2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Exploiting Sparsity in the Sum-of-Squares Approximations to Robust Semidefinite Programs

Tanagorn Jennawasin*

Abstract— **This paper aims to improve computational complexity in the sum-of-squares approximations to robust semidefinite programs whose constraints depend polynomially on uncertain parameters. By exploiting sparsity, the proposed approach constructs sum-of-squares polynomials with smaller number of monomial elements, and hence gives approximate problems with smaller sizes. The sparse structure is extracted by a special graph pattern. The quality of the approximation is improved by dividing the parameter region, and can be expressed in terms of the resolution of the division. This expression shows that the proposed approach is asymptotically exact in the sense that, the quality can be arbitrarily improved by increasing the resolution of the division.**

I. Introduction

A robust semidefinite program (robust SDP in short) is the optimization of a linear objective function subject to linear matrix inequalities (LMIs in short) whose coefficients depend on uncertain parameters. It plays an important role in robust control [4], [20] an nonlinear optimization [9], [15]. A survey about robust SDPs can be found in [1].

A robust SDP is difficult to solve in general [1], [5]. Therefore, we have to consider an approximate approach, that is, a solvable SDP is constructed so that its feasible region is included in that of the original robust SDP. This approach is conservative since there is in general a nonzero gap, called an approximation error, between the optimal value of the approximate problem and that of the original problem. Early results on such an approximate approach are found in [1], [2], [5]. In the case of polynomial parameter dependence, asymptotically exact approximate approaches were proposed with the Kalman-Yakubovich-Popov lemma [14], [3], with Pólya's lemma [19], with the sum-of-squares (SOS) technique [15], [21], [7], and with the matrix dilation [11], [12], [13]. Here, the asymptotic exactness means that the approximation error can be made arbitrary small by considering an approximate problem of larger size, that is, an SDP with more number of variables and/or an LMI constraint of larger size.

Here we pay special interest on the SOS-based approach, which is the most widely used among the approximate approaches mentioned above due to its efficacy in implementation with the software SOSTOOLS [16]. In the SOS-based approach, a sequence of approximate problems which is asymptotically exact to the original one can be constructed by the degree increase of SOS polynomial matrices in [21] or by the division on the parameter region in [7]. The recent work [7] developed by the author showed several advantages of the region-dividing scheme over that based on the degree increase of the SOS polynomials. In spite of its advantages, an approximate problem of large size is required to guarantee convergence of the scheme. It may be beyond capability of the currently available SDP solvers in some situations. However, there is still a room for improvement of the result for a special class of robust SDPs of practical importance.

In this paper, we aim to improve the result of [7] by constructing an approximate problem of reduced size while the convergence of the region-dividing approach is still preserved. This can be done by considering the following structure of a given robust SDP. Here, the LMI constraint of the given robust SDP is supposed to depend polynomially on an uncertain parameter $\theta := (\theta_1, \theta_2, \dots, \theta_p)^{\mathrm{T}} \in \mathbb{R}^p$, and the maximum degree for θ_i in the polynomial representation is d_i for $i = 1, 2, \dots, p$. In the region-dividing approach in [7], it is supposed implicitly that the LMI constraint is represented as a *dense* polynomial in θ, that is, it contains all of monomials whose degree in θ_i is at most d_i for each of $i = 1, 2, \dots, p$. However, such a case hardly appears in many practical problems. Then, we assume in this paper that the constraint is a *sparse* polynomial in θ, i.e., only a few monomials appear in the polynomial representation. We use this structure to construct a reduced-size approximate problem. In particular, a special graph called a rectilinear Steiner arborescence is applied to capture the sparse structure. The resulting approximate problem always has a smaller size than the existing one, and hence computational efficiency is improved. The discrepancy is apparent when the number of monomials is small and the degrees d_i's are large. Note that, this idea has been used in [13] to reduce the size of the approximate problem based on the matrix-dilation approach.

The convergence of our approach is similarly obtained as that of [7]. Precisely, this approach is asymptotically exact in the sense that the approximation error converges to zero as the resolution of the division becomes finer. Moreover, the main feature of the region-dividing approach is still preserved in this setting. Namely, an upper bound on the approximation error can be obtained in terms of the resolution of the division. The existence of such an upper bound is the main result of this paper. This result is important because the tradeoff between the computational complexity and the amount of conservatism can be understood via this bound. The result can be used to construct an efficient division, which attains good approximation with moderate computational cost, along the same line as in [11], [7]. The procedure to obtain the main result in this paper can be

*T. Jennawasin is with Control System Laboratory, Toyota Technological Institute, Hisakata 2-12-1, Tempaku-ku, Nagoya 468-8511, JAPAN. Email: tanagorn@toyota-ti.ac.jp

978-1-4244-4524-0/09/$25.00 ©2009 AACC

performed in a similar fashion to [7], by using a connection between the SOS approach and the matrix-dilation approach.

It is notable that another method for exploiting sparsity in the SOS approach is independently studied in [10], [24]. This method is applied to polynomial optimization, which can be considered as a special class of robust SDPs. However, extension to the class of robust SDPs studied in this paper, as well as the existence of the error bound are not known in this method.

