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A Novel Internal Model-Based Tracking Control for a Class of Linear Time-Varying Plants

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Abstract— **This paper provides a novel method of constructing an internal model for a class of LTV plants driven by known LTI exosystems. It is shown how the realization of a time-varying internal model can be constructed by means of a novel feedback mechanism. The design of the internal model consists of two ingredients: 1) a time-varying system immersion of the exosystem; 2) automatic generation of the desired control input, based on the complete knowledge of the plant model. The important features of the proposed method lie in that the solution does not involve solving Sylvester differential equations, moreover the immersion is guaranteed to hold for the class of plant models under consideration. These features significantly broaden the range of applications of the proposed method and simplify the control implementation process.**

I. Introduction

It has long been recognized that internal model-based design is a powerful approach for tracking and/or rejecting signals governed by an autonomous exogenous system. The problem of this kind, known as *output regulation* or *servomechanism*, has been extensively studied for LTI systems. The extension to linear time-varying systems was introduced in [1] with extension of the regulator equation [2] to the LTV case. Recently a more comprehensive regulation theory for LTV and linear periodic systems has been developed in [3], [4], where the solvability of the problem was characterized. Based on the complete knowledge of the plant and exosystem models, a regulator synthesis was given by solving the Sylvester differential equation. For general LTV systems, however, the construction of a time-varying internal model by means of *system immersion* [5], remained open.

As a special case of the internal model principle, a scheme called *repetitive control* [6], [7], [8], focuses on dealing with periodic exogenous signals and has been applied to many industrial applications [9], [10]. It is worth noting that among the applications a wide class involves tracking and/or rejecting rotational-angle dependent signals, where the signals of interest are periodic with respect to the angular displacement but not periodic with respect to time as the rotational speed varies. For instance, the problem of controlling internal combustion engines, many engine subsystems demonstrate rotational-angle dependent behaviors [11] such as the intake and exhaust valves motion, but the engine rotational speed changes in real-time. To leverage the periodicity of signals in the rotational-angle domain, a feasible approach is to convert the plant model into the rotational-angle domain, which yields the plant model time-varying [12] (actually angle-varying). This motivates us to investigate the extension of the internal model-based design to the case of LTV plant models in this work.

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Based on the complete knowledge of the plant and exosytem models, we provide a new approach of internal model design for a class of LTV plants driven by LTI exosystems. The internal model unit is based on a unique feedback mechanism inspired by repetitive control [6], [7]. The important features of our design lie in that the plant model setup and the feedback mechanism allow us to avoid explicitly calculating the desired input, which keeps the regulated error identically zero. Moreover, the construction of the internal model can be realized by a time-varying system immersion, and the realization is guaranteed to hold for the class of plant models under consideration without assuming any additional property on the augmented system.

The rest of the paper is organized as follows: In Section II, we give the formulation of the asymptotic tracking and/or disturbance rejection problem for a class of LTV plants. The realization of a time-varying system immersion is given in Section III. Based on the unique feedback mechanism, the condition for the internal model design is proposed in Section IV. An illustrative example is presented in Section V, followed by some conclusions.

II. Problem Setting

Consider the tracking control problem for the LTV plant models of the form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y_{\rm p} &= C(t)x \\ y &= C(t)x + d \end{aligned} \tag{1}$$

with the plant state $x \in \mathbb{R}^n$, the plant output $y_{\rm p} \in \mathbb{R}$, the control input $u \in \mathbb{R}$, the regulated error $y \in \mathbb{R}$, and bounded and smooth $(A(\cdot),\, B(\cdot),\, C(\cdot))$. The signal d to be tracked or rejected is generated by an LTI system of the form

$$\begin{aligned} \dot{w} &= Sw \\ d &= Qw \end{aligned} \tag{2}$$

with the state of exosystem $w \in \mathbb{R}^{\rho}$. The plant model (1) satisfies the following assumptions:

Assumption 2.1: The pair $(A(\cdot),\, B(\cdot))$ and $(A(\cdot),\, C(\cdot))$ is uniformly controllable and uniformly observable [13] respectively.

The exosystem (2) satisfies the standing assumption which corresponds to periodic trajectories.

Assumption 2.2: All the eigenvalues of S are on the imaginary axis, and the pair $(S,\, Q)$ is observable.

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In addition, for the ease of s stabilizer design, we assume that the plant model (1) is already stabilized.

Assumption 2.3: Assume that the homogenous part of the plant model (1)

$$\dot{x} = A(t)x$$

is uniformly asymptotically stable.

We consider the tracking control problem given as follows:

Problem 2.1: The tracking control problem consists of finding an output-feedback compensator of the form

$$\begin{aligned} \dot{\zeta} &= G(t)\zeta + F(t)y \\ u &= H(t)\zeta + K(t)y \end{aligned} \tag{3}$$

with state $\zeta \in \mathbb{R}^{n_\zeta}$, such that:

1. The origin of the closed-loop unforced system ($w = 0$) is a uniformly asymptotically stable equilibrium.
2. Trajectories of the closed-loop system originating from any initial condition (w_0, x_0, ζ_0) are bounded and $\lim_{t\to\infty} y(t) = 0$.

