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Robust Feedback Tracking of Autonomous Underwater Vehicles with Disturbance Rejection

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***Abstract*— This paper treats the practical and challenging control problem of tracking a prescribed continuous trajectory for an autonomous underwater vehicle immersed in the presence of gravity, buoyancy, hydrodynamic and other uncertain forces and moments. These uncertain forces and moments are bounded and may be difficult to model accurately, but they act persistently or over long periods of time. The trajectory is specified in terms of desired attitude and translational motion for a rigid body model of the vehicle. For an autonomous underwater vehicle (AUV) in a dynamic environment like the ocean, the presence of changing ocean currents create hydrodynamic forces and moments that are not well-known or predictable, even though they are bounded. In the absence of such forces and moments, it is possible to model the AUV dynamics very accurately and use the model for trajectory generation and tracking, as has been shown in prior research. However, the presence of uncertainties makes it difficult to accurately model the dynamics. This in turn makes the control task of tracking a desired or prescribed trajectory very challenging as an accurate model of the system dynamics is not available. We develop a robust feedback tracking scheme for autonomous underwater vehicles that can track a prescribed trajectory, while rejecting the effects of disturbances due to poorly known inputs with a simple yet general internal model.**

I. INTRODUCTION

We consider robust feedback tracking control of an autonomous underwater vehicle(AUV) that is required to track a desired state trajectory in the presence of uncertainties in the dynamics model. Although the problem of control of AUVs or underwater robots has been extensively studied in the past, our approach to this problem is very different from previous approaches. Three salient features of our approach are: (i) we use a robust feedback geometric control scheme that ensures practically global asymptotic trajectory tracking in motion states, provided that actuator bounds are not exceeded; (ii) we use an internal model for the uncertainties that only assume bounds on the magnitude, direction and rates of change of these uncertain inputs; and (iii) we use estimates of the uncertain inputs, that may be obtained from an a priori known model or a state estimation scheme.

The dynamics of an AUV modeled as a rigid body is given by its translation and attitude motion. The dynamics model that we use is based on available models in standard texts on this subject like [1], [2], [3]. The AUV has to track a desired trajectory that can be generated as a translation and attitude time profile that results in a desired state trajectory. The task of transferring the system state under dynamic constraints, from its initial state to a desired final state, is termed *motion planning* or *dynamic interpolation*.

Motion planning in nonlinear spaces, like the group of rigid body motions $\mathrm{SE}(3)$, under prescribed and well-known dynamics, has been studied extensively in the past. Motion planning for AUVs also has an extensive literature, with application of several control strategies and schemes. Neural net-based controllers for AUVs were reported in [4], [5]. Open-loop geometric control schemes have also been applied to AUVs in [6]-[8], and shown to work well in the absence of model uncertainties. Such open-loop control strategies give trajectories in $\mathrm{TSE}(3)$ that transfer the system from the given initial state to the desired final state, while minimizing a *cost function* that is usually (a combination of) the time taken or control energy expended.

In practice, the presence of dynamic uncertainties like unmodeled external forces and moments, makes it impossible for the desired trajectory to be followed by an open-loop control scheme. Therefore, a *feedback trajectory tracking control scheme that is robust to these dynamic uncertainties* is essential to ensure that the autonomous vehicle tracks the desired trajectory, while rejecting the effect of the uncertain forces and moments on this trajectory. Recent research using the framework of geometric mechanics and geometric control has been successful in demonstrating *almost global* rigid body attitude and angular velocity stabilization and tracking with bounded control inputs. Such results have been applied to orbiting spacecraft in gravity in [9], [10], to autonomous underwater vehicles in [11] and to simple mechanical systems in [12]. However, these results do not take into account uncertainties in the system model.

This paper is organized into five sections besides this introduction. Section II gives the dynamical model of the AUV used in our theoretical development. This model includes the effects of the bounded (but otherwise unknown) disturbance forces and moments. Section III gives the feedback control approach based on the dynamics model without the effect of disturbances. Section IV gives assumptions on the disturbance models for bounded disturbance inputs acting on the system. Section V presents results for the feedback tracking control scheme with disturbance rejection, based on the complete dynamical model given in Section II. Finally, Section VI presents a concluding discussion of results obtained in this paper and possible future work.

II. MODEL OF AUV DYNAMICS

We treat the AUV dynamics within the framework of geometric mechanics, which makes it convenient to deal with

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feedback trajectory tracking in a global setting. We denote the position vector of the AUV by $b = (b_1, b_2, b_3)^{\mathrm{T}} \in \mathbb{R}^3$, and $R \in \mathrm{SO}(3)$ is the rotation matrix describing its orientation. Therefore the configuration space of an AUV is the special Euclidean group $\mathrm{SE}(3)$, the semi-direct product of $\mathrm{SO}(3)$ and $\mathbb{R}^3$, with $(b, R) \in \mathrm{SE}(3)$ denoting the configuration. The state space of an AUV in spatial motion is therefore the tangent bundle $\mathrm{TSE}(3)$, with the translational and angular velocities in the body-fixed frame denoted by $\nu = (\nu_1, \nu_2, \nu_3)^{\mathrm{T}}$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3)^{\mathrm{T}}$ respectively. The kinematic equations for a rigid body are:

$$\begin{aligned} \dot{b} &= R\nu \\ \dot{R} &= R\Omega^{\times} \end{aligned} \tag{1}$$

where the operator $(\cdot)^{\times} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by $y^{\times} z = y \times z$, $\mathfrak{so}(3)$ being the Lie algebra of $\mathrm{SO}(3)$ or equivalently, the space of skew-symmetric 3×3 matrices.