This paper is structured as follows. Section II introduces the concept of SOS matrices, and provides a robust SDP as well as an overview on the region-dividing approach. Section III is the main section, which gives a reduced-size approximate problem with an upper bound on the approximation error. A numerical example is provided in Section IV. Section V concludes this paper.

The notation used in this paper is rather standard. The symbol $\mathbb{Z}_+^p$ denotes the set of p-dimensional vectors of nonnegative integers. For $\theta = (\theta_1, \theta_2, \ldots, \theta_p)^{\mathrm{T}} \in \mathbb{R}^p$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)^{\mathrm{T}} \in \mathbb{Z}_+^p$, the symbol θ^α means the product $\theta_1^{\alpha_1}\theta_2^{\alpha_2}\cdots\theta_p^{\alpha_p}$. We let I_m denote the $m \times m$ identity matrix. For a real symmetric matrix A, the inequality $A \succeq O$ means that A is positive semi definite. Similarly, $A \succ O$ indicates that A is positive definite. For two real symmetric matrices A and B, the inequalities $A \succeq B$ and $A \succ B$ mean $A - B \succeq O$ and $A - B \succ O$, respectively. Finally, $C \otimes D$ stands for the Kronecker product of matrices C and D.

II. Preliminaries

A. Sum-of-squares polynomial matrices

Let $\mathbb{R}[\theta]^{m\times n}$ denote the set of $m \times n$ polynomial matrices in $\theta \in \mathbb{R}^p$ and $\mathbb{S}[\theta]^n$ denote the set of $n \times n$ symmetric polynomial matrices. We define the notion of *sum-of-squares (SOS)* polynomial matrices as follows.

Definition 1: [8], [21] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is said to be a sum of squares (SOS) if there exists a polynomial matrix $T \in \mathbb{R}[\theta]^{q\times m}$ such that

$$S(\theta) = T(\theta)^{\mathrm{T}} T(\theta).$$

This is a generalization of the SOS representation for scalars [9], [15]. We use $\Sigma[\theta]^m$ to represent the set of $m \times m$ SOS polynomial matrices. It is clear that any polynomial matrix $S \in \Sigma[\theta]^m$ is globally positive semidefinite, i.e., $S(\theta) \succeq 0$, $\forall \theta \in \mathbb{R}^p$, but the converse is not true in general.

A computational procedure for verifying whether $S(\theta)$ is an SOS proceeds as follows. Choose pairwise different monomials $u_1(\theta), \ldots, u_{n_u}(\theta)$ and search for the coefficient matrix Y in the representation

$$T(\theta) = Y(u(\theta) \otimes I_m)$$

with $Y = (Y_1, \ldots, Y_{n_u})$ and $u(\theta) = (u_1(\theta), \ldots, u_{n_u}(\theta))^{\mathrm{T}}$. The matrix $S(\theta)$ is said to be an SOS with respect to $u(\theta)$ if there exists some Y satisfying $S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}}(Y^{\mathrm{T}}Y)(u(\theta) \otimes I_m)$. Substituting $Z = Y^{\mathrm{T}}Y$ yields the following result.

Proposition 1: [8], [21] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is an SOS with respect to the monomial basis $u(\theta)$ if and only if there exists a symmetric matrix $Z \succeq O$ with

$$S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}} Z (u(\theta) \otimes I_m). \tag{1}$$

Expanding the right-hand side of (1) yields a polynomial whose coefficients depend affinely on elements of Z. As an identity in θ, we can match coefficients of the polynomials in both sides of (1). Hence the condition (1) can be interpreted as an affine constraint in Z. This implies that the problem to find $Z \succeq O$ with (1) can be formulated as an SDP. In other words, we can check whether $S \in \Sigma[\theta]^m$ with respect to some monomial basis by solving an SDP.

B. A robust SDP and its SOS approximation

A robust SDP is considered in this section. We briefly present construction of an approximate problem using the SOS approach, as well as improvement of the approximation by the region-dividing approach proposed in [11], [7].

A robust SDP is the following optimization problem:

$$\left.\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & F_0(\theta) + \displaystyle\sum_{i=1}^{n} x_i F_i(\theta) \succeq O, \quad \forall \theta \in \Theta, \end{array}\right\} \tag{2}$$

where $c \in \mathbb{R}^n$ is given, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ is an optimization variable, and $\theta := (\theta_1, \ldots, \theta_p)^{\mathrm{T}}$ is an uncertain parameter which can take any value in the p-dimensional interval $\Theta = \{\theta \in \mathbb{R}^p \mid \underline{\theta}_i \le \theta_i \le \overline{\theta}_i,\ \ i = 1, \ldots, p\}$. The coefficients $F_0(\theta), \ldots, F_n(\theta)$ are given $m \times m$ symmetric-matrix-valued functions which depend polynomially on the parameter θ. Let $F(x, \theta) \triangleq F_0(\theta) + \sum_{i=1}^{n} x_i F_i(\theta)$, we denote by d_i the maximum degree of $F(x, \theta)$ in θ_i for $i = 1, 2, \ldots, p$. We can assume without loss of generality that $\exists i$, $d_i \neq 0$. Otherwise, $F(x, \theta)$ is independent of θ and the problem (2) is just a standard SDP.