The solvability of Problem 2.1 is shown as follows (with a minor extension of the result for periodic systems in [4]).

Proposition 2.1: A stabilizing controller (3) is a regulator if and only if there exist smooth mappings $\Pi : \mathbb{R} \mapsto \mathbb{R}^{n\times\rho}$, $\Sigma : \mathbb{R} \mapsto \mathbb{R}^{n_\zeta\times\rho}$, and $R : \mathbb{R} \mapsto \mathbb{R}^{1\times\rho}$ satisfying the differential equations:

$$\begin{aligned} \dot{\Pi}(t) + \Pi(t)S &= A(t)\Pi(t) + B(t)R(t) \\ 0 &= C(t)\Pi(t) + Q \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{\Sigma}(t) + \Sigma(t)S &= G(t)\Sigma(t) \\ R(t) &= H(t)\Sigma(t)\,. \end{aligned} \tag{5}$$

Equation (5) characterizes the internal model principle in the time-varying setting. The role of the internal model is to generate the desired input $u_{\mathrm{ff}} = R(t)w$, which keeps the regulated output identically zero.

III. Time-Varying System Immersion

Based on the above solvability conditions, the existing research activities of the output regulation problem were spent on constructing the required input $u_{\mathrm{ff}} = R(t)w$ for the exosystem with an output of the form

$$\begin{aligned} \dot{w} &= Sw \\ u_{\mathrm{ff}} &= R(t)w\,. \end{aligned} \tag{6}$$

Note that exosystem state w is not measurable. If there is no (parametric) uncertainties in the plant model, the desired input u_{ff} can be constructed by solving the inverse of a Sylvester differential equation (SDE) [4, Proposition 5.1], which is very difficult to obtain a closed form solution. This difficulty could be overcome by resorting to the concept of system immersion. The usefulness of a system immersion is to reconstruct u_{ff} by finding another system whose output includes every output of system (6).

Definition 3.1: The LTV system (6) is said to be immersed into another LTV system

$$\begin{aligned} \dot{\xi} &= \Phi(t)\xi \\ u_{\mathrm{ff}} &= \Gamma(t)\xi\,. \end{aligned} \tag{7}$$

where $\Phi : \mathbb{R} \mapsto \mathbb{R}^{n_\xi\times n_\xi}$ and $\Gamma : \mathbb{R} \mapsto \mathbb{R}^{1\times n_\xi}$ are smooth functions of their arguments, if there exists a smooth mapping $U : \mathbb{R} \mapsto \mathbb{R}^{n_\xi\times\rho}$, satisfying

$$\begin{aligned} \dot{U}(t) + U(t)S &= \Phi(t)U(t) \\ R(t) &= \Gamma(t)U(t)\,. \end{aligned}$$

If such an immersion exists, the pair $(\Phi(\cdot), \Gamma(\cdot))$ is termed as the *internal model pair* of (6). Note that for a robust design, it means that immersion (7) can be found, and it is independent of parametric uncertainties in $R(t)$. The difficulty lies in the fact that an explicit design method for finding such an immersion of a time-varying pair $(S, R(\cdot))$ is not available yet due to the absence of Cayley-Hamilton like theorem.

In this study, we present an explicit design method of the internal model unit, whose role is to keep the regulated error identically zero if the plant is appropriately initialized. Inspired by the unique structure of repetitive control [6], [7], we propose a feedback mechanism (Figure 1) to realize such a persistent input. The idea is that a self-excitation mechanism is embedded in the feedback loop so that it drives the plant to compensate the persistent and bounded signal $d(t)$ when $y(t) = 0$. The specific steps are: first, by designing controllers in the feedback loop, the exogenous signal $d(t)$ is embedded in the place of u_d (see Section III); second, by finding certain conditions between the internal model candidate and the plant model, the required control input u_{ff} is automatically generated, which keeps the regulated error $y(t)$ identically zero (see Section IV). In short we would like to find a certain condition, under which the exosystem (2) is immersed into a time-varying system, and u_{ff} can be automatically generated by leveraging on the setup (1) to avoid explicitly computing $R(t)$. The overall controller is given as the cascade connection of an internal model unit and the stabilizer of the interconnected plant model and the internal model. The stabilizer is of the form

$$\begin{aligned} \dot{\xi}_{\mathrm{st}} &= G_{\mathrm{st}}(t)\xi_{\mathrm{st}} + F_{\mathrm{st}}(t)y \\ u_{\mathrm{st}} &= H_{\mathrm{st}}(t)\xi_{\mathrm{st}} + K_{\mathrm{st}}(t)y \end{aligned} \tag{8}$$

with state ξ_{st} of the stabilizer of the interconnected system. The internal model candidate is of the form

$$\begin{aligned} \dot{\xi}_1 &= \Phi_1(t)\xi_1 + \Psi_1(t)u \\ u_d &= \Gamma_1(t)\xi_1 \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dot{\xi}_2 &= \Phi_2(t)\xi_2 + \Psi_2(t)(u_{\mathrm{st}} - u_d) \\ u &= \Gamma_2(t)\xi_2 + D_2(t)(u_{\mathrm{st}} - u_d) \end{aligned} \tag{10}$$

with the state $\xi = \mathrm{col}(\xi_1, \xi_2)$. In what follows, we show how to construct the internal model (9)–(10), by first showing in Section III that how to immerse exosystem (2) into (9)–(10)

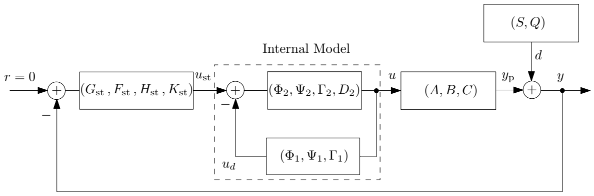

Fig. 1. Block diagram of the error-feedback compensator.

when $u_{\rm st} = 0$, then showing how the desired input $u_{\rm ff}$ can be automatically generated in Section IV.

