2. *If $h(k)$ does not change sign, then there is a real positive root z_0 of $D(z,K)$ of maximum modulus. Moreover, if there are more than one root that is of maximum modulus, the multiplicity of the real positive root is maximum.*

We will also need the following result of Markov-Lucaks based on Feijer's theorem:

Lemma 3. *A polynomial $N(x)=\sum_{l=0}^{m}n_lx^l$ be non-negative in the interval $[1,\infty)$ if and only if there exist polynomials $f_1(x)$ of degree at most $\frac{m}{2}$, and $f_2(x)$ of degree at most $\frac{m-1}{2}$ such that $N(x)=f_1(x)^2+(x-1)f_2(x)^2$.*

A consequence of the result of Markov-Lucaks theorem is that the non-negativity of a polynomial on the interval $[1,\infty)$ can be checked using an appropriate semi-definite program. Let R be the smallest integer greater than or equal to $\frac{m}{2}$ and S be the smallest integer greater than or equal to $\frac{R-1}{2}$. Let $M_l(x)$ be a vector of monomials $[1\ x\ x^2\ \ldots x^l]$. Then, for some appropriate semi-definite matrices $Y_1,Y_2\geq$

0 of dimensions R and S, one may express $f_1(x)^2 = M_R(x)Y_1M_R(x)^T$ and $f_2(x)^2 = M_S(x)Y_2M_S(x)^T$. Comparing the coefficients of $f_1(x)^2+(x-1)f_2(x)^2$ and equating them to that of $N(x)$, one arrives at a set of linear equations relating the coefficients of $N(x)$ to the entries of Y_1, Y_2. In essence, one can establish the non-negativity of a polynomial $N(x)$ on $[1,\infty)$ by checking the feasibility of a semi-definite program $Y_1 \geq 0,\ Y_2 \geq 0$ and the linear system of equations that relate the entries of the matrices Y_1, Y_2 with the coefficients of the given polynomial.

We now state a lemma that is useful in computing the set of stabilizing controllers guaranteeing a non-negative impulse response:

Lemma 4. *Let $N(z,K) = \sum_{l=0}^{n} a_l(K)z^l$. If K is a stabilizing controller in a polyhedral set $\mathcal{F}_i$ such that $D(1,K) > 0$ for all $K \in \mathcal{F}_i$, and K renders the impulse response of $\frac{N(z,K)}{D(z,K)}$ non-negative, then K is in the feasible set of the semi-definite program that enforces the requirement that $N(z,K)$ is non-negative on $[1,\infty)$.*

Proof. If K renders the impulse response of $\frac{N(z,K)}{D(z,K)}$ non-negative, clearly, it cannot have any real, positive zeros greater than 1 and hence must be non-negative. As we have seen, the non-negativity of $N(z,K)$ can be verified using a semi-definite program. Since $K \in \mathcal{P}_i$, a polyhedral set, and an Linear Program can always be expressed as a semi-definite program and furthermore, the intersection of two semi-definite programs is a semi-definite program, one can get a first approximation of the set of the stabilizing controllers guaranteeing the non-negativity of the impulse response of $\frac{N(z,K)}{D(z,K)}$ as a union of semi-definite programs with one obtaining one semi-definite program corresponding to each $\mathcal{P}_i$. □

One may sometimes be able to refine the approximation by considering the transfer function $H_1(z,K) = -z\frac{d}{dz}\frac{N(z,K)}{D(z,K)} = \frac{N_1(z,K)}{D(z,K)^2}$. Clearly, the numerator, $N_1(z,K)$ of this transfer function must also be non-negative. However, the numerator of this transfer function has coefficients that are quadratic in K leading one to a quadratic matrix inequality of the form:

$$\sum_{\beta\in\mathcal{I}} K^\beta S_\beta + \sum_{ij} T_{ij}Y_{ij} \geq 0,$$

where $\mathcal{I}$ is the index set of all the monomials of K of degree at most two, K^β is a monomial of degree at most two and S_β, T_{ij} are symmetric matrices of appropriate dimension and Y_{ij} are the entries of the semi-definite matrices associated with the semi-definite program obtained by applying the Markov-Lucaks theorem. The constant term is the zeroth degree term (i.e., K^0).