With the notion of the SOS matrices, an approximate SDP for (2) can be constructed. Quality of the approximation is improved by dividing the parameter region. Precisely, the approximation error decreases as the maximum size of the subregions becomes smaller. We first explain this idea by preparing some terminology for the region-dividing approach. A *division* $\Delta = \{\Theta^{[j]}\}_{j=1}^{J}$ of Θ is a set of closed p-dimensional intervals such that $\Theta = \cup_{j=1}^{J}\Theta^{[j]}$ holds and $\Theta^{[j]} \cap \Theta^{[k]}$ has no interior point whenever $j \neq k$. Each element $\Theta^{[j]}$ of a division Δ is called a *subregion*. Write $\Theta^{[j]}$ as $\Pi_{i=1}^{p}[\underline{\theta}_i^{[j]}, \overline{\theta}_i^{[j]}]$ for each $j = 1, 2, \ldots, J$. The *radius* of the subregion $\Theta^{[j]}$ is defined as $\mathrm{rad}\ \Theta^{[j]} := \max_i \frac{\overline{\theta}_i^{[j]} - \underline{\theta}_i^{[j]}}{2}$. The *maximum radius* of a division Δ is defined as $\overline{\mathrm{rad}}\ \Delta := \max_j \mathrm{rad}\ \Theta^{[j]}$.

For a given division Δ, we then construct the following approximate problem:

$$
\left. \begin{array}{lll} P(\Delta): & \text{minimize} & c^{\mathrm{T}}x \\ & \text{subject to} & F(x,\theta) = S_0^{[j]}(\theta) \\ & & + \displaystyle\sum_{i=1}^{p} (\theta_i - \underline{\theta}_i^{[j]})(\overline{\theta}_i^{[j]} - \theta_i) S_i^{[j]}(\theta), \\ & & \forall j = 1,2,\ldots,J, \end{array} \right\}
$$

where $S_0^{[j]}, S_1^{[j]}, \ldots, S_p^{[j]} \in \Sigma[\theta]^m$ are SOS matrices guaranteeing $F(x,\theta) \succeq O,\ \forall \theta \in \Theta^{[j]}$. We use the same monomial basis, say $u_i(\theta)$, for the SOS matrices $S_i^{[j]}(\theta)$, for all $j = 1,2,\ldots,J$. This means that $S_i^{[j]} = (u_i(\theta) \otimes I_m)^{\mathrm{T}} Z_i^{[j]} (u_i(\theta) \otimes I_m)$, $\forall j = 1,2,\ldots,J$, for some matrices $Z_i^{[j]}$'s. It is not difficult to express the problem $P(\Delta)$ as an SDP in the decision variables x and $Z_0^{[j]}, Z_1^{[j]}, \ldots, Z_p^{[j]}$ using the idea discussed at the end of Section II-A.

The existence of the SOS matrices $S_0^{[j]}(\theta), S_1^{[j]}(\theta), \ldots, S_p^{[j]}(\theta),\ j = 1,2,\ldots,J$ implies $F(x,\theta) \succeq O,\ \forall \theta \in \Theta$. This is immediately obtained from the definition of SOS matrices and the expression of $\Theta^{[j]}$. Hence the feasible region of $P(\Delta)$ projected into the space of x is included in that of (2). This implies $\inf P(\Delta) \geq v_{\mathrm{opt}}$, where $\inf P(\Delta)$ denotes the optimal value of $P(\Delta)$ and v_{opt} denotes the optimal value of (2).

The approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$ can be made smaller by subdivision on each subregion $\Theta^{[j]}$. It has been proved in [7] that $|\inf P(\Delta) - v_{\mathrm{opt}}|$ converges to zero as $\overline{\mathrm{rad}}\ \Delta \to 0$, if the monomial bases $u_i(\theta),\ i = 1,2,\ldots,p$ contain all monomials whose degree in θ_i is not exceed d_i for each $i = 1,2,\ldots,p$. Moreover, an upper bound on the approximation error is available under such choice of monomial bases. The size of the approximate problem, however, tends to become large even with a coarse division when the degrees d_i's and the dimension p are high. This drawback arises from the assumption that $F(x,\theta)$ contain all monomials whose degree in each θ_i is not exceed d_i, which is not suitable for practical problems.

In the next section, we assume that $F(x,\theta)$ contains only a small number of monomials and use this sparse structure to construct monomial bases with a few elements. This results in an approximate problem whose size is reduced from that in [7]. We will also prove that the convergence of the region-dividing approach as well as an upper bound on the approximation error are still available, even with these monomial bases of reduced size.

III. The Proposed Approach

A. Constructing a reduced-size approximate problem

We first state some graph theoretic concepts recapped from [13]. These concepts are necessary for constructing small-size monomial bases, which leads to construction of a small-size approximate problem. A *directed graph* consists of the set of vertices and the set of arcs, where an arc connects two vertices in either direction. A vertex α is said to be *reachable* from a vertex β if either $\alpha = \beta$ or the directed graph has a path emanating from β to α through the arcs in the directed way.