To begin with, assume that the initial condition is appropriately set such that $y(t_0) = 0$ and hence $u_{\rm st} = 0$. Then our goal is to construct an internal model which produces a self-excited $u = u_{\rm ff}$, which keeps $y(t) = 0$, that is, the exosystem with output (6) is immersed to

$$\begin{aligned} \dot{\xi} &= \Phi(t)\xi \\ u &= \Gamma(t)\xi\,, \end{aligned} \tag{11}$$

where $\Phi(t) = \begin{pmatrix} \Phi_1(t) - \Psi_1(t)D_2(t)\Gamma_1(t) & \Psi_1(t)\Gamma_2(t) \\ -\Psi_2(t)\Gamma_1(t) & \Phi_2(t) \end{pmatrix}$. $\Gamma(t) = \begin{pmatrix} -D_2(t)\Gamma_1(t) & \Gamma_2(t) \end{pmatrix}$. If there exist such pairs $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ and $(\Phi_2(\cdot),\ \Psi_2(\cdot),\ \Gamma_2(\cdot),\ D_2(\cdot))$, then for the given feedback structure there exists a realization of the time-varying internal pair $(\Phi(\cdot), \Gamma(\cdot))$ (11):

$$\begin{aligned} \dot{\xi} &= G_{\rm im}(t)\xi + F_{\rm im}(t)\Gamma_{\rm im}(t)\xi \\ u &= \Gamma_{\rm im}(t)\xi\,, \end{aligned} \tag{12}$$

where $G_{\rm im}(t) = \begin{pmatrix} \Phi_1(t) & 0 \\ -\Psi_2(t)\Gamma_1(t) & \Phi_2(t) \end{pmatrix}$, $F_{\rm im}(t) = \begin{pmatrix} \Psi_1(t) \\ 0 \end{pmatrix}$, $\Gamma_{\rm im}(t) = \Gamma(t)$.

Since in general the pair $(\Phi(\cdot), \Gamma(\cdot))$ is not uniformly observable, how to design the controllers in (11) such that exosystem with output (6) is immersed into (11) is not clear. In order to make the realization (11) real, we propose an approach in twofold:

T1. Find the condition for $(\Phi_1(\cdot)\,, \Psi_1(\cdot)\,, \Gamma_1(\cdot))$, $(\Phi_2(\cdot)\,, \Psi_2(\cdot)\,,\ \Gamma_2(\cdot)\,, D_2(\cdot))$ and (S, Q) under which signal d can be embedded.

T2. Without explicit calculation, generate $u_{\rm ff} = R(t)w$ by matching the I/O maps between the plant model and subsystem $(\Phi_1(\cdot)\,, \Psi_1(\cdot)\,, \Gamma_1(\cdot))$ (see Figure 1).

Task 1 is inspired by the fact that a suitable copy of exosystem is required to be embedded into the feedback controller [2] to achieve asymptotic regulation. Moreover if signal d can be embedded, then it will facilitate the generation of $u_{\rm ff}$ in Task 2. From Figure 1, it is clear that when the regulation is achieved, $y(t) = 0$ for all $t \geq t_0$, the output of the plant model $y_{\rm p}(t)$ is identically $-d(t)$. This observation is one reason why we are interested in seeing how signal $d(t)$ can be embedded inside the feedback loop.

Specifically, we show how to accomplish Task 1 in what follows, and leave Task 2 in Section IV. For Task 1 to see how to embed d in the feedback loop, note that controllers (9) and (10) are in the cascade connection. If exogenous signal d can be embedded in the place of u_d inside the feedback loop, then the loop connection can be put in the following form

$$\begin{aligned} \dot{\xi} &= \Phi(t)\xi \\ d &= \Gamma_1^{\rm a}(t)\xi\,, \end{aligned} \tag{13}$$

with $\Gamma_1^{\rm a}(t) = \begin{pmatrix} -\Gamma_1(t) & 0 \end{pmatrix}$.

