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On Semi-global Stabilization of Minimum Phase Nonlinear Systems without Vector Relative Degrees

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***Abstract*— Recently, we developed a structural decomposition for multiple input multiple output nonlinear systems that are affine in control but otherwise general. This structural decomposition simplifies the conventional backstepping design and allows a new backstepping design procedure that is able to stabilize some systems on which the conventional backstepping is not applicable. In this paper we further exploit the properties of such a decomposition for the purpose of solving the semi-global stabilization problem for minimum phase nonlinear systems without vector relative degrees. By taking advantage of special structure of the decomposed system, we first apply the low gain design to the part of system that possesses a linear dynamics. The low gain design results in an augmented zero dynamics that is locally stable at the origin with a domain of attraction that can be made arbitrarily large by lowering the gain. With this augmented zero dynamics, backstepping design is then apply to achieve semi-global stabilization of the overall system.**

I. Introduction and Problem Statement

In this paper, we consider the problem of semi-globally stabilizing a nonlinear system of the affine-in-control form

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x)+g(x)u, \\ y &=& h(x), \end{array}\right. \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and $y\in\mathbb{R}^p$ are the state, input and output, respectively, and the mappings f, g and h are smooth with $f(0)=0$ and $h(0)=0$. In a semi-global stabilization problem, we are to construct, for any given, *arbitrarily large*, bounded set of the state space $\mathcal{X}_0$, a smooth feedback law, say $u=v_{\mathcal{X}_0}(x)$, with $v(0)=0$, such that the closed-loop system is asymptotically stable at the origin with $\mathcal{X}_0$ contained in the domain of attraction.

The non-local stabilization of nonlinear systems of the form (1) has been made possible by the structural decomposition, in the form of various normal forms, of these systems. Indeed, there is a vast literature on the development of the normal forms for affine-in-control nonlinear systems ([1–13]), which explores the nonlinear analogous of linear systems structural properties, establishes the nonlinear equivalence of linear system structures, and identifies more intricate structural properties that linear systems do not display. There is also a vast literature on the applications of the discovered structural properties to solve nonlinear control problems (see, e.g., [14–23]).

The development of nonlinear system structural decomposition started with the definition of relative degrees, the nonlinear equivalence of infinite zeros, and the normal form decomposition for the single input single output case, *i.e.*, $m=p=1$ [4]. This definition of relative degrees was soon generalized to the case with $m=p>1$. In general, the system (1) with $m=p\geq 1$ has a vector relative degree [6, 14] $\{r_1,r_2,\cdots,r_m\}$ at $x=0$ if

$$L_{g_j}L_f^k h_i(x)=0,\ 0\leq k<r_i-1,\ 1\leq i,j\leq m$$

in a neighborhood of $x=0$, and $\det\{L_{g_j}L_f^{r_i-1}h_i(0)\}_{m\times m}\neq 0$. If the system (1) has a vector relative degree $\{r_1,r_2,\cdots,r_m\}$ at $x=0$, and with the assumption of the distribution spanned by the row vectors of $g(x)$ being involutive in a neighborhood of $x=0$, it can be described by

$$\left\{\begin{array}{rcll} \dot{\eta} &=& f_0(x), & \\ \dot{\xi}_{i,j} &=& \xi_{i,j+1}, & j=1,2,\cdots,r_i-1, \\ \dot{\xi}_{i,r_i} &=& v_i, & \\ y_i &=& \xi_{i,1}, & i=1,2,\cdots,m, \end{array}\right. \tag{2}$$

where $v_i=a_i(x)+b_i(x)u,\ \ i=1,2,\cdots,m$, with the matrix $\mathrm{col}\{b_1(x),b_2(x),\cdots,b_m(x)\}$ being smooth and nonsingular.

Even though the definition of relative degree and the resulting normal form are nonlinear equivalence of the notion of infinite zeros and the related canonical form for single input single output systems, the vector relative degree for multiple input multiple output systems is a rather strong structural property that not even all square invertible linear systems, with the freedom of choosing coordinates for the state, output and input spaces, could possess [24].

A major generalization of the form (2) was made in [9, 12, 13], where square invertible systems are considered. By using the *Zero Dynamics Algorithm*, under the assumptions that the ranks of certain matrices are constant and that the distribution spanned by the row vectors of $g(x)$ is involutive, the system can be transformed into the following form

$$\left\{\begin{array}{rcll} \dot{\eta} &=& f_0(x), & \\ \dot{\xi}_{i,j} &=& \xi_{i,j+1}+\displaystyle\sum_{l=1}^{i-1}\delta_{i,j,l}(x)v_l, & j=1,2,\cdots,n_i-1, \\ \dot{\xi}_{i,n_i} &=& v_i, & \\ y_i &=& \xi_{i,1}, & i=1,2,\cdots,m, \end{array}\right. \tag{3}$$

where $n_1\leq n_2\leq\cdots\leq n_m$, $v_i=a_i(x)+b_i(x)u,\ i=1,2,\cdots,m$, with the matrix $\mathrm{col}\{b_1(x),b_2(x),\cdots,b_m(x)\}$ being smooth and nonsingular.