Taking the origin of the body-fixed frame for the AUV at its center of gravity C_G, the only moment due to the restoring buoyancy force is the righting moment $-r_{C_B} \times (\rho g \mathcal{V}) R^{\mathrm{T}} e_3$, where r_{C_B} is the vector from C_G to the center of buoyancy C_B, ρ is the fluid density, g the acceleration of gravity, $\mathcal{V}$ the volume of displaced fluid and $e_3 = [0 \;\; 0 \;\; 1]^{\mathrm{T}}$ be the inertial unit vector pointing in the direction of gravity. The dynamics with uncertain forces and moments is given by:

$$\begin{aligned} M\dot{\nu} =& M\nu \times \Omega + D_{\nu}(\nu)\nu + \rho g \mathcal{V} R^{\mathrm{T}} e_3 + \varphi_c + \varphi_u, \\ J\dot{\Omega} =& J\Omega \times \Omega + M\nu \times \nu + D_{\Omega}(\Omega)\Omega \\ & - r_{C_B} \times (\rho g \mathcal{V}) R^{\mathrm{T}} k + \tau_c + \tau_u, \end{aligned} \tag{2}$$

where M accounts for the mass and added mass, J accounts for the body moments of inertia and the added moments of inertia. The matrices $D_{\nu}(\nu)$ and $D_{\Omega}(\Omega)$ represent the drag force and drag momentum, respectively. The vectors $\varphi_u \in \mathbb{R}^3$ and $\tau_u \in \mathbb{R}^3$ are bounded uncertain force and uncertain moment, respectively, expressed in the body-fixed frame. Finally, φ_c and τ_c are the control forces and moments on the AUV, respectively.

Our previous research on the time and energy consumption minimization problem resulted in open loop motion planning for an AUV based on a theoretical model that neglected many factors affecting the experiments. These factors include drag associated with the attached tether, thruster dynamics, underwater currents, etc. This implementation on the testbed AUV was carried out in a controlled environment (a swimming pool, see [6]-[8]). This research is an improvement of our prior research, motivated by the observed failure of such open loop schemes in following a path prescribed by a motion planning algorithm. This research is necessary to translate our AUV experiments from a controlled environment to an ocean environment.

III. Asymptotic Feedback Tracking of AUV Motion

Prior literature on local nonlinear control methods for tracking desired attitude motion for a rigid body in the presence of disturbance moments exists, for example [13], [14]. These local methods are not suitable for controlling a highly maneuverable AUV that has to implement large motions. Additionally, it is impossible to obtain a globally asymptotically stable tracking control scheme for the motion of a rigid body or multibody system; as is known in some circles since the early 1980s (see [15] for instance). The first correct treatment of this problem was in [16], which introduced the concept of almost global asymptotic stabilzation of such systems by continuous feedback. This was followed by a few correct treatments of this problem, like [17].

The almost global property of the control schemes in [9], [10] ensures that a desired attitude or attitude motion trajectory is tracked starting from almost any initial state modulo a set of measure zero in the state space. In [10], we also included the effects of additional drag-type disturbance moments (that are bounded but unknown) on the attitude dynamics, and showed that the desired trajectory in $\mathrm{TSO}(3)$ could be tracked almost globally even in the presence of such disturbances.

A. Trajectory Tracking for AUV

The reference trajectory to be tracked by the AUV can be obtained from an open loop scheme like that in [6]-[8]. It can be specified in terms of the initial desired position vector in inertial frame $b_r(0)$,the initial desired attitude $R_r(0)$, and the desired translational and angular velocity time profiles in body frame, $\nu_r^0(t)$ and $\Omega_r(t)$ respectively. The reference trajectory satisfies the kinematic equation in $\mathrm{SE}(3)$:

$$\dot{g}_r = g_r \zeta_r, \quad g_r = \begin{bmatrix} R_r & b_r \\ 0 & 1 \end{bmatrix} \text{ and } \zeta_r = \begin{bmatrix} \Omega_r^{\times} & \nu_r^0 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

Next we define the trajectory tracking errors, as follows:
$a(t) \triangleq b(t) - b_r(t) =$ error in inertial position,
$x(t) \triangleq R_r^T(t) a(t) =$ error in position expressed in reference body frame,
$Q(t) \triangleq R_r^T(t) R(t) =$ error in body attitude (orientation),
$\upsilon(t) \triangleq \nu(t) - Q^T(t)(\nu_r^0(t) + \Omega_r(t)^{\times} x(t)) =$ error in body translational velocity,
$\omega(t) \triangleq \Omega(t) - Q^T(t)\Omega_r(t) =$ error in body angular velocity.
We also define $\nu_r \triangleq \nu_r^0 + \Omega_r^{\times} x$. Thus, we can express the tracking error kinematics in left-invariant form on $\mathrm{SE}(3)$:

$$\left.\begin{aligned} \dot{x} &= Q\upsilon \\ \dot{Q} &= Q\omega^{\times} \end{aligned}\right\} \Leftrightarrow \dot{h} = h\xi, \tag{4}$$

where

$$h = \begin{bmatrix} Q & x \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \quad \xi = \begin{bmatrix} \omega^{\times} & \upsilon \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