We next consider the following directed graph (V_0, A_0) in $\mathbb{R}^p$ for a given $(d_1, d_2, \ldots, d_p) \in \mathbb{Z}_+^p$: the set of vertices is $V_0 = \{\alpha \in \mathbb{Z}_+^p | 0 \leq \alpha_i \leq d_i \text{ for } i = 1,2,\ldots,p\}$; the set of arcs is $A_0 = \{(\alpha,\beta) | \alpha,\beta \in V_0, \alpha_i + 1 = \beta_i \text{ for some } i = 1,2,\ldots,p, \text{ and } \alpha_j = \beta_j, \forall j \neq i\}$. Namely, the arcs are the line segments of length one connecting two vertices and directed away from the origin. An arc (α,β) is said to be *parallel* to the ith axis, if it satisfies $\alpha_i + 1 = \beta_i$.

For a given nonempty set $S \subseteq V_0$, consider a subgraph (V, A) of (V_0, A_0) such that (i) V contains any vertex in S and the origin; (ii) any vertex in V is reachable from the origin through a unique path. Such a graph (V, A) is called *a rectilinear Steiner arborescence* [17], [22] for S. An example of a rectilinear Steiner arborescence is given in Section IV.

Let d_i's in the directed graph (V_0, A_0) being the degree of $F(x,\theta)$ in θ_i for $i = 1,2,\ldots,p$. We write $F(x,\theta) = \sum_{\alpha \in S} F_\alpha(x)\theta^\alpha$, and let S be the *support* of $F(x,\theta)$ defined as $S := \{\alpha \in V_0 | F_\alpha(x) \not\equiv O\}$. Since $F(x,\theta)$ is not independent of θ, the support S contains at least one element different from the origin.

For construction of the desired monomial bases, we consider a rectilinear Steiner arborescence (V, A) for S, which is desired to have a small length. It is obvious that $2 \leq |V| \leq |V_0|$. We number the vertices in V as $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(|V|)}$ in the way that a vertex $\alpha^{(r)}$ is reachable from a vertex $\alpha^{(q)}$ only if $q \leq r$. It follows that $\alpha^{(1)}$ is the origin. We now define

$$
m(\theta) = \begin{bmatrix} \theta^{\alpha^{(1)}} & \theta^{\alpha^{(2)}} & \cdots & \theta^{\alpha^{(|V|)}} \end{bmatrix}^{\mathrm{T}}
$$

and $\tilde{m}(\theta)$ as a vector constructed by removing redundant elements in $\begin{bmatrix} m^{\mathrm{T}}(\theta) & \theta_1 m^{\mathrm{T}}(\theta) & \cdots & \theta_p m^{\mathrm{T}}(\theta) \end{bmatrix}^{\mathrm{T}}$. We then consider the following monomial bases for $P(\Delta)$:

$$
u_0(\theta) = \tilde{m}(\theta), \quad u_i(\theta) = m(\theta),\ i = 1,\ldots,p. \tag{3}
$$

In a word, all the elements of the above monomial bases correspond to the vertices in V.

Here, we discuss on the size of the approximate problem. For a fixed division Δ, the size of the approximate problem $P(\Delta)$ depends on the size of chosen monomial bases. It can be seen that the size of the new monomial bases in (3) is $|V|$, while the size of the monomial bases in [7] is $|V_0|$. Since $|V| \leq |V_0|$, the size of the new approximate problem is not larger than that of the conventional one. Note here that the difference between $|V|$ and $|V_0|$ is apparent when $|S|$ is small and d_i's are large [13].

The remaining task is to verify the quantitative property of $P(\Delta)$ with the monomial bases in (3): there exists an upper bound on the approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$. This property leads to the convergence of the region-dividing approach with the reduced-size approximate problem. The details are discussed in the next subsection.

B. An upper bound on the approximation error

In this subsection, we provide an upper bound on the approximation error of the reduced-size approximate problem. This result is a generalization of that in [7], which is for the full-size approximate problem.

We first need the following assumption [11] in order to obtain the result.

Assumption 1:

(a) The robust SDP (2) is strictly feasible, that is, there exists $x \in \mathbb{R}^n$ such that

$$F(x,\theta) \succ O, \quad \forall \theta \in \Theta.$$

(b) For any v, the level set

$$\{x \in \mathbb{R}^n \mid c^{\mathrm{T}}x \le v, \;\; F(x,\theta) \succeq O, \;\; \forall\theta \in \Theta\}$$

is bounded.

Our main result, which provides the desired upper bound, is given in the following theorem.

Theorem 1: Suppose that Assumption 1 holds. Then, with the monomial bases in (3), the approximate problem $P(\Delta)$ satisfies

$$|\inf P(\Delta) - v_{\mathrm{opt}}| \le C\overline{\mathrm{rad}}\;\Delta \tag{4}$$

for any division Δ with $\overline{\mathrm{rad}}\;\Delta \le C_1$, where C_1 and C are positive numbers independent of Δ.

Remark 1: The inequality (4) implies that if the original robust SDP (2) is feasible, then $P(\Delta)$ is always feasible when the maximum radius $\overline{\mathrm{rad}}\;\Delta$ is small enough.