As pointed out in [9], when all $\delta_{i,j,l}(x)=0$, the set of integers $\{n_1,n_2,\cdots,n_m\}$ in (3) corresponds to the vector relative degrees, which in this case, represent the infinite zero structure if the system is linear. These integers however are not related to the infinite zero structure of linear systems

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when $\delta_{i,j,l}(x) \neq 0$, and thus cannot be viewed as the nonlinear equivalence of and expected to play a similar role as infinite zeros (see [24] for an example showing this).

In a recent paper [24], we study the structural properties of affine-in-control nonlinear systems beyond the case of square invertible systems. We propose an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form representation that corresponds to these integers as well as to the system invertibility structure. This new normal form representation takes the following form

$$\begin{cases} \dot{\eta} = f_\vartriangle(\eta, z_\mathrm{d}) + g_\vartriangle(\eta, z_\mathrm{d}) u_\vartriangle, \\ \dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_{\mathrm{d},l}, \ j = 1, 2, \cdots, q_i - 1, \\ \dot{\xi}_{i,q_i} = v_{\mathrm{d},i}, \\ y_\vartriangle = h_\vartriangle(\eta, z_\mathrm{d}), \\ y_{\mathrm{d},i} = \xi_{i,1}, \qquad i = 1, 2, \cdots, m_\mathrm{d}, \end{cases} \tag{4}$$

where $q_1 \leq q_2 \leq \cdots \leq q_{m_\mathrm{d}}$, $\xi_i = \{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $z_\mathrm{d} = \{\xi_1, \xi_2, \cdots, \xi_{m_\mathrm{d}}\}$, $v_{\mathrm{d},i} = a_i(x) + b_i(x) u$, with the matrix col $\{b_1(x), b_2(x), \cdots, b_{m_\mathrm{d}}(x)\}$ being of full row rank and smooth, and

$$\delta_{i,j,l}(x) = 0, \quad \text{for } \ j < q_l, \ i = 1, 2, \cdots, m_\mathrm{d}. \tag{5}$$

We note here that m_d is the largest integer for which the system assumes the above form. The system is left invertible if $u_\vartriangle$ is non-existent, right invertible if $y_\vartriangle$ is non-existent, and invertible if both are non-existent. In the case that the system is square and invertible, *i.e.*, the system that was considered in [9, 12, 13], $m = p = m_\mathrm{d}$ and the parts containing $y_\vartriangle$ and $u_\vartriangle$ drop off. Thus, the normal form (4) simplifies to

$$\begin{cases} \dot{\eta} = f_\vartriangle(\eta, \xi), \\ \dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_l, \ j = 1, 2, \cdots, q_i - 1, \\ \dot{\xi}_{i,q_i} = v_i, \\ y_i = \xi_{i,1}, \qquad i = 1, 2, \cdots, m, \end{cases} \tag{6}$$

where $q_1 \leq q_2 \leq \cdots \leq q_m$, $\xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $\xi = \text{col}\{\xi_1, \xi_2, \cdots, \xi_m\}$, and

$$\delta_{i,j,l}(x) = 0, \quad \text{for } \ j < q_l, \ i = 1, 2, \cdots, m. \tag{7}$$

We note that the normal form (6) is the same as (3) except for the additional structural property (7). The $\dot{\xi}_{i,j}$ equation in (6) displays a triangular structure of the control inputs that enter the system. The property (7) imposes additional structure within each chain of integrators on how control inputs enter the system. With this additional structural property, the set of integers $\{q_1, q_2, \cdots, q_m\}$ indeed represent infinite zero structure when the system is specialized to a linear one.