For the ease of our derivation, we now choose specific forms of exosystem (2), controllers $(\Phi_1(\cdot),\ \Psi_1(\cdot),\ \Gamma_1(\cdot))$ and $(\Phi_2(\cdot), \Psi_2(\cdot), \Gamma_2(\cdot), D_2(\cdot))$. Since the LTI pair (S, Q) is observable, we consider that the exosystem is of the observer canonical form $(S_{\rm o}, Q_{\rm o})$, with a constant vector $\alpha = {\rm col}(\alpha_{\rho-1}, \cdots, \alpha_0) \in \mathbb{R}^\rho$ collecting the first column of $S_{\rm o}$. Also we choose (13) as follows: Since $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ is uniform observable, it is topologically equivalent to the observer canonical form $(\Phi_{1\rm o}(t), \Psi_{1\rm o}(\cdot), \Gamma_{1\rm o})$, with $g(t) = {\rm col}(g_{\rho-1}(t), \cdots, g_0(t)) \in \mathbb{R}^\rho$ collecting the first column of $\Phi_{1\rm o}(\cdot)$ and $\Psi_{1\rm o}(t) = f(t) = {\rm col}(f_{\rho-1}(t), \cdots, f_0(t)) \in \mathbb{R}^\rho$. Since $(\Phi_2(\cdot),\ \Psi_2(\cdot),\ \Gamma_2(\cdot),\ D_2(\cdot))$ is uniform controllable, it is topologically equivalent to the controllability canonical form $(\Phi_{2\rm c}(\cdot),\ \Psi_{2\rm c},\ \Gamma_{2\rm c}(\cdot),\ D_{2\rm c}(\cdot))$ and the phase-variable (controller) form $(\Phi_{2\rm p}(\cdot), \Psi_{2\rm p}, \Gamma_{2\rm p}(\cdot), D_{2\rm p}(\cdot))$. Denote $q(t) = {\rm col}(q_{\rho-2}(t), \cdots, q_0(t)) \in \mathbb{R}^{\rho-1}$ the first column of $\Phi_{2\rm c}(\cdot)$, and $p(t) = {\rm col}(p_0(t), \cdots, p_{\rho-1}(t)) \in \mathbb{R}^\rho$ the coefficients of $\Gamma_{2\rm c}(t) = (p_0(t) - p_{\rho-1}(t)q_0(t)\,, \cdots, p_{\rho-2}(t) - p_{\rho-1}(t)q_{\rho-2}(t))$, and $D_{2\rm c}(t) = p_{\rho-1}(t)$. The coefficients of $(\Phi_{2\rm p}(\cdot),\ \Psi_{2\rm p}(\cdot),\ \Gamma_{2\rm p}(\cdot),\ D_{2\rm p}(\cdot))$, $\bar{p}(t)$ and $\bar{q}(t)$ can be represented by the derivatives of $p(t)$ and $q(t)$.

Now we are in position to find conditions to enforce (13) by resorting to a suitable output mapping, which yields the resulting system uniformly observable. Define an auxiliary signal $v = \Gamma_{2\rm o}\xi_2$ with $\Gamma_{2\rm o} = (1\ 0\ \cdots\ 0)$ and observe that the system is of the form

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_2 \\ \dot{\xi}_1 \end{pmatrix} &= \Phi_{\rm t}(t) \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix} \\ v &= \begin{pmatrix} \Gamma_{2\rm o} & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix}, \end{aligned} \tag{14}$$

where

$$\Phi_{\rm t}(t) = \begin{pmatrix} \Phi_{2\rm p}(t) & -\Psi_{2\rm p}\Gamma_{1\rm o} \\ \Psi_{1\rm o}(t)\Gamma_{2\rm p}(t) & \Phi_{1\rm o}(t) - \Psi_{1\rm o}(t)D_{2\rm p}(t)\Gamma_{1\rm o}(t) \end{pmatrix}$$

is in the lower triangular form. It can be shown that this fact and the output mapping in the form of $\Gamma_{\rm o} = (\Gamma_{2\rm o}\ 0)$ makes the pair $(\Phi_{\rm t}(\cdot),\ \Gamma_{\rm o})$ uniformly observable. Therefore, it is topologically equivalent to the system in observer canonical form as

$$\begin{aligned} \dot{\bar{\xi}} &= \Phi_{\rm o}(t)\bar{\xi} \\ v &= \Gamma_{\rm o}\,\bar{\xi}\,, \end{aligned} \tag{15}$$

where it can be verified that

$$\Phi_{\rm o}(t) = \begin{pmatrix} \mathcal{O}_{\Phi_1}(t) \cdot \begin{pmatrix} 1 \\ q(t) \end{pmatrix} + \mathcal{C}_{\Psi_1}(t)\,p(t) & \begin{matrix} I \\ 0 \end{matrix} \end{pmatrix}_{2\rho-1}, \tag{16}$$

with

$$\mathcal{O}_{\Phi_1}(t) = \begin{pmatrix} G_\rho(t) & \cdots & \begin{pmatrix} 0_{\rho-2} \\ 1 \\ G_1(t) \end{pmatrix} & \begin{pmatrix} 0_{\rho-1} \\ 1 \\ G_0(t) \end{pmatrix} \end{pmatrix}_{2\rho\times\rho} ,$$

and

$$\mathcal{C}_{\Psi_1}(t) = \begin{pmatrix} F_{\rho-1}(t) & \cdots & \begin{pmatrix} 0_{\rho-1} \\ F_1(t) \end{pmatrix} & \begin{pmatrix} 0_{\rho} \\ F_0(t) \end{pmatrix} \end{pmatrix}_{2\rho\times\rho} ,$$

with

$$G_{k+1}(t) = \begin{pmatrix} G_k(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{G}_k(t) \end{pmatrix} , \quad G_0(t) = g(t) ,$$

$k = 0, \cdots, \rho - 1$, and

$$F_{k+1}(t) = \begin{pmatrix} F_k(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{F}_k(t) \end{pmatrix} , \quad F_0(t) = f(t) ,$$

$k = 0, \cdots, \rho - 2$.