Remark 2: When the support S is equal to the whole vertex set V_0, a rectilinear Steiner arborescence for S is actually the same as the directed graph (V_0, A_0). In this case, we obtain the full-size monomial bases presented in [7]. In this sense, the result of Theorem 1 is a generalization of the main result in [7].

A direct consequence of this theorem is the asymptotic exactness of our approach. As we can see from (4), the approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$ converges to zero as the maximum radius of the division goes to zero. Evaluation of C_1 and C is available in [11], [12], though the resulting bound is often conservative. Once the value of C is obtained, we can compute a priori a value of $\overline{\mathrm{rad}}\;\;\Delta$ which can guarantee a sufficiently accurate optimal value. Recall that the existing sparse SOS approach does not provide such an explicit quantitative result.

The upper bound (4) also gives a relationship between the approximation error and the size of the approximate problem. Namely, in order to reduce the approximation error, we need to decrease the maximum radius. This increases the number of subregions and, then, the number of the LMI constraints. Especially when the parameter dimension is high, this increase is rapid and makes the approximate problem more difficult to solve. In order to reduce the computational complexity, however, it is possible to apply the adaptive division of the parameter region parallelly to that in [7]. The details are omitted.

C. A proof on the main theorem

This subsection is devoted to prove Theorem 1. In order to prove the statement, we need some results of the matrix-dilation approach [11], [12], [13]. The key idea of the proof is based on a relationship between a reduced-size version of the SOS approach and that of the matrix-dilation approach. Here, the procedure to compute the upper bound is performed in the similar line as [7]. We first construct an auxiliary approximate problem, say $P_1(\Delta)$, by utilizing the matrix-dilation approach in [12]. Then we prove the existence of an upper bound on the approximation error $|\inf P_1(\Delta) - v_{\mathrm{opt}}|$. In the second step, we show some connection between $P_1(\Delta)$ and $P(\Delta)$, which gives a way to compute an upper bound on $|\inf P(\Delta) - v_{\mathrm{opt}}|$ from that on $|\inf P_1(\Delta) - v_{\mathrm{opt}}|$.

We now give an overview on the matrix-dilation approach, in order to construct the auxiliary approximate problem $P_1(\Delta)$. First, we write $F(x,\theta) = \sum_{\alpha\in S} F_\alpha(x)\theta^\alpha$ and consider its structure discussed in Section III-A. We next consider a decomposition $F(x,\theta) = M(\theta)^{\mathrm{T}}G(x)M(\theta)$. The matrix $G(x)$ contains coefficient matrices of $F(x,\theta)$, while

$$\begin{aligned} M(\theta) &= \begin{bmatrix} \theta^{\alpha^{(1)}} I_m & \theta^{\alpha^{(2)}} I_m & \cdots & \theta^{\alpha^{(|V|)}} I_m \end{bmatrix}^{\mathrm{T}} \\ &= m(\theta) \otimes I_m. \end{aligned}$$

Moreover, we consider a matrix $H(\theta)$ such that the matrix $[M(\theta) \;\; H(\theta)]$ is nonsingular and the relation $M(\theta)^{\mathrm{T}}H(\theta) = O$ holds for all $\theta \in \mathbb{R}^p$. Such $H(\theta)$ is called an *orthogonal complement* of $M(\theta)$. An important fact is that the orthogonal complement $H(\theta)$ can be chosen to be affine in θ. Construction of $H(\theta)$ is discussed in [13].

For a given division Δ, pick up one subregion $\Theta^{[j]}$, which is a multi-dimensional interval $\Pi_{i=1}^{p}[\underline{\theta}_i^{[j]}, \overline{\theta}_i^{[j]}]$ by assumption. We define θ^{c} as the center of $\theta^{[j]}$, that is $\theta^{\mathrm{c}} := \begin{bmatrix} \frac{\underline{\theta}_1^{[j]}+\overline{\theta}_1^{[j]}}{2} & \frac{\underline{\theta}_2^{[j]}+\overline{\theta}_2^{[j]}}{2} & \cdots & \frac{\underline{\theta}_p^{[j]}+\overline{\theta}_p^{[j]}}{2} \end{bmatrix}$. Since $H(\theta)$ is affine in θ, it can be expanded around θ^{c} as

$$H(\theta) = H(\theta^{\mathrm{c}}) + (\theta_1 - \theta_1^{\mathrm{c}})H_1 + (\theta_2 - \theta_2^{\mathrm{c}})H_2 + \cdots + (\theta_p - \theta_p^{\mathrm{c}})H_p,$$

where $H_1, \ldots, H_p$ are constant matrices. We now consider the constraints

$$G(x) + H(\theta^{\mathrm{c}})(W^{[j]})^{\mathrm{T}} + W^{[j]}H(\theta^{\mathrm{c}})^{\mathrm{T}} - \sum_{i=1}^{p} V_i^{[j]} \succeq O, \tag{5}$$

$$V_i^{[j]} \succeq -(\overline{\theta}_i^{[j]} - \theta_i^{\mathrm{c}})(H_i(W^{[j]})^{\mathrm{T}} + W^{[j]}H_i^{\mathrm{T}}),\; i = 1, \ldots, p, \tag{6}$$

$$V_i^{[j]} \succeq (\overline{\theta}_i^{[j]} - \theta_i^{\mathrm{c}})(H_i(W^{[j]})^{\mathrm{T}} + W^{[j]}H_i^{\mathrm{T}}),\; i = 1, \ldots, p. \tag{7}$$