In this paper, we would like to explore the application of the normal form (6)-(7) in solving the problem of semi-global stabilization for nonlinear systems (1). The normal form (6)-(7) does not require a vector relative degree. The problem of semi-global stabilization of system (1) with a vector relative degree has been well-studied in the literature. For example, the work of [14, 15] solved the semi-global stabilization problem for nonlinear systems with vector relative degrees, *i.e.*, in the form of (2), but the zero dynamics is driven only by $\xi_{i,1}$, $i = 1, 2, \cdots, m$, the states at the top of the m chains of integrators. The works of [16, 18] generalized this result of [14, 15] by allowing f_0 to be dependent on any one state of each of the m chains of integrators. More specifically, the system considered in [16, 18] can be represented as follows,

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi_{1,\ell_1}, \xi_{2,\ell_2}, \cdots, \xi_{m,\ell_m}), \\ \dot{\xi}_{i,j} = \xi_{i,j+1}, \ j = 1, 2, \cdots, r_i - 1, \\ \dot{\xi}_{i,q_i} = v_i, \\ y_i = \xi_{i,1}, \ i = 1, 2, \cdots, m, \end{cases} \tag{8}$$

where $1 \leq \ell_i \leq r_i + 1$, $i = 1, 2, \cdots, m$, and $\xi_{i,q_i+1} \equiv v_i$. The peaking phenomenon, which was identified in [15] as a main obstacle to semi-global stabilization, in such systems is eliminated by stabilizing part of linear system with a high-gain linear control and the remaining part of the linear subsystem with a small, bound nonlinear control [16]. The reference [18] shows that the same problem can be done by linear state feedback laws, which, of course, depend only on the linear states. The fundamental issue in design of such a linear state feedback law is to induce a specific time-scale structure in the linear part of the closed-loop system. This time-scale structure consists of a very slow and a very fast time scale, which are the results of a linear state feedback of the high-and-low-gain nature.

In this paper, we consider semi-global stabilization problem for the following nonlinear system,

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi_{1,\ell_1}, \xi_{2,\ell_2}, \cdots, \xi_{m,\ell_m}), \\ \dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(\eta, \xi) v_l, \ j = 1, 2, \cdots, q_i - 1, \\ \dot{\xi}_{i,q_i} = v_i, \\ y_{\mathrm{d},i} = \xi_{i,1}, \ i = 1, 2, \cdots, m, \end{cases} \tag{9}$$

where $q_1 \leq q_2 \leq \cdots \leq q_m$, $\xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $\xi = \text{col}\{\xi_1, \xi_2, \cdots, \xi_m\}$, and

$$\delta_{i,j,l}(\eta, \xi) = 0, \quad \text{for } \ j < q_l, \ i = 1, 2, \cdots, m. \tag{10}$$

$$\ell_i \leq q_1 + 1, \quad i = 1, 2, \cdots, m, \tag{11}$$

with $\xi_{i,q_i+1} \equiv v_i$.

As explained earlier, no vector relative degree is required for systems to be decomposed into the above normal form.

Note that in [16, 18], $\delta_{i,j,l} = 0$. That is, the systems considered in [16, 18] are a cascade of a linear subsystem with the zero dynamic, which is the only source of nonlinearity.

The remainder of this paper is organized as follows. In Section II, we recall the conventional backstepping design methodology. We will also describe the level-by-level backstepping approach as well as the mixed chain-by-chain and level-by-level backstepping, both of which have been developed in [25] to solve the global stabilization problem for nonlinear systems. Section III presents our solution to the semi-global stabilization problem for nonlinear systems without vector relative degrees. Some examples are used to illustrate how the proposed design approach works. A brief conclusion to the paper is drawn in Section IV.

II. PRELIMINARY RESULTS

In the section, we recall some results on the backstepping design methodology [8, 12, 20, 25]. The backstepping design method is readily applicable to systems that have vector relative degrees and are represented in the form (2), which contains m chains of integrators. Each of these chains independently controlled by a separate input. If the zero dynamics is only dependent on the states of the leading integrators of each chain, *i.e.*,

$$\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}), \tag{12}$$

and there exist smooth functions, $v_i^\star(\eta)$, with $v_i^\star(0) = 0,\ i = 1, 2, \cdots, m,$ such that $\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta), \cdots, v_m^\star(\eta))$ is globally asymptotically stable at $\eta = 0$, then it is straightforward to design a globally stabilizing feedback law $v_1(x), v_2(x), \cdots, v_m(x)$, recursively, by viewing the next integrators as a new intermediate input. Such a design procedure is thus referred to as "backstepping."

The technique of backstepping, however, cannot as easily be implemented if the system does not have a vector relative degree. An additional assumption is required. In what follows, we recall from [12] this additional assumption on the normal form (3) and the backstepping design procedure that is implemented under these assumptions.