On the other hand, if (13) holds, then based on the given structure it is realized as

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} &= \begin{pmatrix} \Phi_{1\mathrm{o}}(t) - \Psi_{1\mathrm{o}}(t) D_{2\mathrm{p}}(t)\Gamma_{1\mathrm{o}}(t) & \Psi_{1\mathrm{o}}(t)\Gamma_{2\mathrm{p}}(t) \\ 0 & \Phi_{2\mathrm{p}}(t) \end{pmatrix} \\ &\quad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Psi_{2\mathrm{p}} \end{pmatrix} d \\ d &= \begin{pmatrix} -\Gamma_{1\mathrm{o}} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} . \end{aligned} \tag{17}$$

From (17) and (14), it is clear that

$$\begin{aligned} \dot{\xi}_2 &= \Phi_{2\mathrm{p}}(t)\xi_2 + \Psi_{2\mathrm{p}}\, d \\ v &= \Gamma_{2\mathrm{o}}\, \xi_2 \end{aligned} \tag{18}$$

where the triplet $(\Phi_{2\mathrm{p}}(\cdot), \Psi_{2\mathrm{p}}, \Gamma_{2\mathrm{o}})$ is in both observer and controller canonical form. Then the cascade connection of exosystem (2) and (18) reads as

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_2 \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} \Phi_{2\mathrm{p}}(t) & \Psi_{2\mathrm{p}} Q_\mathrm{o} \\ 0 & S_\mathrm{o} \end{pmatrix} \begin{pmatrix} \xi_2 \\ w \end{pmatrix} \\ v &= \begin{pmatrix} \Gamma_{1\mathrm{o}} & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ w \end{pmatrix} . \end{aligned} \tag{19}$$

Again the above system is uniformly observable. Therefore it can be transformed into a system in observer canonical form $(\bar{\Phi}_\mathrm{o}(\cdot), \Gamma_\mathrm{o})$, where it can be verified that

$$\bar{\Phi}_\mathrm{o}(t) = \begin{pmatrix} \mathcal{O}_S \cdot \begin{pmatrix} 1 \\ q(t) \end{pmatrix} & \begin{matrix} I \\ 0 \end{matrix} \end{pmatrix}_{2\rho} , \tag{20}$$

with

$$\mathcal{O}_S = \begin{pmatrix} \alpha & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 \\ \vdots & \cdots & \alpha & 1 \\ 0 & \cdots & 0 & \alpha \end{pmatrix}_{2\rho\times\rho} .$$

Equating (20) to (16) yields the following form

$$\begin{pmatrix} \mathcal{O}_{\Phi_1}(t) & \mathcal{C}_{\Psi_1}(t) \end{pmatrix} \begin{pmatrix} 1 \\ q(t) \\ p(t) \end{pmatrix} = \mathcal{O}_S \begin{pmatrix} 1 \\ q(t) \end{pmatrix} . \tag{21}$$

The solvability of the proposed realization (17) depends on the nonsingularity of the matrix $(\mathcal{O}_{\Phi_1}(t) - \mathcal{O}_S, \mathcal{C}_{\Psi_1}(t))$, and the result is characterized by the following lemma.

Lemma 3.1: If the pair $(\Phi_{1\mathrm{o}}(\cdot) - \alpha\Gamma_{1\mathrm{o}}, \Psi_{1\mathrm{o}}(\cdot))$ is uniformly controllable, there exists a unique solution for (21).

Proof: It is easy to check that the pair $(\Phi_1(\cdot) - \alpha\Gamma_1, \Gamma_1(\cdot))$ is uniformly observable. The proof follows the same lines in [14, Corollary 2.16, Lemma 2.33]. ■

By solving (21), we have obtained a realization for system (14) and hence the realization (17) holds as well by noting (18) and (19), that is:

Proposition 3.2: If the condition in Lemma 3.1 holds, and the following differential equation

$$\dot{\Xi}_2(t) + \Xi_2(t) S_\mathrm{o} = \Phi_{2\mathrm{p}}(t)\Xi_2(t) + \Psi_{2\mathrm{p}} Q_\mathrm{o} , \tag{22}$$

admits a solution $\Xi_2(t)$, then the exosystem (2) is immersed into (13), and there exists a mapping $\Xi : \mathbb{R} \mapsto \mathbb{R}^{2\rho-1\times\rho}$ such that

$$\begin{aligned} \dot{\Xi}(t) + \Xi(t) S_\mathrm{o} &= \Phi(t)\Xi(t) \\ Q_\mathrm{o} &= \Gamma_1^\mathrm{a}(t)\Xi(t) . \end{aligned} \tag{23}$$