Suppose $\mathcal{P}_{outer}$ is an outer approximation of the desired set of controllers. If it is empty, we know that there cannot be any stabilizing controller that satisfies the desired transient specification. To make it non-trivial, let $\mathcal{P}_{outer}$ be non-empty. Pick a $K_0 \in \mathcal{P}_{outer}$. If it is such that $\sum_{\beta\in\mathcal{I}} K_0^\beta S_\beta + \sum_{i,j} T_{ij}Y_{ij}$ is not positive semi-definite for any Y_{ij} entries, then there is a semi-definite $R \geq 0$ such that

$$\sum_{\beta\in\mathcal{I}} K_0^\beta \underbrace{Trace(RS_\beta)}_{\gamma_\beta} < 0,$$

and $Trace(RT_{ij}) = 0$ for all i,j. One can therefore enforce the constraint

$$\sum_{\beta\in\mathcal{I}} K^\beta\gamma_\beta \geq 0,$$

to eliminate K_0 in the approximation thereby refining it.

A characterization of the convex hull of a quadratic inequality through a semi-definite program was provided by Lasserre through the use of the theory of moments [16]. Let $\mathcal{P}_i$ be bounded, i.e., there is a $M > 0$ such that $M - \|K\|^2 \geq 0$. Consider the set $\mathcal{K} \subset \mathcal{P}_i$ such that for all $K \in \mathcal{K}$, we have

$$\begin{aligned} P_1(K) &= \sum_{\beta\in\mathcal{I}} K^\beta\gamma_\beta \geq 0, \\ P_2(K) &= M - \|K\|^2 \geq 0. \end{aligned}$$

Let $y := \{y_\beta,\ \beta \in \mathcal{I}\}$ and let $\mathcal{L}_y : \Re[K] \to \Re$ be the linear functional that maps a polynomial $P = \sum_\beta p_\beta K^\beta \in \Re[K]$ to $L_y(P) = s\sum_\beta p_\beta y_\beta$. One can then introduce a moment matrix, $M_r(y)$ with rows and columns also indexed according to the monomials so that

$$M_r(y)(i,j) := \mathcal{L}_y(K^{i+j}) = y_{i+j},\ 0 \leq i,j \leq r.$$

For example, if one is dealing with inequalities in two variables,

$$M_1(y) = \begin{bmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix}.$$

Theorem 3. *The convex hull of $\mathcal{K}$ (given by $\Omega \subset \Re^n$) is representable as the feasible set of the lifted following semi-definite program:*

$$\Omega = \{(x,y) : \begin{array}{ccc} M_1(y) & \geq & 0, \\ L_y(P_j(K) & \geq & 0, j = 1,2 \\ L_y(K_i) & = & x_i,\ i = 1,\ldots,n, \\ y_{00} = 1 & & \end{array} \}.$$

One can now refine any outer approximation of the set of stabilizing controllers guaranteeing the transient performance by intersecting it with the convex hull of $\mathcal{K}$ given by the semi-definite program above.

To use Lemma 2, we construct a polynomial $\tilde{D}(s,K) = (1-s)^n D(\frac{s+1}{1-s}, K)$ and we claim the following:

Lemma 5. *Let $D(z,K)$ be Schur and of degree n. If a real, positive root, ρ, of $D(z,K)$ is of maximum modulus then the polynomial $\tilde{D}(s,K)$ has roots with real parts less than or equal to $\frac{\rho-1}{\rho+1}$.*