By following the idea of Ben-Tal and Nemirovski [2], it is easy to see that if there exist $V_1^{[j]}, V_2^{[j]}, \ldots, V_p^{[j]}$ satisfying the inequalities (5)–(7), then $(x, W^{[j]})$ satisfies the constraint

$$G(x) + H(\theta)(W^{[j]})^{\mathrm{T}} + W^{[j]}H(\theta)^{\mathrm{T}} \succeq O \tag{8}$$

for any vertex of $\Theta^{[j]}$. Hence, $(x, W^{[j]})$ satisfies (8) for any point in $\Theta^{[j]}$. Premultiplication of $M(\theta)^{\mathrm{T}}$ and postmultiplication of $M(\theta)$ to this inequality give $F(x,\theta) \succeq O$, $\forall\theta \in \Theta^{[j]}$.

Define $S_1(\Theta^{[j]})$ as the set of all $(x, W^{[j]}, \{V_i^{[j]}\}_{i=1}^{p})$ such that (5)–(7) hold. We now obtain the following new approximate problem:

$$\left.\begin{array}{lll} P_1(\Delta): & \text{minimize} & c^{\mathrm{T}}x \\ & \text{subject to} & (x, W^{[j]}, \{V_i^{[j]}\}_{i=1}^{p}) \in S_1(\Theta^{[j]}) \\ & & \forall j = 1, 2, \ldots, J. \end{array}\right\}$$

By construction, the feasible region of $P_1(\Delta)$ is included in that of (2), which implies $\inf P_1(\Delta) \ge v_{\mathrm{opt}}$.

The result on the approximation error of the problem $P_1(\Delta)$, i.e., $|\inf P_1(\Delta) - v_{\rm opt}|$ is given in the following proposition, which is a reduced-size counterpart of Theorem 7 in [12].

Proposition 2: Under Assumption 1, there exist constants C_1 and C such that, if a given division Δ satisfies $\overline{\rm rad}\ \Delta \le C_1$, then

$$|\inf P_1(\Delta) - v_{\rm opt}| \le C\overline{\rm rad}\ \Delta.$$

Proof: The result of Theorem 11 in [13] is needed for the proof of this proposition. An upper bound on $|\inf P_1(\Delta) - v_{\rm opt}|$ can be derived from that in [13] using a similar technique to the proof of Theorem 7 in [12]. This technique is based on magnification of each subregion $\Theta^{[j]}$ by some factor dependent of the size of $G(x)$, together with a connection between the constraints (5)–(7) and the constraint (8) with respect to the magnified subregion. In this case, we choose the magnification factor of $\pi\sqrt{|V|m}/2$, and obtain the result along the same line as in Theorem 7 in [12]. ■

We next prepare the following two lemmas to explain the relationship between the problem $P_1(\Delta)$ and the problem $P(\Delta)$. The subscript $[j]$ is omitted in the following lemmas for convenience.

Lemma 1: [7] If there exist x, W and $\{V_i\}_{i=1}^p$ such that

$$\begin{aligned} &G(x) + H(\theta^{\rm c})W^{\rm T} + WH(\theta^{\rm c})^{\rm T} - \sum_{i=1}^p V_i \succeq O, \\ &V_i + (\overline{\theta}_i - \theta_i^{\rm c})(H_iW^{\rm T} + WH_i^{\rm T}) \succeq O, \quad i = 1, \dots, p, \\ &V_i - (\overline{\theta}_i - \theta_i^{\rm c})(H_iW^{\rm T} + WH_i^{\rm T}) \succeq O, \quad i = 1, \dots, p, \end{aligned} \tag{9}$$

then there exist an SOS matrix $\hat{S}_0(\theta)$ of degree two, and constant matrices $\hat{S}_i \succeq O,\ i = 1, 2, \dots, p$ with

$$G(x){+}H(\theta)W^{\rm T}{+}WH(\theta)^{\rm T} = \hat{S}_0(\theta){+}\sum_{i=1}^p(\theta_i{-}\underline{\theta}_i)(\overline{\theta}_i{-}\theta_i)\hat{S}_i. \tag{10}$$

■

The SOS representation of $G(x) + H(\theta)W^{\rm T} + WH(\theta)^{\rm T}$ leads to that of $F(x, \theta)$ as shown in the next lemma.

Lemma 2: If there exist an SOS matrix $\hat{S}_0(\theta)$, and constant matrices $\hat{S}_i$'s such that the condition (10) holds, then there exist SOS matrices $S_0(\theta), S_1(\theta), \dots, S_p(\theta)$ satisfying

$$F(x, \theta) = S_0(\theta) + \sum_{i=1}^p(\theta_i - \underline{\theta}_i)(\overline{\theta}_i - \theta_i)S_i(\theta). \tag{11}$$

Moreover, the monomial bases $u_0(\theta), u_1(\theta), \dots, u_p(\theta)$ for the SOS matrices are expressed as in (3).