Proof: The existence of $\Xi_2(t)$ in (22) implies that $\xi_2 = \Xi_2(t) w$. Also note that both system (14) and (19) are of the same dimension and are uniformly observable, so there exists a mapping such that

$$\begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ M_1(t) & M_2(t) \end{pmatrix} \begin{pmatrix} \xi_2 \\ w \end{pmatrix} ,$$

which implies that $\xi_1 = [M_1(t)\Xi_2(t) + M_2(t)] w := \Xi_1(t) w$. Therefore the proof follows by noting that

$$\Xi(t) = \begin{pmatrix} \Xi_1(t) \\ \Xi_2(t) \end{pmatrix} , \quad \Gamma_1^\mathrm{a}(t) = \begin{pmatrix} -\Gamma_{1\mathrm{o}} & 0 \end{pmatrix} .$$

In particular,

$$\begin{aligned} \dot{\Xi}_1(t) + \Xi_1(t) S_\mathrm{o} &= (\Phi_{1\mathrm{o}}(t) - \Psi_{1\mathrm{o}}(t) D_{2\mathrm{p}}(t)\Gamma_{1\mathrm{o}})\Xi_1(t) \\ &\quad + \Psi_{1\mathrm{o}}(t)\Gamma_{2\mathrm{p}}(t)\Xi_2(t) \\ Q_\mathrm{o} &= -\Gamma_{1\mathrm{o}}\Xi_1(t) . \end{aligned} \tag{24}$$

■

For LTV plant models, condition (21) alone cannot make system (12) the internal model, as the signal u generated in equation (12) is not necessarily the desired input u_{ff}. So we need to provide an extra condition under which the above construction is indeed an internal model unit.

IV. Design of Internal Model

We have shown that the exosystem (2) can be immersed into system (13) by solving an algebraic equation (21), which yields the coefficients, $p_i(t)$ and $q_i(t)$, of $(\Phi_{2\mathrm{c}}(\cdot), \Gamma_{2\mathrm{c}}(\cdot), D_{2\mathrm{c}}(\cdot))$ (10) in terms of the coefficients, $f_i(t)$

and $g_i(t)$, and their derivatives of system (9). In order to make system (13) the internal model, we also need to find the condition under which the control input $u(t)$ is exactly the desired one $u_{\rm ff}$ by choosing the remaining degree of freedom in (13), i.e., $g_i(t)$ and $f_i(t)$ in $(\Phi_{1\rm o}(\cdot), \Psi_{1\rm o}(\cdot))$ (9).

Recalling equation (12), the following condition holds based on the above design

$$\begin{aligned}\begin{pmatrix}\dot\xi_1\\ \dot\xi_2\end{pmatrix} &= \begin{pmatrix}\Phi_{1\rm o}(t) & 0\\ -\Psi_{2\rm p}\Gamma_{1\rm o} & \Phi_{2\rm p}(t)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} + \begin{pmatrix}\Psi_{1\rm o}(t)\\ 0\end{pmatrix}u\\ u &= \begin{pmatrix}-D_{2\rm p}(t)\Gamma_{1\rm o} & \Gamma_{2\rm p}(t)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},\end{aligned} \tag{25}$$

where u may not be the desired input $u_{\rm ff}$. As system $(\Phi_{2\rm p}(\cdot)\,, \Psi_{2\rm p}\,, \Gamma_{2\rm p}(\cdot)\,, D_{2\rm p}(\cdot))$ is fixed (by solving (21) to relate $(\Phi_{1\rm o}(\cdot),\ \Psi_{1\rm o}(\cdot),\ \Gamma_{1\rm o})$, we need to find a condition between $(\Phi_{1\rm o}(\cdot),\ \Psi_{1\rm o}(\cdot),\ \Gamma_{1\rm o})$ and $(A(\cdot),\ B(\cdot),\ C(\cdot))$ under which the input $u = u_{\rm ff}$. For the design of $(\Phi_{1\rm o}(\cdot),\ \Psi_{1\rm o}(\cdot),\ \Gamma_{1\rm o})$ we make the following assumption:

Assumption 4.1: The dimensions of ξ_1 and x are the same.

Assumption 4.1 will be removed and the more general situation will be discussed later but the idea used for the design remains the same.

Lemma 4.1: Suppose that condition (21) and Assumptions 2.1, 4.1 hold and if the I/O maps of $(A(\cdot), B(\cdot), C(\cdot))$ and $(\Phi_{1\rm o}(\cdot), \Psi_{1\rm o}(\cdot), \Gamma_{1\rm o}(\cdot))$ are the same, then system (12) is indeed an internal model unit, i.e., there exists a mapping $\Sigma : \mathbb{R} \mapsto \mathbb{R}^{2\rho-1\times\rho}$ such that

$$\begin{aligned}\dot\Sigma(t) + \Sigma(t)S &= \Phi(t)\Sigma(t)\\ R(t) &= \Gamma(t)\Sigma(t)\,.\end{aligned}$$

Proof: From Proposition 3.2, equation (23) holds. By assumption $(A(\cdot),\ B(\cdot),\ C(\cdot))$ and $(\Phi_{1\rm o}(\cdot),\ \Psi_{1\rm o}(\cdot),\ \Gamma_{1\rm o})$ are uniform realizations of the same impulse response, hence they are related by a Lyapunov transformation $T : \mathbb{R} \mapsto \mathbb{R}^{\rho\times\rho}$ such that

$$\Phi_{1\rm o}(t) = T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot T(t)\,,\ \Gamma_{1\rm o} = C(t)T(t)\,,$$
$$\Psi_{1\rm o}(t) = T^{-1}(t)B(t)\,.$$