The dynamics of the AUV can be expressed in terms of the trajectory tracking errors as follows:

$$\begin{aligned} & M\dot{\upsilon} = M\{\omega^{\times} Q^T \nu_r - Q^T \dot{\nu}_r\} \\ & + M(\upsilon + Q^T \nu_r)^{\times}(\omega + Q^T \Omega_r) + D_{\nu}(\nu)(\upsilon + Q^T \nu_r) \\ & + \rho g \mathcal{V} Q^T R_r^T e_3 + \varphi_c, \end{aligned} \tag{5}$$

$$\begin{aligned} & J\dot{\omega} = J(\omega^{\times} Q^T \Omega_r - Q^T \dot{\Omega}_r) - (\omega + Q^T \Omega_r)^{\times} J(\omega \\ & + Q^T \Omega_r) - (\upsilon + Q^T \nu_r)^{\times} M(\upsilon + Q^T \nu_r) \\ & + D_{\Omega}(\Omega)(\omega + Q^T \Omega_r) - r_{CB} \times (\rho g \mathcal{V}) Q^T R_r^T e_3 + \tau_c, \end{aligned} \tag{6}$$

as obtained from equation (2) in the absence of uncertain inputs ($\varphi_u = \tau_u = 0$). The control laws for the inputs φ_c and τ_c are created to asymptotically track the reference trajectory.

B. Asymptotic Tracking Control Laws in TSE(3)

We design a feedback tracking control scheme, based on Lyapunov-type analysis and full-state feedback, to achieve this task. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{C}^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(u) > 0$ for all $u \in \mathbb{R}^+$. Furthermore, let $\Phi'(\cdot) \le \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$\mathcal{K}$ function [18]. Let K, L_ν, L_Ω and N be positive definite control gain matrices, with $K = \text{diag}(k_1, k_2, k_3)$ such that $0 < k_1 < k_2 < k_3$. Therefore $\Phi(\text{trace}(K - KQ))$ is a Morse function on SO(3) (where $Q \in \text{SO}(3)$) whose critical points are non-degenerate and hence isolated, according to the Morse lemma [19]. Along the kinematics (4), the time derivative of this function is (see [9] and [10]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\text{trace}(K - KQ)) = -\Phi'(\text{trace}(K - KQ))\omega^{\mathrm{T}} \times \left[k_1 e_1^\times Q^{\mathrm{T}} e_1 + k_2 e_2^\times Q^{\mathrm{T}} e_2 + k_3 e_3^\times Q^{\mathrm{T}} e_3\right]. \tag{7}$$

We propose the following control laws for φ_c and τ_c to asymptotically track the reference trajectory in TSE(3) in the absence of disturbance inputs:

$$\begin{aligned}\varphi_c = &-L_\nu \upsilon + MQ^{\mathrm{T}}\dot{\nu}_r + (Q^{\mathrm{T}}\Omega_r)^\times\big(M(\upsilon + Q^{\mathrm{T}}\nu_r)\big)\\ &- D_\nu(\nu)\big(\upsilon + Q^{\mathrm{T}}\nu_r\big) - \rho g\mathcal{V}Q^{\mathrm{T}}R_r^{\mathrm{T}}e_3 - Q^{\mathrm{T}}Nx,\end{aligned} \tag{8}$$

$$\begin{aligned}\tau_c = &-L_\Omega\omega + JQ^{\mathrm{T}}\dot{\Omega}_r + (Q^{\mathrm{T}}\Omega_r)^\times(JQ^{\mathrm{T}}\Omega_r)\\ &+ (Q^{\mathrm{T}}\nu_r)^\times(MQ^{\mathrm{T}}\nu_r) - D_\Omega(\Omega)(\omega + Q^T\Omega_r)\\ &+ (\rho g\mathcal{V})r_{CB}^\times(Q^{\mathrm{T}}R_r^{\mathrm{T}}e_3) + \Phi'(\text{trace}(K - KQ))\\ &\times \left[k_1 e_1^\times Q^{\mathrm{T}} e_1 + k_2 e_2^\times Q^{\mathrm{T}} e_2 + k_3 e_3^\times Q^{\mathrm{T}} e_3\right].\end{aligned} \tag{9}$$

Note that this control law, and hence the trajectories of the closed-loop system, are continuous with respect to the error variables x, υ, Q and ω. We next show the almost global asymptotic tracking properties of the closed-loop system (5)-(6) and (8)-(9) when there are no disturbance inputs.

C. Critical Points for Feedback Attitude Dynamics

Let $\langle\cdot,\cdot\rangle$ denote the trace inner product on the vector space $\mathbb{R}^{n\times n}$, given by

$$\langle A, B\rangle \triangleq \text{trace}\big(A^{\mathrm{T}}B\big).$$

We first present a couple of lemmas that are used to prove the main result on asymptotic trajectory tracking.

Lemma 1. *The function $\Phi\big(\text{trace}(K - KQ)\big)$ on SO(3) has the set of non-degenerate critical points*

$$E_c \triangleq \{I,\ \textit{diag}(-1,1,-1),\ \textit{diag}(1,-1,-1),\ \textit{diag}(-1,-1,1)\}. \tag{10}$$

Further, the unique minimum point of this function is $Q = I \triangleq \textit{diag}(1,1,1)$.

The proof of this result is given in [10]. A function Φ of this type has the minimum number of critical points for a Morse function on the nonlinear space SO(3). This lemma is also a corollary of Proposition 1 of [20], which treats Wahba's problem in attitude determination using similar techniques. For the closed-loop attitude dynamics of the AUV, we first state a result on the local asymptotic stability of the equilibrium $(I, 0) \in \text{TSO}(3)$. The proof of this result is also provided in [10], which uses a result in [21].