This assumption is that the coefficient functions $\delta_{i,j,l}$ to display a certain "triangular" dependency on the state variables [9]. Under this "triangular" dependency, a feedback law $v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1};\ \xi_1, \xi_2, \cdots, \xi_i), i = 1, 2, \cdots, m$ that globally stabilizes the whole system can be constructed from $v_i^\star(\eta), i = 1, 2, \cdots, m,$ through a backstepping procedure. The procedure commences with the subsystem (12), and is followed by backstepping n_1 times through the variables in first chain of integrators to obtain $u_1^\star(\eta; \xi_{1,1}; v_2^\star(\eta), v_3^\star(\eta), \cdots, v_m^\star(\eta); \xi_1)$, and backstepping n_2 times through the variables in the second chain of integrators to obtain the feedback law $u_2^\star(\eta;\ \xi_{1,1}, \xi_{2,1};\ v_3^\star(\eta), v_4^\star(\eta), \cdots, v_m^\star(\eta); \xi_1, \xi_2)$. This procedure is continued chain by chain for $i = 1$ through m, each backstepping n_i times through i-th chain of integrators to discover the feedback law $u_i^\star(\eta;\ \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}; v_{i+1}^\star(\eta), v_{i+2}^\star(\eta), \cdots, v_m^\star(\eta); \xi_1, \xi_2, \cdots, \xi_i)$.

As the backstepping is implemented on the integrators chain by chain, we will refer to the above backstepping procedure as the chain-by-chain backstepping, and correspondingly, the "triangular" dependency of $\delta_{i,j,l}$ on the state variables the chain-by-chain triangular dependency.

Let us call all $\xi_{i,1}$, *i.e.*, the "leading" variables in each chain of integrators which connect an input to an output, the first level integrators, and call all $\xi_{i,2}$ the second level integrators, and so on. As an alternative to the chain-by-chain backstepping, in [25], we proposed to carry out the backstepping on all first level integrators, and then repeat the procedure on all second level integrators until we reach to last level of integrators. We will refer to such a backstepping procedure as the level-by-level backstepping, in contrast with the chain-by-chain backstepping procedure.

To make the level-by-level backstepping possible, the coefficients $\delta_{i,j,l}$ in (6) should satisfy the level-by-level triangular dependency [25]. Suppose that the level-by-level triangular dependency is satisfied, the level-by-level backstepping procedure for (6) can be described as follows. We will start with

$$\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta), \cdots, v_m^\star(\eta)).$$

After the first-level backstepping, we obtain the feedback laws

$$v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}),\ \ i = 1, 2, \cdots, \alpha_1,$$

where α_1 is the number of chains that contain exactly one integrator, *i.e.*, $q_1 = q_2 = \cdots = q_{\alpha_1} = 1$. For chains that contain more than one integrator, we have

$$\xi_{i,2} = \phi_{i,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}),\ i = \alpha_1 + 1, \alpha_1 + 2, \cdots, m.$$

Here, $\xi_{i,2}$ are viewed as inputs. We next proceed with backstepping on the second level integrators. After the second level backstepping, we obtain the feedback laws

$$v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2}, \cdots, \xi_{i,2}),$$
$$i = \alpha_1 + 1, \alpha_1 + 2, \cdots, \alpha_2,$$

where $\alpha_2 - \alpha_1$ is the number of chains that contain exactly two integrators, *i.e.*, $q_{\alpha_1+1} = q_{\alpha_1+2} = \cdots = q_{\alpha_2} = 2$. For chains with lengths greater than 2, we obtain

$$\xi_{i,3} = \phi_{i,3}^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2} \cdots, \xi_{i,2}),$$
$$i = \alpha_2 + 1, \alpha_2 + 2, \cdots, m.$$

Here, $\xi_{i,3}$ are viewed as inputs. Continuing in this way, we finally obtain

$$v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2}, \cdots, \xi_{m,2};$$
$$\cdots; \xi_{1,q_m-1}, \xi_{2,q_m-1}, \cdots, \xi_{m,q_m-1}; \xi_{i,q_m}),$$

for chains that contain q_m integrators.

The level-by-level backstepping will allow the backstepping to be implemented on some systems for which the chain-by-chain backstepping procedure is not applicable. As pointed out in [25], the triangular dependency requirement can be further weakened if we mix the chain-by-chain backstepping and the level-by-level backstepping and implement it on a same system, allowing the stabilization of a larger class of systems.

Consider

$$\begin{cases} \dot{\eta} &=& f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}), \\ \dot{\xi}_{i,j} &=& \xi_{i,j+1} + \displaystyle\sum_{l=1}^{i-1} \delta_{i,j,l}(\eta, \xi) v_l, \ \ j = 1, 2, \cdots, q_i - 1, \\ \dot{\xi}_{i,q_i} &=& v_i, \\ y_{\mathrm{d},i} &=& \xi_{i,1}, \ \ i = 1, 2, \cdots, m, \end{cases} \tag{13}$$

where $q_1 \leq q_2 \leq \cdots \leq q_m,\ \xi = \mathrm{col}\,\{\xi_1, \xi_2, \cdots, \xi_m\},\ \xi_i = \mathrm{col}\,\{\xi_{i,1}, \cdots, \xi_{i,q_i}\}$,

$$\delta_{i,j,l} = 0, \quad \text{for } j < q_l,\ i = 1, 2, \cdots, m. \tag{14}$$

Following [25], we have the following theorem, which is crucial for the results in next section.