Condition (24) can be rewritten as

$$\begin{aligned}\dot\Pi(t) + \Pi(t)S &= A(t)\Pi(t) + B(t)(\Gamma_{2\rm p}(t)\Xi_2(t)\\ &\quad - D_{2\rm p}(t)\Gamma_{1\rm o}T^{-1}(t)\Pi(t))\\ 0 &= C(t)\Pi(t) + Q\,,\end{aligned}$$

by setting $\Pi(t) = T(t)\Xi_1(t)$. The above condition is nothing else but the regulator equation (4) which implies $R(t) = -D_{2\rm p}(t)\Gamma_{1\rm o}\Xi_1(t) + \Gamma_{2\rm p}(t)\Xi_2(t)$. The proof follows by noting that

$$\Sigma(t) = \begin{pmatrix}\Xi_1(t)\\ \Xi_2(t)\end{pmatrix}, \quad \Gamma(t) = \begin{pmatrix}-D_{2\rm p}(t)\Gamma_{1\rm o} & \Gamma_{2\rm p}(t)\end{pmatrix}.$$

■

The easiest design of $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ can be chosen as $(\Phi_{1\rm o}(\cdot), \Psi_{1\rm o}(\cdot), \Gamma_{1\rm o}) = (A_{\rm o}(\cdot), B_{\rm o}(\cdot), C_{\rm o})$, where $(A_{\rm o}(\cdot), B_{\rm o}(\cdot), C_{\rm o})$ is in observer canonical form.

Now we consider the general case where $\dim(\xi_1) \neq \dim(x)$. Without loss of generality, assume that $\dim(\xi_1) > \dim(x)$, as for the case that $\dim(\xi_1) < \dim(x)$, one can always augment the exosystem (2) such that $\dim(\xi_1) = \dim(x)$. Augment system $(A(\cdot), B(\cdot), C(\cdot))$ with an exponentially stable LTI filter, the resulting augmented plant model $(\bar A(\cdot), \bar B(\cdot), \bar C(\cdot))$ shares the same I/O map with $(A(\cdot), B(\cdot), C(\cdot))$, and $(\bar A(\cdot),\ \bar C(\cdot))$ is uniformly observable.

Example 4.1: Consider the case that $\rho = 4$ and $n = 2$. Augment $(A(\cdot), B(\cdot), C(\cdot))$ with an exponential stable LTI filter $\dfrac{s^2 + c_1 s + c_0}{s^2 + c_1 s + c_0}$, the resulting augmented plant model $(\bar A(\cdot), \bar B(\cdot), \bar C(\cdot))$ can be put in the following observer canonical form where

$$\bar A_{\rm o}(t) = \begin{pmatrix}-c_1 + a_1(t) & 1 & 0 & 0\\ -(c_0 + c_1 a_1(t) + a_0(t)) & 0 & 1 & 0\\ -(c_0 a_1(t) + c_1 a_0(t)) & 0 & 0 & 1\\ -c_0 a_0(t) & 0 & 0 & 0\end{pmatrix},$$

$$\bar B_{\rm o}(t) = \begin{pmatrix}b_1(t)\\ c_1 b_1(t) + b_0(t)\\ c_0 b_1(t) + c_1 b_0(t)\\ c_0 b_0(t)\end{pmatrix}, \ \bar C_{\rm o}(t) = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}.$$

The result in Lemma 4.1 holds by replacing $(A(\cdot), B(\cdot), C(\cdot))$ with $(\bar A(\cdot), \bar B(\cdot), \bar C(\cdot))$ and using Kalman decomposition arguments. The proof is straightforward and hence is omitted. The corresponding design of $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ can be determined as

$$(\Phi_{1\rm o}(\cdot), \Psi_{1\rm o}(\cdot), \Gamma_{1\rm o}) = (\bar A_{\rm o}(\cdot), \bar B_{\rm o}(\cdot), \bar C_{\rm o})\,.$$

where pair $(\bar A_{\rm o}(\cdot), \bar B_{\rm o}(\cdot), \bar C_{\rm o})$ is in observer canonical form.

Note that system (12) remains the internal model if $(\Phi_1(\cdot), \Psi_1(\cdot), \Gamma_1(\cdot))$ is a realization of the same impulse response of the plant model with parametric uncertainties.

Once the internal model is designed as shown in the preceding sections, Problem 2.1 is then converted to a stabilization problem. If the plant model is uniform minimum-phase and the internal model is a canonical realization (i.e., in (12) $\dot\xi = G_{\rm im}(t)\xi$ is uniformly asymptotically stable), a static output feedback stabilizer can be invoked (see [4]). We do not make such assumptions in this work. With uniform observability and controllability assumptions on the plant model (1) time-varying "pole-placement" technique [14] can be applied to the stabilization of the interconnection of the plant (1) and the internal model (9)– (10). Due the space limitation, the derivation is omitted.