Lemma 2. *The equilibrium $(I, 0)$ of the closed-loop attitude dynamics given by* (6) *and the control law* (9) *is locally asymptotically stable when $(\upsilon, \omega) = (0, 0)$. The other equilibria given by $(Q_e, 0)$, where $Q_e \in E_c \setminus \{I\}$, of the closed-loop attitude dynamics under these conditions are unstable. Furthermore, under these conditions the set of all initial conditions converging to the equilibrium $(Q_e, 0)$, where $Q_e \in E_c \setminus \{I\}$ form a lower dimensional manifold.*

D. Asymptotic Convergence Results

We now present our main result on asymptotic convergence of the tracking error dynamics for the closed-loop dynamics (5)-(6) and (8)-(9) to the desired equilibrium $(x_e, Q_e, \upsilon_e, \omega_e) = (0, I, 0, 0)$.

Theorem 1. *In the absence of uncertain inputs ($\varphi_u = 0$ and $\tau_u = 0$), the trajectories of the closed-loop tracking error system given by the* (5)-(6) *and control laws* (8)-(9) *converge to the set*

$$\mathcal{E} = \{(x, Q, \upsilon, \omega) \in \text{TSE}(3) \,:\, \upsilon = 0, \omega = 0, x = 0, Q \in E_c\}, \tag{11}$$

where E_c is as defined in (10). Further, the equilibrium $(x_e, Q_e, \upsilon_e, \omega_e) = (0, I, 0, 0)$ of the closed-loop system is asymptotically stable in this case and its domain of attraction is almost global.

Proof: For the closed-loop tracking error dynamics given by (5)-(6) and control laws (8)-(9), we propose the following candidate Lyapunov function:

$$\begin{aligned}V(x, Q, \upsilon, \omega) &= V_T(x, \upsilon) + V_A(Q, \omega),\\ V_T(x, \upsilon) &= \frac{1}{2}\upsilon^{\mathrm{T}}M\upsilon + \frac{1}{2}x^{\mathrm{T}}Nx,\\ V_A(Q, \omega) &= \frac{1}{2}\omega^{\mathrm{T}}J\omega + \Phi\big(\text{trace}(K - KQ)\big).\end{aligned} \tag{12}$$

Note that $V(x, Q, \upsilon, \omega) \ge 0$ and its "attitude component" $V_A(Q, \omega) = 0$ if and only if $(Q, \omega) = (I, 0)$. Thus $V(x, Q, \upsilon, \omega)$ is a positive definite function on TSE(3) that is zero only at the desired equilibrium.

We evaluate the time derivative of $V(x, Q, \upsilon, \omega)$ along the trajectories of the closed-loop system, using the time derivative of $\Phi\big(\text{trace}(K - KQ)\big)$ given by equation (7). The time derivative of $V_T(x, \upsilon)$ along (5) and (8) is:

$$\begin{aligned}\dot{V}_T &= \upsilon^{\mathrm{T}}M\dot{\upsilon} + x^{\mathrm{T}}N\dot{x} \qquad (13)\\ &= \upsilon^{\mathrm{T}}[M(\omega^\times Q^{\mathrm{T}}\nu_r) + \big(M(\upsilon + Q^{\mathrm{T}}\nu_r)\big)^\times\omega - L_\nu\upsilon].\end{aligned}$$

The time derivative of $V_A(Q, \omega)$ along (6), (7) and (9) is:

$$\begin{aligned}\dot{V}_A =&\,\omega^{\mathrm{T}}J\dot{\omega} - \Phi'(\text{trace}(K - KQ))\omega^{\mathrm{T}}\big[k_1 e_1^\times Q^{\mathrm{T}} e_1\\ &+ k_2 e_2^\times Q^{\mathrm{T}} e_2 + k_3 e_3^\times Q^{\mathrm{T}} e_3\big] \qquad (14)\\ &= -\omega^{\mathrm{T}}[\upsilon^\times M(\upsilon + Q^{\mathrm{T}}\nu_r) + (Q^{\mathrm{T}}\nu_r)^\times M\upsilon + L_\Omega\omega].\end{aligned}$$

Therefore, combining (13) and (14) and using the scalar triple product identity, we get

$$\dot{V} = \dot{V}_T + \dot{V}_A = -\upsilon^{\mathrm{T}} L_\nu \upsilon - \omega^{\mathrm{T}} L_\Omega \omega \leq 0,$$

and $\dot{V} = 0$ if and only if $\upsilon = 0$ and $\omega = 0$.

Recall that $\Phi(\cdot)$ is a strictly increasing monotone function. Hence, for any $(x(0), Q(0), \upsilon(0), \omega(0)) \in \mathrm{TSE}(3)$, the set

$$\mathcal{I} = \{(x, Q, \upsilon, \omega) \in \mathrm{TSE}(3) \;:\; V(x, Q, \upsilon, \omega) \leq V(x(0), Q(0), \upsilon(0), \omega(0))\},$$

is an invariant set of the closed-loop system [9], [10]. By LaSalle's invariant set theorem, it follows that all solutions that begin in $\mathcal{I}$ converge to the largest invariant subset of $\dot{V}^{-1}(0)$ contained in $\mathcal{I}$. Since $\dot{V}(x, Q, \upsilon, \omega) \equiv 0$ implies $\upsilon = \omega \equiv 0$, we substitute this into the closed-loop system equations to get:

$$\begin{aligned} \dot{V}^{-1}(0) = \{&(x, Q, \upsilon, \omega) \in \mathrm{TSE}(3) \,:\, \upsilon \equiv 0,\ \omega \equiv 0, \\ &x = 0,\ k_1 e_1^\times Q^{\mathrm{T}} e_1 + k_2 e_2^\times Q^{\mathrm{T}} e_2 + k_3 e_3^\times Q^{\mathrm{T}} e_3 \equiv 0\Big\} \\ = \{&(x, Q, \upsilon, \omega) \in \mathrm{TSE}(3) \,:\, x \equiv 0,\ Q \in E_c, \\ &\upsilon \equiv 0, \omega \equiv 0\}, \end{aligned}$$

since $Q^{\mathrm{T}} N x = 0 \Rightarrow x = 0$ as $Q \in \mathrm{SO}(3)$ is invertible and N is positive definite. In this case, each of the four points given by (11) are an equilibrium of the closed-loop dynamics in $\mathrm{TSE}(3)$. Therefore, by LaSalle's theorem, all solutions of the closed-loop system converge to one of the equilibria in $\mathcal{E} \cap \mathcal{I}$, where $\mathcal{E}$ is given by (11).