Theorem 2.1: Suppose that the system (13) satisfies (14) and there exist smooth functions, $v_i^\star(\eta)$, with $v_i^\star(0)=0,\ i=1,2,\cdots,m$, such that $\dot\eta = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta),\cdots,v_m^\star(\eta))$ is globally asymptotically stable at $\eta=0$. Then there exists a state feedback v that globally asymptotically stabilizes the system at the origin if there exists an ordered list κ containing all variables of ξ such that

1) $\xi_{i,1}$ are the first m elements of κ; $\xi_{i,j}$ appears earlier than $\xi_{i,j+1}$ in κ, for $j=2,3,\cdots,q_i-1$, $i=1,2,\cdots,m$;
2) The function $\delta_{i,j,l}$ depends only on η, and variables that appear earlier than $\xi_{i,j+1}$ in κ, for $j=2,3,\cdots,q_i-1$, $i=1,2,\cdots,m$.

III. Main Results

Definition 3.1: The system (9) is semi-globally stabilizable by state feedback if, for any compact set of initial conditions $\mathcal{X}_0$ of the state space, there exists a smooth state feedback

$$u = \alpha_{\mathcal{X}_0}(\eta,\xi) \tag{15}$$

such that the equilibrium $(0,0)$ of the closed-loop system (9) and (15) is locally asymptotically stable and $\mathcal{X}_0$ is contained in its domain of attraction.

In what follows, we will present an algorithm for constructing a family of feedback laws that semi-globally stabilize the system (9). This algorithm consists of two steps.

We first find positive constants $c_{i,k}$s, such that the polynomials

$$p_i(s) = s^{\ell_i-1} + c_{i,\ell_i-2}\, s^{\ell_i-2} + \cdots + c_{i,1}\, s + c_{i,0},$$

$$i = 1,2,\cdots,m,$$

have all roots with negative real parts. Define

$$v_i^\star = -\varepsilon^{\ell_i-1}c_{i,0}\xi_{i,1} - \varepsilon^{\ell_i-2}c_{i,1}\xi_{i,2} - \cdots - \varepsilon c_{i,\ell_i-2}\xi_{i,\ell_i-1},$$

$$i=1,2,\cdots,m.$$

where $\varepsilon>0$. Consider

$$\begin{cases} \dot\eta &=& f_0(\eta, v_1^\star, v_2^\star,\cdots,v_m^\star),\\ \dot\xi_{i,j} &=& \xi_{i,j+1},\ \ j=1,2,\cdots,\ell_i-2,\\ \dot\xi_{i,\ell_i-1} &=& v_i^\star,\ \ i=1,2,\cdots,m. \end{cases} \tag{16}$$

Following [18], the dynamics of (16) has a locally asymptotically stable equilibrium at the origin of

$$(\eta;\xi_{i,1},\cdots,\xi_{i,\ell_i-1}, i=1,2,\cdots,m).$$

Moreover, the domain of attraction of this equilibrium can be made arbitrarily large by decreasing the value of the low gain parameter ε.

Lemma 3.1: Consider the system (16). Suppose that its zero dynamics have a globally asymptotically stable equilibrium at the origin. For any $R>0$, there exists $\varepsilon^\star>0$ such that, for any $0<\varepsilon\le\varepsilon^\star$, the system (16) is locally asymptotically stable and, moreover,

$$\begin{cases} \|\eta(0)\|\le R,\\ \|\xi_{i,j}(0)\|\le R,\\ j=1,2,\cdots,\ell_i-1,\ \ i=1,2,\cdots,m \end{cases}$$

$$\Longrightarrow \begin{cases} \lim_{t\to\infty}\eta(t)=0,\\ \lim_{t\to\infty}\xi_{i,j}(t)=0,\\ j=1,2,\cdots,\ell_i-1,\ \ i=1,2,\cdots,m. \end{cases}$$

Once the intermediate inputs $v_i^\star,\ i=1,2,\cdots,m$, have been obtained, both [16] and [18] design the overall controller by using linear high-gain state feedback. This is possible because the systems considered there are linear except the zero dynamics. In our situation, the system is in the form of (9). Because of the nonlinearities $\delta_{i,j,l}(\eta,\xi)v_l$, we have to resort to backstepping procedure as described in [25], where a special case of (9), *i.e.*, $\ell_i=1,\ i=1,2,\cdots,m$, is considered.