V. Illustrative example

Consider the plant model (1) and the exosystem (2) as

$$A(t) = \phi(t)\begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}, \ B(t) = \phi(t)\begin{pmatrix}b_1\\ b_0\end{pmatrix},$$

$$S_{\rm o} = \begin{pmatrix}0 & 1\\ -\omega^2 & 0\end{pmatrix}, \quad Q_{\rm o} = C = \begin{pmatrix}1 & 0\end{pmatrix},$$

where $a_1 = 20$, $a_0 = 200$, $b_1 = 30$, and $b_0 = -100$, and $\phi \in \mathbb{R}$ is a smooth and bounded function of time.

Also it is easy to verify that $(A(\cdot), B(\cdot), C(\cdot))$ is uniformly controllable and observable and the plant is not minimum-phase. The plant model can be transformed to observer form

$$A_{\mathrm{o}}(t) = \begin{pmatrix} -a_1(t) & 1 \\ -a_0(t) & 0 \end{pmatrix}, \; B_{\mathrm{o}}(t) = \begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix}, \; C_{\mathrm{o}} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where

$$a_1(t) = a_1\phi(t) - \frac{\dot{\phi}(t)}{\phi(t)}, \quad a_0(t) = a_0\phi^2(t) - \dot{a}_1(t),$$

$$b_1(t) = b_1\phi(t), \quad b_0(t) = b_0\phi^2(t) + b_1 a_1(t)\phi(t).$$

The internal model is chosen in the form (9)–(10), where

$$\Phi_{1\mathrm{o}}(t) = \begin{pmatrix} -g_1(t) & 1 \\ -g_0(t) & 0 \end{pmatrix}, \quad \Psi_{1\mathrm{o}}(t) = \begin{pmatrix} f_1(t) \\ f_0(t) \end{pmatrix},$$

$$\Gamma_{1\mathrm{o}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \Phi_{2\mathrm{p}}(t) = -\bar{q}_0(t) = -q_0(t), \quad \Psi_{2\mathrm{p}} = 1,$$

$$\Gamma_{2\mathrm{p}}(t) = \bar{p}_0(t) - \bar{p}_1(t)\bar{q}_0(t), \quad D_{2\mathrm{p}}(t) = \bar{p}_1(t) = p_1(t),$$

with $\bar{p}_0(t) = \dot{p}_1(t) + p_0(t)$. Since $\dim(\xi_1) = \dim(x)$, the coefficients of $f_i(t)$ and $g_i(t)$ can be chosen by $g_i(t) = a_i(t)$, $f_i(t) = b_i(t)$, $i = 0, 1$. The coefficients of $q_0(t)$, $p_1(t)$, and $p_0(t)$ can be determined by solving equation (21), where

$$\begin{pmatrix} 0 & b_1(t) & 0 \\ a_1(t) & b_0 - \dot{b}_1(t) & b_1(t) \\ a_0(t) - \omega^2 & -\dot{b}_0(t) & b_0(t) \end{pmatrix} \begin{pmatrix} q_0(t) \\ p_1(t) \\ p_0(t) \end{pmatrix} = \begin{pmatrix} -a_1(t) \\ -a_0(t) + \dot{a}_1(t) + \omega^2 \\ \dot{a}_0(t) \end{pmatrix}.$$

The stabilizer (8) can be obtained by solving a similar algebraic equation, which is omitted duce to the space limitation. Figure 2 shows respectively the tracking error and the control input for a simulation with $\omega = 10$ and the initial conditions $y(0) = 0.5$, $w(0) = (1, -0.5)'$. Figure 3 shows respectively time-varying $\phi(t)$ and the time-varying coefficients of the plant model in observer form.

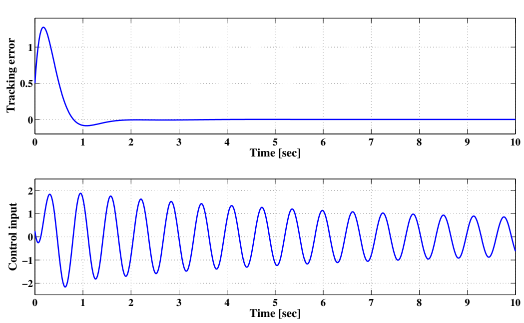

Fig. 2. Tracking error and control input.

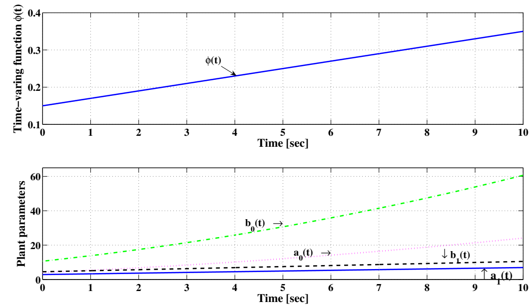

Fig. 3. Plant parameters.

VI. Conclusions

In this paper, we have proposed a new internal model-based design of asymptotic trajectory tracking and/or disturbance rejection for a class of LTV plants. Based on the unique feedback mechanism of the internal model, we have given the condition of the realization of the internal model, where the calculation of the desired input is not required a priori. The solution has been obtained without solving differential equations. The construction of a robust regulator remains open and is currently under investigation.

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