From Lemma 2, the only stable equilibrium is $(x, Q, \upsilon, \omega) = (0, I, 0, 0)$, and all solutions that converge to the other three equilibria form a lower dimensional manifold. Thus, this set of solutions has measure zero in $\mathrm{TSE}(3)$ (see also [9], [10]). Solutions of the closed-loop system that do not start in this manifold, converge aymptotically to the stable equilibrium $(x, Q, \upsilon, \omega) = (0, I, 0, 0)$. Therefore, the domain of attraction of this equilibrium is almost global. □

IV. Internal Model for Bounded Disturbance Inputs

Here we consider the AUV system with model uncertainties, given by the uncertain force φ_u and moment τ_u in equations (2). We add disturbance rejection control inputs to the control laws to ensure that the effect of these bounded uncertainties on the dynamics of the feedback system vanishes asymptotically. These uncertain inputs are described by an internal model that prescribes bounds in the direction, magnitude and time rates of these uncertainties.

For ease of notation, we use $g \in \mathrm{SE}(3)$ for the configuration variables and $\xi \in \mathfrak{se}(3)$ for the velocity variables as in the kinematic equation (1). It is reasonable to assume that we know the bounds in magnitude and direction of φ_u and τ_u at any given time. We also have estimates of these uncertain quantities, which could be obtained from a known a priori model or from real-time state measurements. An ellipsoidal bound on magnitude would result in an ellipsoidal neighborhood of the origin in $\mathbb{R}^3$, while a bound on direction would result in a solid cone with vertex at the origin. The intersection of these regions will be a compact subset of $\mathbb{R}^3$ in which the uncertain input will lie, as shown in Figure 1.

The ellipsoidal bounds on magnitudes are given by:

$$\begin{aligned} \varphi_u(g, \xi, t)^{\mathrm{T}} P_\varphi \varphi_u(g, \xi, t) \leq 1, \\ \tau_u(g, \xi, t)^{\mathrm{T}} P_\tau \tau_u(g, \xi, t) \leq 1, \end{aligned} \tag{15}$$

where P_φ and P_τ are constant symmetric positive definite matrices defining the shape and size of the ellipsoidal bounds. The internal model of these uncertain quantities is therefore of the form:

$$P_\varphi^{\frac{1}{2}} \varphi_u(g, \xi, t) = \big(I + \Xi_\varphi(t)^\times\big) P_\varphi^{\frac{1}{2}} \bar{\varphi}_u(g, \xi, t), \tag{16}$$

$$P_\tau^{\frac{1}{2}} \tau_u(g, \xi, t) = \big(I + \Xi_\tau(t)^\times\big) P_\tau^{\frac{1}{2}} \bar{\tau}_u(g, \xi, t), \tag{17}$$

where $\bar{\varphi}_u(g, \xi, t)$ and $\bar{\tau}_u(g, \xi, t)$ are the known estimates of the uncertain force and moment respectively. $P_\varphi^{\frac{1}{2}}$ and $P_\tau^{\frac{1}{2}}$ are the positive definite square root matrices of P_φ and P_τ.

The uncertainty in directions of these quantities are characterized by $\Xi_\varphi(t) \in \mathbb{R}^3$ and $\Xi_\tau(t) \in \mathbb{R}^3$, which we assume have known bounds. The above model implies that $P_\varphi^{\frac{1}{2}} \varphi_u(g, \xi, t)$ ($P_\tau^{\frac{1}{2}} \tau_u(g, \xi, t)$) is coning about $P_\varphi^{\frac{1}{2}} \bar{\varphi}_u(g, \xi, t)$ ($P_\tau^{\frac{1}{2}} \bar{\tau}_u(g, \xi, t)$, respectively). The cone angles give the undertainty bounds in direction. The tangent of the half-cone angle is given by the maximum value of the ratio of the norm of $\Xi_\varphi(t)^\times P_\varphi^{\frac{1}{2}} \bar{\varphi}_u(g, \xi, t)$ to $P_\varphi^{\frac{1}{2}} \bar{\varphi}_u(g, \xi, t)$ for the force uncertainty. A similar relation holds for the uncertainty in the direction of the moment $P_\tau^{\frac{1}{2}} \tau_u(g, \xi, t)$. Thus, the uncertainties in the directions of $P_\varphi^{\frac{1}{2}} \varphi_u(g, \xi, t)$ and $P_\tau^{\frac{1}{2}} \tau_u(g, \xi, t)$ are given by the half-cone angles

$$\tan \alpha_\varphi = \max_t \|\Xi_\varphi(t)\| \text{ and } \tan \alpha_\tau = \max_t \|\Xi_\tau(t)\|.$$