By Lemma 3.1 and Theorem 2.1, we have

Theorem 3.1: Let the system (9) satisfy (10) and (11), and there exist smooth functions, $v_i^\star(\eta)$, with $v_i^\star(0)=0,\ i=1,2,\cdots,m$, such that $\dot\eta = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta),\cdots,v_m^\star(\eta))$ is globally asymptotically stable at $\eta=0$. If there exists an ordered list κ containing $\{\xi_{i,j},\ j=\ell_i,\ell_i+1,2,\cdots,q_i,\ i=1,2,\cdots,m\}$, such that

1) $\xi_{i,j}$ appears earlier than $\xi_{i,j+1}$ and in κ;
2) The function $\delta_{i,j,l}$ depends only on η, $\{\xi_{i,j},\ \ j=1,2,\ldots,\ell_i-1,\ i=1,2,\cdots,m\}$, and variables that appear earlier than $\xi_{i,j+1}$ in κ, for $j=2,3,\cdots,q_i-1$, $i=1,2,\cdots,m$.

Then the system (9) is semi-globally stabilizable. That is, there exists a state feedback v that locally asymptotically stabilizes the system (9) and the basin of attraction of the closed-loop system contains any compact set X of the state space (η,ξ).

Example 3.1: Consider a three input three output system in the form of (9) with three chains of integrators of lengths $\{3,4,4\}$,

$$\begin{cases} \dot\eta &=& f_0(\eta,\xi_{1,2},\xi_{2,1},\xi_{3,3}),\\ \dot\xi_{1,j} &=& \xi_{1,j+1},\\ \dot\xi_{1,3} &=& v_1,\\ \dot\xi_{2,j} &=& \xi_{2,j+1},\\ \dot\xi_{2,3} &=& \xi_{2,4}+\delta_{2,3,1}(\eta;\xi_1;\xi_{2,1},\xi_{3,1};\xi_{2,2},\xi_{3,2};\xi_{2,3})v_1,\\ \dot\xi_{2,4} &=& v_2,\\ \dot\xi_{3,j} &=& \xi_{3,j+1},\quad j=1,2,\\ \dot\xi_{3,3} &=& \xi_{3,4}+\delta_{3,3,1}(\eta;\xi_1;\xi_{2,1},\xi_{3,1};\xi_{2,2},\xi_{3,2};\xi_{2,3},\xi_{3,3})v_1,\\ \dot\xi_{3,4} &=& v_3. \end{cases} \tag{17}$$

It is obvious that $\ell_1=2,\quad \ell_2=1,\quad \ell_3=3$. Suppose there exist smooth functions, $v_i^\star(\eta)$, with $v_i^\star(0)=0,\ i=1,2,3$, such that $\dot\eta = f_0(\eta,v_1^\star(\eta),v_2^\star(\eta),v_3^\star(\eta))$ is globally asymptotically stable at $\eta=0$. Clearly, this system satisfies the conditions in Theorem 3.1.

Let

$$v_1^\star = -\varepsilon\xi_{1,1} - \xi_{1,2},$$

$$v_3^\star = -\varepsilon^2\xi_{3,1} - \varepsilon\xi_{3,2} - \xi_{3,3},$$

Note that the equilibrium $\eta = 0$ of the subsystem

$$\dot{\eta} = f_0(\eta, 0, v_2^\star(\eta), 0)$$

is globally asymptotically stable. In what follows, we will illustrate how to implement the level-by-level backstepping on this system. The backstepping procedure starts with the following subsystem,

$$\dot{\eta} = f_0(\eta, v_1^\star, v_2^\star(\eta), v_2^\star).$$

The variable $\xi_{2,1}$ is the only first level variable in κ. To carry out the backstepping on the first level variables, we consider

$$\left\{\begin{array}{rcl} \dot{\eta} &=& f_0(\eta, v_1^\star, \xi_{2,1}, v_3^\star), \\ \dot{\xi}_{1,1} &=& \xi_{1,2}, \\ \dot{\xi}_{1,2} &=& v_1^\star, \\ \dot{\xi}_{3,1} &=& \xi_{3,2}, \\ \dot{\xi}_{3,2} &=& \xi_{3,3}, \\ \dot{\xi}_{3,3} &=& v_3^\star, \\ \dot{\xi}_{2,1} &=& \xi_{2,2}, \end{array}\right. \tag{18}$$

with $\xi_{2,2}$ as the input. This subsystem can be asymptotically stabilized by a control of the form

$$\xi_{2,2} = \phi_{2,2}^\star(\eta; \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}). \tag{19}$$

The subsystem (18) can be written as

$$\dot{\eta}_{\text{I}} = f_{\text{I}}(\eta_{\text{I}}, \xi_{2,2}), \tag{20}$$

where $\eta_{\text{I}} = \text{col}\{\eta, \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}\}$. The equilibrium $\eta_{\text{I}} = 0$ of this system (20) is asymptotically stabilized by the virtual inputs $\xi_{2,2}$ as given by (19).