Proof: The proof is rather straightforward and proceeded in the same line as in Lemma 2 in [7]. Details are omitted due to space limitation. ■

The proof of our theorem is now given here.

Proof of Theorem 1. Suppose in the problem $P(\Delta)$ that the monomial bases $u_0(\theta), u_1(\theta), \dots, u_p(\theta)$ of the SOS matrices $S_0(\theta), S_1(\theta), \dots, S_p(\theta)$ are chosen as (3). Lemmas 1 and 2 imply that, for each subregion $\Theta^{[j]}$, the constraint of the problem $P_1(\Delta)$ is just a sufficient condition of the constraint of $P(\Delta)$. Therefore, the feasible region of $P_1(\Delta)$ is included in that of $P(\Delta)$, and thus $v_{\rm opt} \le \inf P(\Delta) \le \inf P_1(\Delta)$. If a given division Δ satisfies $\overline{\rm rad}\ \Delta \le C_1$, then we obtain from Proposition 2 that

$$|\inf P(\Delta) - v_{\rm opt}| \le |\inf P_1(\Delta) - v_{\rm opt}| \le C\overline{\rm rad}\ \Delta,$$

and the proof is complete. ■

IV. Numerical Example

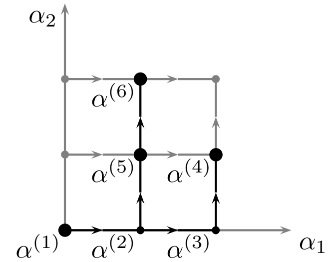

Fig. 1. The directed graph (V_0, A_0) (gray) in the case of $p = 2$, $d_1 = d_2 = 2$. The set S is shown by the large circles. A rectilinear Steiner arborescence (V, A) for S is shown in black.

An example on polynomial optimization is exhibited in this section. We apply the proposed approach to solve the problem and compare the results with those from the conventional approach. The software SOSTOOLS [16] with SeDuMi [23] as an SDP solver is used for the computation.

Example: We maximize

$$f(\theta) = -5\theta_1^2\theta_2 - 5\theta_1\theta_2^2 + 9\theta_1\theta_2$$

over $\Theta = [0, \gamma]^2$, with $\gamma = 1, 2, 3$. The global maximum 1.08 is attained at $\theta = [0.6\ \ 0.6]^{\rm T}$ for all values of γ. This problem can be formulated into the following robust SDP:

$$\left.\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & x - f(\theta) \ge 0, \quad \forall \theta \in \Theta. \end{array}\right\} \tag{12}$$

In this case, $F(x, \theta) = x - f(\theta) = x + 5\theta_1^2\theta_2 + 5\theta_1\theta_2^2 - 9\theta_1\theta_2$ with $n = 1$, $m = 1$, $p = 2$, $d_1 = d_2 = 2$.

The SOS-approximate problem for (12) is constructed as follows:

$$\left.\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & x - f(\theta) = s_0(\theta) + \displaystyle\sum_{i=1}^2 \theta_i(\gamma - \theta_i)s_i(\theta), \end{array}\right\} \tag{13}$$

where $s_0(\theta), s_1(\theta)$ and $s_2(\theta)$ are SOS polynomials with monomial bases $u_0(\theta), u_1(\theta)$ and $u_2(\theta)$ respectively.

In order to choose the monomial bases of small sizes, we consider the support S of $F(x, \theta)$, as well as a rectilinear Steiner arborescence (V, A) for S as presented in Fig. 1. Using the rectilinear Steiner arborescence, we obtain the following monomial bases:

$$\begin{aligned} u_1(\theta) = u_2(\theta) &= \begin{bmatrix} 1 & \theta_1 & \theta_1^2 & \theta_1^2\theta_2 & \theta_1\theta_2 & \theta_1\theta_2^2 \end{bmatrix}, \\ u_0(\theta) &= \begin{bmatrix} u_1(\theta) & \theta_1^3 & \theta_1^3\theta_2 & \theta_1^2\theta_2^2 & \theta_2 & \theta_1\theta_2^3 \end{bmatrix}. \end{aligned}$$

We apply the above monomial bases to the approximate problem (13). For comparison, an approximate problem with full-size monomial bases is also constructed. For each approximate problem, the exact optimal value is attained with the coarsest division $\Delta = \{\Theta\}$, for all values of γ. The results are summarized in Table I.

We can see from Table I that the proposed approach with the reduced-size monomial bases achieves the same quality of approximation as the conventional approach with the full-sized monomial bases. However, the computational cost in the proposed approach is less than that in the conventional approach as expected. This confirms the effectiveness of the proposed approach.

TABLE I

COMPARISON BETWEEN THE TWO APPROACHES

Monomial bases	Upper bound	SDP size	
Proposed	1.0800	194	33
[7]	1.0800	419	49

V. CONCLUSION

Reduction of the size of the approximate problem is considered in the SOS approach to robust SDPs. The reduce-sized monomial bases is constructed by exploitation of the sparse structure of a given robust SDP. The asymptotic exactness of the region-dividing scheme, as well as the existence of an error bound are proved for the reduced-size approximate problem. Application to control problems will be a possible research direction in the near future.