The uncertain vectors $\Xi_\varphi(t)$ and $\Xi_\tau(t)$ also satisfy

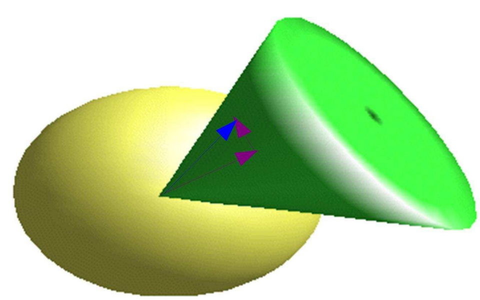

Fig. 1. *Region of $\mathbb{R}^3$ in which an uncertain force or moment may lie, is depicted as an intersection of an ellipsoid and a solid cone.*

$$\begin{aligned} \Xi_\varphi(t) =& \bar{\Xi}_\varphi(t) + b_\varphi(t) P_\varphi^{\frac{1}{2}} \bar{\varphi}_u(g, \xi, t), \\ \Xi_\tau(t) =& \bar{\Xi}_\tau(t) + b_\tau(t) P_\tau^{\frac{1}{2}} \bar{\tau}_u(g, \xi, t), \end{aligned} \tag{18}$$

where $\bar{\Xi}_\varphi(t)$ ($\bar{\Xi}_\tau(t)$) are estimates of $\Xi_\varphi(t)$ ($\Xi_\tau(t)$) that are not parallel to $P_\varphi^{\frac{1}{2}}\bar{\varphi}_u$ ($P_\tau^{\frac{1}{2}}\bar{\tau}_u$, respectively). The scalar quantities $b_\varphi(t)$ and $b_\tau(t)$ are unknown but in a small bounded neighborhood of 0.

The internal model (16)-(17) implies that the ellipsoidal magnitude bounds given by (15) are also satisfied by the force estimate $\bar{\varphi}_u(g,\xi,t)$ and moment estimate $\bar{\tau}_u(g,\xi,t)$. We also assume that $\bar{\varphi}_u(g,\xi,t)$, $\bar{\tau}_u(g,\xi,t)$, $\Xi_\varphi(t)$ and $\Xi_\tau(t)$ are slowly time-varying. This is made explicit in the disturbance rejection control law given in the next section.

V. Asymptotic Feedback Tracking with Disturbance Rejection

A. Rejection of Disturbances from Bounded Uncertainties

The statement below gives sufficient conditions under which a control input u rejects an uncertain disturbance input v acting on a system evolving on $\mathrm{SE}(3)$.

Proposition 1. *Let $v(g,\xi,t)\in\mathbb{R}^3$ and $\Xi(t)\in\mathbb{R}^3$ be uncertain quantities that satisfy*

$$P^{\frac{1}{2}}v(g,\xi,t)=\big(I+\Xi(t)^\times\big)P^{\frac{1}{2}}\bar{v}(g,\xi,t), \tag{19}$$

where P is positive definite and $\bar{v}(g,\xi,t)$ is an estimate of $v(g,\xi,t)$. Also let $u(g,\xi,t)\in\mathbb{R}^3$ and $\bar{\Xi}(t)\in\mathbb{R}^3$ satisfy

$$P^{\frac{1}{2}}u(g,\xi,t)=\big(I+\bar{\Xi}(t)^\times\big)P^{\frac{1}{2}}\bar{v}(g,\xi,t), \tag{20}$$

where $\bar{\Xi}(t)$ is an estimate of $\Xi(t)$. We define

$$\Delta(t)\triangleq\Xi(t)-\bar{\Xi}(t).$$

If in addition the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta^\times P^{\frac{1}{2}}\bar{v}\Big)=-\epsilon\big(\Delta^\times P^{\frac{1}{2}}\bar{v}\big)+\zeta^\times\big(\Delta^\times P^{\frac{1}{2}}\bar{v}\big) \tag{21}$$

is satisfied, where $0<\epsilon(t)\in[\gamma,\delta]$ and $\|\zeta(t)\|$ is bounded, then the control input $u(t)$ rejects the disturbance input $v(t)$. Further, $\dot{V}_{P^{\frac{1}{2}}(v-u)}\le-\gamma V_{P^{\frac{1}{2}}(v-u)}$, where $V_\eta\triangleq\frac{1}{2}\eta^T\eta$.

Proof: For ease of notation, we define $\eta(t)\triangleq P^{\frac{1}{2}}v(t)$, $\bar{\eta}(t)\triangleq P^{\frac{1}{2}}\bar{v}(t)$ and $\eta^c(t)\triangleq P^{\frac{1}{2}}u(t)$. Then we construct the energy-like function

$$V_{P^{\frac{1}{2}}(v-u)}(t)=V_{(\eta-\eta^c)}(t)\triangleq\frac{1}{2}\big(\eta(t)-\eta^c(t)\big)^{\mathrm{T}}\big(\eta(t)-\eta^c(t)\big).$$