To start the second level backstepping, consider

$$\left\{\begin{array}{rcl} \dot{\eta}_{\text{I}} &=& f_{\text{I}}(\eta_{\text{I}}, \xi_{2,2}), \\ \dot{\xi}_{1,2} &=& \xi_{1,3}, \\ \dot{\xi}_{2,2} &=& \xi_{2,3}, \end{array}\right. \tag{21}$$

and view $\xi_{1,3}$ and $\xi_{2,3}$ as its inputs. Following the same procedure as in the first level backstepping, we find the controls of the form

$$\left\{\begin{array}{rcl} \xi_{1,3} &=& \phi_{1,3}^\star(\eta_{\text{I}}; \xi_{1,2}), \\ \xi_{2,3} &=& \phi_{2,3}^\star(\eta_{\text{I}}; \xi_{1,2}, \xi_{2,2}) \end{array}\right. \tag{22}$$

that asymptotically stabilize the equilibrium $\eta_{\text{II}} = \text{col}\{\eta_{\text{I}}, \xi_{1,2}, \xi_{2,2}\} = 0$ of the subsystem (21). The subsystem (21) can be written as

$$\dot{\eta}_{\text{II}} = f_{\text{II}}(\eta_{\text{II}}, \xi_{1,3}, \xi_{2,3}),$$

whose equilibrium $\eta_{\text{II}} = 0$ is asymptotically stabilized by the virtual inputs $\xi_{1,3}$ and $\xi_{2,3}$ given by (22).

For the third level backstepping, we define

$$\left\{\begin{array}{rcl} \dot{\eta}_{\text{II}} &=& f_{\text{II}}(\eta_{\text{II}}, \xi_{1,3}, \xi_{2,3}), \\ \dot{\xi}_{1,3} &=& v_1, \\ \dot{\xi}_{2,3} &=& \xi_{2,4} + \delta_{2,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3})v_1, \\ \dot{\xi}_{3,3} &=& \xi_{3,4} + \delta_{3,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}, \xi_{3,3})v_1, \end{array}\right. \tag{23}$$

with v_1, $\xi_{2,4}$ and $\xi_{3,4}$ as its inputs. This system can be asymptotically stabilized by the controls of the form

$$v_1 = u_1^\star(\eta; \xi_1, \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}). \tag{24}$$

The subsystem (23) under the control (24) can be written as $\dot{\eta}_{\text{III}} = f_{\text{III}}(\eta_{\text{III}}; \xi_{2,4}, \xi_{3,4})$, and its equilibrium $\eta_{\text{III}} = \text{col}\{\eta_{\text{II}}, \xi_{1,3}\} = 0$ is asymptotically stabilized by the virtual inputs $\xi_{2,4}$ and $\xi_{3,4}$ as given by

$$\left\{\begin{array}{rcl} \xi_{2,4} &=& \phi_{2,4}^\star(\eta_{\text{II}}; \xi_{1,3}, \xi_{2,3}), \\ \xi_{3,4} &=& \phi_{3,4}^\star(\eta_{\text{II}}; \xi_{1,3}, \xi_{2,3}, \xi_{3,3}). \end{array}\right.$$

Finally, define

$$\left\{\begin{array}{rcl} \dot{\eta}_{\text{III}} &=& f_{\text{III}}(\eta_{\text{III}}; \xi_{2,4}, \xi_{3,4}), \\ \dot{\xi}_{2,4} &=& v_2, \\ \dot{\xi}_{3,4} &=& v_3, \end{array}\right.$$

on which we carry out the last level of backstepping to obtain

$$\begin{array}{rcl} v_2 &=& u_2^\star(\eta; \xi_1; \xi_2; \xi_{3,1}, \xi_{3,2}, \xi_{3,3}), \\ v_3 &=& u_3^\star(\eta; \xi_1; \xi_2; \xi_3). \end{array}$$

The inputs v_1, v_2 and v_3 semi-globally asymptotically stabilize the equilibrium $\text{col}\{\eta, \xi_1, \xi_2, \xi_3\} = 0$ of the system (17).

In what follows, we give an example which requires the mixed chain-by-chain and level-by-level backstepping design procedure.