Proof. Let $re^{j\theta}$ be a root of $D(z,K)$. The roots of $D(z,K)$ and $\tilde{D}(s,K)$ are related through $r(\cos\theta + j\sin\theta) = \frac{s+1}{1-s}$. Then $\Re(s) = \frac{r^2-1}{r^2+2r\cos\theta+1} \leq \frac{r-1}{r+1}$. Since $r \leq \rho < 1$, the real part of s is a maximum, when $\theta = 0$. Moreover, for $\theta = 0$, the real part of s is an increasing function of r. Therefore, a real positive root of $D(z,K)$ is of maximum modulus if and only if $\Re(s) \leq \frac{\rho-1}{\rho+1}$. $\square$

One can now pose the problem of determining the set of controllers in $\mathcal{F}_i$ that have a dominant real pole as a search for α so that a linear program parameterized in α is feasible. The following lemma deals with this problem:

Lemma 6. *Let $\mathcal{P}_i$ be a set of controllers K such that $\tilde{D}(s,K)$ is Hurwitz. If $\mathcal{L} \subset \mathcal{P}_i$ is the set of controllers such that $\tilde{D}(s,K)$ has a generically dominant real root, then there exists an $\alpha < 0$ such that any $K \in \mathcal{L}$ must satisfy one of the following additional set of linear constraints. Let $D_\alpha(s,K) := \tilde{D}(s+\alpha,K) := d_0(\alpha,K)+d_1(\alpha,K)s+\cdots+d_n(\alpha,K)s^n$.*
LP1:

$$d_0(\alpha,K) < 0, d_1(\alpha,K) > 0, d_2(\alpha,K) > 0, \cdots, d_n(\alpha,K) > 0.$$

LP2:

$$d_0(\alpha,K) > 0, d_1(\alpha,K) < 0, d_2(\alpha,K) < 0, \cdots, d_n(\alpha,K) < 0.$$

One can determine a union of these polyhedral sets which will contain the desired set of controllers. Intersecting this union with the union of feasible sets of the semi-definite programs constructed earlier, one can find an outer approximation for the set of stabilizing controllers K that render the impulse response of $\frac{N(z,K)}{D(z,K)}$ non-negative.

III. RESULTS

Example 1. Consider the plant:

$$G(z) = \frac{1}{z^2 - 0.25}.$$

The controller is considered to be of the following PID structure:

$$C(z) = \frac{k_3 z^2 + k_2 z + k_1}{z^2 - z}.$$

The characteristic polynomial is

$$z^4 - z^3 + (k_3 - 0.25)z^2 + (k_2 + 0.25)z + k_1.$$

The inner approximation of the set of stabilizing controllers is found using the algorithm developed by the authors in [15]. This approximation of the set of stabilizing controllers is shown in Fig. 3 in the lighter shade (red color).

Within this set of stabilizing controllers lies the set of controllers which renders the impulse response of the corresponding closed loop system to be non-negative. This set is generated using the method developed in this paper. It is shown in Fig. 3 in the darker color (blue). This set of controllers is an outer approximation of the required set and can be refined using the results of Theorem 3.

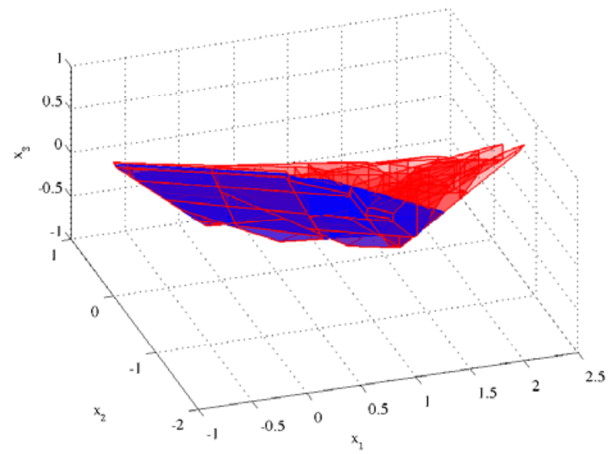

Fig. 3. Set of controllers.

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