Note that $\eta(t)-\eta^c(t)=\Delta(t)^\times\bar{\eta}(t)$. The time derivative of this function along (19)-(20) is:

$$\begin{aligned}\dot{V}_{(\eta-\eta^c)}&=\big(\eta-\eta^c\big)^{\mathrm{T}}\big(\dot{\eta}-\dot{\eta}^c\big)\\&=\big(\Delta^\times\bar{\eta}\big)^{\mathrm{T}}\big(\dot{\Delta}^\times\bar{\eta}+\Delta^\times\dot{\bar{\eta}}\big)\\&=-\epsilon\big(\Delta^\times\bar{\eta}\big)^{\mathrm{T}}\big(\Delta^\times\bar{\eta}\big)\le-\gamma V_{(\eta-\eta^c)}\end{aligned}$$

using equation (21). Therefore, this time derivative is negative semi-definite. Note that $\dot{V}_{(\eta-\eta^c)}(t)$ is uniformly continuous and bounded due to the assumptions on $\eta(t),\bar{\eta}(t),\Xi(t)$ and $\bar{\Xi}(t)$. Therefore, by Barbalat's lemma, $\dot{V}_{(\eta-\eta^c)}(t)\to0$ as $t\to\infty$ [18]. This in turn implies $V_{(\eta-\eta^c)}(t)\to0$ since

$$\dot{V}_{(\eta-\eta^c)}(t)\le-\gamma V_{(\eta-\eta^c)}(t)\le0.$$

Thus, under the condition (21), $\eta-\eta^c(t)$ converges to 0, and therefore $v(t)-u(t)\to0$ as $t\to\infty$. Thus, the control input $u(t)$ rejects the disturbance $v(t)$. □

Further sufficient conditions for disturbance rejection, obtained from condition (21), can be obtained when bounds on the time rate of $\Xi(t)$ are known. The following corollary of Proposition 1 gives a set of sufficient conditions under which the control input rejects the disturbance by satisfying condition (21).

Corollary 1. *Let the uncertain vector $\Xi(t)$ and its time rate satisfy*

$$\Xi=\bar{\Xi}+b\bar{\eta},\ \ \dot{\Xi}=\beta\bar{\eta}+\zeta^\times\Xi-b\bar{\Xi}, \tag{22}$$

where $b(t)$ and $\beta(t)$ are in a bounded neighborhood of 0. Further, let the time rates of change of $\bar{\Xi}(t)$ and $\bar{\eta}(t)$ satisfy

$$\dot{\bar{\Xi}}=\zeta\times\bar{\Xi},\ \ \dot{\bar{\eta}}=-\epsilon\bar{\eta}-\bar{\Xi}, \tag{23}$$

where $\zeta(t)\in\mathbb{R}^3$ and $\epsilon(t)$ are as specified in Proposition 1. Then we can guarantee that $\lim_{t\to0}\eta(t)-\bar{\eta}(t)=0$, i.e., the control input $u(t)$ rejects the disturbance $v(t)$.

Proof: Equations (22) and (23) together imply that

$$\dot{\Delta}=\dot{\Xi}-\dot{\bar{\Xi}}=\beta\bar{\eta}+\zeta^\times\Delta-b\bar{\Xi}. \tag{24}$$

Thereafter, we verify that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta^\times\bar{\eta}\Big)&=\dot{\Delta}\times\bar{\eta}+\Delta\times\dot{\bar{\eta}}\\&=\zeta^\times\Big(\Delta^\times\bar{\eta}\Big)-b\bar{\Xi}^\times\bar{\eta}-\epsilon\Big(\Delta^\times\bar{\eta}\Big)-\Xi^\times\bar{\Xi}\\&=\zeta^\times\Big(\Delta^\times\bar{\eta}\Big)-b\bar{\Xi}^\times\bar{\eta}-\epsilon\Big(\Delta^\times\bar{\eta}\Big)-b\bar{\eta}^\times\bar{\Xi}\\&=-\epsilon\Big(\Delta^\times\bar{\eta}\Big)+\zeta^\times\Big(\Delta^\times\bar{\eta}\Big),\end{aligned}$$

which is equivalent to condition (21). □

The disturbance rejection control $u(t)$ is thus given by equations (20) and (23), with initial conditions for $\bar{\eta}$ and $\bar{\Xi}$.

B. Disturbance Rejection Control Laws

These results can now be applied to the system (5)-(6) with uncertain inputs $\varphi_u(t)$ and $\tau_u(t)$ satisfying the bounds given earlier. We first write down the control inputs to the system:

$$\begin{aligned}\varphi_c=&-L_\nu\upsilon+MQ^{\mathrm{T}}\dot{\nu}_r+(Q^{\mathrm{T}}\Omega_r)^\times\big(M(\upsilon+Q^{\mathrm{T}}\nu_r)\big)\\&-D_\nu(\nu)\big(\upsilon+Q^{\mathrm{T}}\nu_r\big)-\rho g\mathcal{V}Q^{\mathrm{T}}R_r^{\mathrm{T}}e_3-Q^{\mathrm{T}}Nx-\varphi_u^c,\end{aligned} \tag{25}$$

$$\begin{aligned}\tau_c=&-L_\Omega\omega+JQ^{\mathrm{T}}\dot{\Omega}_r+(Q^{\mathrm{T}}\Omega_r)^\times(JQ^{\mathrm{T}}\Omega_r)\\&+(Q^{\mathrm{T}}\nu_r)^\times(MQ^{\mathrm{T}}\nu_r)-D_\Omega(\Omega)(\omega+Q^T\Omega_r)\\&+(\rho g\mathcal{V})r_{CB}^\times(Q^{\mathrm{T}}R_r^{\mathrm{T}}e_3)+\Phi'(\mathrm{trace}(K-KQ))\\&\times\big[k_1e_1^\times Q^{\mathrm{T}}e_1+k_2e_2^\times Q^{\mathrm{T}}e_2+k_3e_3^\times Q^{\mathrm{T}}e_3\big]-\tau_u^c,\end{aligned} \tag{26}$$

where φ_u^c and τ_u^c are disturbance rejection inputs for translation and attitude control respectively.