Example 3.2: Consider a system in the form of (9) with three chains of integrators of lengths $\{2, 4, 4\}$,

$$\left\{\begin{array}{rcl} \dot{\eta} &=& f_0(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \\ \dot{\xi}_{1,1} &=& \xi_{1,2}, \\ \dot{\xi}_{1,2} &=& v_1, \\ \dot{\xi}_{2,1} &=& \xi_{2,2}, \\ \dot{\xi}_{2,2} &=& \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2})v_1, \\ \dot{\xi}_{2,3} &=& \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{2,3}, \xi_{3,1}, \xi_{3,2})v_1, \\ \dot{\xi}_{2,4} &=& v_2, \\ \dot{\xi}_{3,1} &=& \xi_{3,2}, \\ \dot{\xi}_{3,2} &=& \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2})v_1, \\ \dot{\xi}_{3,3} &=& \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}, \xi_{3,3})v_1, \\ \dot{\xi}_{3,4} &=& v_3, \end{array}\right. \tag{25}$$

where $\ell_1 = \ell_2 = \ell_3 = 2$.

It is obvious that the system satisfies the conditions in Theorem 3.1 with $\kappa = \{\xi_{1,1}, \xi_{2,1}, \xi_{3,1}; \xi_{1,2}, \xi_{2,2}, \xi_{3,2}, \xi_{2,3}, \xi_{2,4}, \xi_{3,3}, \xi_{3,4}\}$. We first find the low-gain control. Let

$$v_1^\star = -\varepsilon\xi_{1,1} - \xi_{1,2},$$

$$v_2^\star = -\varepsilon\xi_{2,1} - \xi_{2,2},$$

$$v_3^\star = -\varepsilon\xi_{3,1} - \xi_{3,2}.$$

Then we carry out a mixed chain-by-chain and level-by-level backstepping in the order of $\xi_{1,2}, v_1, \xi_{2,2}, \xi_{3,2}, \xi_{2,3}, \xi_{2,4}, v_2, \xi_{3,3}, \xi_{3,4}, v_3$ to obtain

$$\begin{array}{rcl} v_1 &=& u_1^\star(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}), \\ v_2 &=& u_2^\star(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}), \\ v_3 &=& u_3^\star(\eta, \xi_1, \xi_2, \xi_3). \end{array}$$

Example 3.3: Consider

$$\dot{\eta} = -\eta + \eta^2(v_1 + \xi_{2,2}),$$
$$\dot{\xi}_{1,1} = v_1,$$
$$\dot{\xi}_{2,1} = \xi_{2,2} + \eta v_1,$$
$$\dot{\xi}_{2,2} = v_2.$$

Obviously the system satisfies the conditions in Theorem 3.1 with

$$q_1 = 1, \ \ q_2 = 2, \ \ \ell_1 = 2, \ \ \ell_2 = 2.$$

Choosing both poles of linear slow subsystem to be $-\varepsilon$, we obtain

$$v_2^\star = -\varepsilon \xi_{2,1},$$
$$v_1 = -\varepsilon \xi_{1,1}.$$

By backstepping, we obtain

$$v_2 = -\xi_{2,1} - \frac{1}{\varepsilon}\xi_{2,2}.$$

Shown in Fig. 1 and Fig. 2 are some simulation results of the closed-loop system.

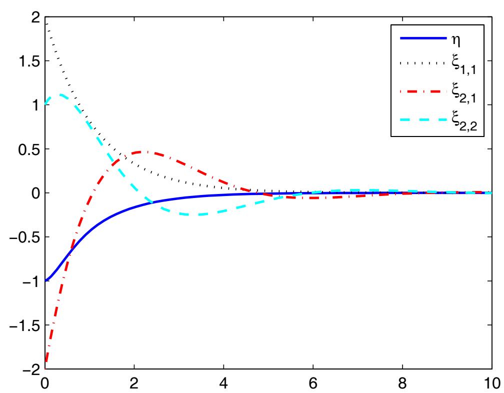

Fig. 1. State trajectories with $\varepsilon = 0.9$

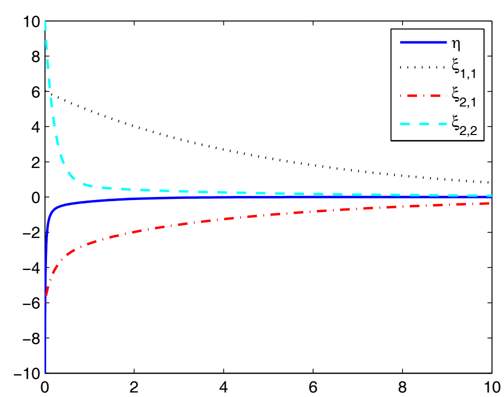

Fig. 2. State trajectories with $\varepsilon = 0.2$

IV. Conclusions

In this paper, we showed how a recently developed structural decomposition can be used to solve the semi-global stabilization of a class of MIMO systems without vector relative degrees. The design procedure involved several existing design techniques in nonlinear stabilization, including low gain feedback and different forms of backstepping design procedures.

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