ACKNOWLEDGEMENT

The author is very obliged to Yasuaki Oishi for helpful suggestions and valuable comments on this paper.

REFERENCES

[1] A. Ben-Tal and A. Nemirovski, Robust Convex Optimization, *Mathematics of Operation Research*, vol. 23, no. 4, pp. 769–805, 1998.

[2] A. Ben-Tal and A. Nemirovski, On Tractable Approximations of Uncertain Linear Matrix Inequality Affected by Interval Uncertainty, *SIAM Journal on Optimization*, vol. 12, no. 3, pp. 811–833, 2002.

[3] P.-A. Bliman, On Robust Semidefinite Programming, in *Proceeding of the 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2004)*, Leuven, Belgium, July 2004.

[4] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, Polynomially Parameter-Dependent Lyapunov Functions for Robust Stability of Polytopic Systems: An LMI Approach, *IEEE transactions on Automatic Control*, vol. 50, no. 3, pp. 365–370, 2005.

[5] L. El Ghaoui, F. Oustry, and H. Lebret, Robust Solutions to Uncertain Semidefinite Programs, *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 33–52, 1998.

[6] D. Henrion and J. B. Lasserre, Gloptipoly: Global Optimization over Polynomials with Matlab and SeDuMi, *ACM Transactions on Mathematical Software*, vol. 29, no. 2, pp. 165–194, 2003.

[7] T. Jennawasin and Y. Oishi, A Region-Dividing Technique for Constructing the Sum-of-Squares Approximations to Robust Semidefinite Programs, to appear in *IEEE transactions on Automatic Control*, Special Issue on "Positive Polynomials in Control", 2008.

[8] M. Kojima, Sum of Squares Relaxations in Polynomial Semidefinite Programs, *Research Reports on Mathematical and Computing Sciences*, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2003.

[9] J. B. Lasserre, Global Optimization with Polynomials and the Problems of Moments, *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2001.

[10] J. B. Lasserre, Convergent SDP-Relaxations in Polynomial Optimization with Sparsity, *SIAM Journal on Optimization*, vol. 17(3), pp. 822–843, 2006.

[11] Y. Oishi, A Region-Dividing Approach to Robust Semidefinite Programming and Its Error Bound, in *Proceedings of the 2006 American Control Conference*, Minneapolis, USA, June 2006, pp. 123–129.

[12] Y. Oishi, Reduction of the Number of Constraints in the Matrix-Dilation Approach to Robust Semidefinite Programming, in *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, USA, December 2006, pp. 5790–5795.

[13] Y. Oishi and Y. Isaka, Exploiting Sparsity in the Matrix-Dilation Approach to Robust Semidefinite Programming, in *Proceeding of the 2007 American Control Conference*, New York, USA, July 2007, pp. 6169–6176.

[14] A. Ohara and Y. Sasaki, On Solvability and Numerical Solutions of Parameter-Dependent Differential Matrix Inequality, in *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, USA, December 2001, pp. 3593–3594.

[15] P. A. Parrilo, Semidefinite Programming Relaxations for Semialgebraic Problems, *Mathematical Programming Series B*, vol. 96, no. 2, pp. 293–320, 2003.

[16] S. Prajna, A. Papachristodoulou and P. A. Parrilo, Introducing SOSTOOLS: A General Purpose Sum of Squares Programming Solver, in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, USA, December 2002, pp. 741–746.

[17] S. K. Rao, P. Sadayappan, F. K. Hwang, and P. W. Shor, The Rectilinear Steiner Arborescence Problem, *Algorithmica*, vol. 7, nos. 2–3, pp. 277–288, 1992.

[18] C. W. Scherer, Robust Performance Analysis for Parameter Dependent Systems Using Tensor Product Splines, in *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, USA, December 1998, pp. 2216–2221.

[19] C. W. Scherer, Relaxations for Robust Linear Matrix Inequality Problems with Verifications for Exactness, *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 2, pp. 365–395, 2005.

[20] C. W. Scherer, LMI Relaxations in Robust Control, *European Journal of Control*, vol. 12, no. 1, pp. 3–29, 2006.

[21] C. W. Scherer and C. W. J. Hol, Matrix Sum-of-Squares Relaxations for Robust Semi-Definite Programs, *Mathematical Programming, Series B*, vol. 107, nos. 1–2, pp. 189–211, 2006.

[22] W. Shi and C. Su, The Rectilinear Steiner Arborescence Problem is NP-Complete, *SIAM Journal on Computing*, vol. 35, no. 3, pp. 729–740, 2006.

[23] J. F. Sturm, Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones, *Optimization Methods and Software*, vols. 11–12, pp. 625–653, 1999.

[24] H. Waki, S. Kim, M. Kojima, and M. Muramatsu, Sums of Squares and Semidefinite Program Relaxations for Polynomial Optimization Problems with Structured Sparsity, *SIAM Journal on Optimization*, vol. 17, no. 1, pp. 218–242, 2006.