The disturbance rejection inputs φ_u^c and τ_u^c satisfy the following equations:

$$P_\varphi^{\frac{1}{2}}\varphi_u^c(g,\xi,t)=\big(I+\bar{\Xi}_\varphi(t)^\times\big)P_\varphi^{\frac{1}{2}}\bar{\varphi}_u(g,\xi,t), \tag{27}$$

$$P_\tau^{\frac{1}{2}}\tau_u^c(g,\xi,t)=\big(I+\bar{\Xi}_\tau(t)^\times\big)P_\tau^{\frac{1}{2}}\bar{\tau}_u(g,\xi,t). \tag{28}$$

The estimates $\bar{\varphi}_u$, $\bar{\tau}_u$, $\bar{\Xi}_\varphi$ and $\bar{\Xi}_\tau$ are updated by

$$P_\varphi^{\frac{1}{2}}\dot{\bar{\varphi}}_u = -\epsilon_\varphi P_\varphi^{\frac{1}{2}}\bar{\varphi}_u - \bar{\Xi}_\varphi, \quad \dot{\bar{\Xi}}_\varphi = \zeta_\varphi \times \bar{\Xi}_\varphi, \tag{29}$$

$$P_\tau^{\frac{1}{2}}\dot{\bar{\tau}}_u = -\epsilon_\tau P_\tau^{\frac{1}{2}}\bar{\tau}_u - \bar{\Xi}_\tau, \quad \dot{\bar{\Xi}}_\tau = \zeta_\tau \times \bar{\Xi}_\tau. \tag{30}$$

The main result on trajectory tracking with disturbance rejection using continuous control inputs now follows.

Theorem 2. *Let the AUV system given by* (5)-(6) *have uncertain inputs satisfying equations* (16)-(18) *and control inputs given by* (25)-(26)*, with disturbance rejection inputs satisfying equations* (27)-(30)*. Further, let*

$$\begin{aligned} 0 < \epsilon_\varphi \in [\gamma_\varphi, \delta_\varphi], \quad L_\nu = 4\gamma_\varphi P_\varphi, \\ 0 < \epsilon_\tau \in [\gamma_\tau, \delta_\tau], \quad L_\Omega = 4\gamma_\tau P_\tau. \end{aligned} \tag{31}$$

Then the equilibrium

$$(x_e, Q_e, \upsilon_e, \omega_e) = (0, I, 0, 0)$$

of the closed-loop dynamics is asymptotically stable in the presence of the disturbance force φ_ν^d and the disturbance torque τ_Ω^d. Further, the domain of attraction of this equilibrium is almost global in TSE(3).

Proof: The proof of this result is obtained after combining the earlier results Theorem 1, Proposition 1 and Corollary 1. For ease of notation, we define

$$\tilde{\varphi} \triangleq \varphi_u - \varphi_u^c, \quad \tilde{\tau} \triangleq \tau_u - \tau_u^c.$$

The time derivative of the Lyapunov function defined by (12) in the proof of Theorem 1 is:

$$\dot{V} = -\upsilon^{\mathrm{T}}\big(L_\nu \upsilon + \tilde{\varphi}\big) - \omega^{\mathrm{T}}\big(L_\Omega \omega + \tilde{\tau}\big), \tag{32}$$

which has additional terms depending on $\tilde{\varphi}$ and $\tilde{\tau}$. For this system, we define the candidate Lyapunov function:

$$\bar{V}(x, Q, \upsilon, \omega, \tilde{\varphi}, \tilde{\tau}) = V(x, Q, \upsilon, \omega) + \frac{1}{2}\tilde{\varphi}^{\mathrm{T}} P_\varphi \tilde{\varphi} + \frac{1}{2}\tilde{\tau}^{\mathrm{T}} P_\tau \tilde{\tau}. \tag{33}$$

Then using equations (25)-(32), we can show that the time derivative of the above function is:

$$\begin{aligned} \dot{\bar{V}} = &-(L_\nu^{\frac{1}{2}}\upsilon + \sqrt{\gamma_\varphi}P_\varphi^{\frac{1}{2}}\tilde{\varphi})^{\mathrm{T}}(L_\nu^{\frac{1}{2}}\upsilon + \sqrt{\gamma_\varphi}P_\varphi^{\frac{1}{2}}\tilde{\varphi}) \\ &-(L_\Omega^{\frac{1}{2}}\omega + \sqrt{\gamma_\tau}P_\tau^{\frac{1}{2}}\tilde{\tau})^{\mathrm{T}}(L_\Omega^{\frac{1}{2}}\omega + \sqrt{\gamma_\tau}P_\tau^{\frac{1}{2}}\tilde{\tau}), \end{aligned}$$

which is negative semi-definite. Using Proposition 1 and its Corollary 1, we can also show that $\tilde{\varphi} = \varphi_\nu^d - \varphi_\nu^u \to 0$ and $\tilde{\tau} = \tau_\Omega^d - \tau_\Omega^u \to 0$. Thereafter, one may use the arguments in the proof of Theorem 1 and the invariance principle to show that we get almost global convergence to the desired equilibrium in the feedback tracking errors. □

VI. Conclusions

This paper deals with the challenging control problem of tracking a reference trajectory in translation and attitude by an autonomous underwater vehicle in the presence of dynamic uncertainties. We first obtain a continuous tracking control law that can track the desired trajectory almost globally over the state space in the absence of any uncertain inputs in the dynamics model. Then we propose a very general internal model of the uncertainties based on known bounds on their magnitude, direction and time rates. With this internal model, we design a disturbance rejection control law that ensures almost global convergence to the desired trajectory in state space while rejecting disturbance due to the uncertain inputs. In forthcoming work, we will use these analytical control laws in numerical simulations to demonstrate their effectiveness with respect to trajectory tracking and disturbance attenuation.

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