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A parameter-dependent Lyapunov approach for the control of nonstationary and hybrid LPV systems

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Abstract— **The paper presents a parameter-dependent Lyapunov approach for the control of nonstationary and hybrid linear-parameter varying (LPV) systems. The work is motivated by the challenges encountered in controlling nonlinear systems about aggressive trajectories, specifically pre-specified eventually periodic ones.**

I. INTRODUCTION

The paper deals with the control of nonlinear systems along trajectories, particularly pre-specified eventually periodic ones. Such trajectories can be arbitrary for a finite amount of time before setting into periodic orbits. Linear parameter-varying (LPV) models will be used to capture the nonlinear dynamics of the system. Specifically, the types of plant models we consider are of the form

$$\begin{aligned} x(k+1) &= A(\delta(k),k)x(k) + B(\delta(k),k)w(k) \\ z(k) &= C(\delta(k),k)x(k) + D(\delta(k),k)w(k), \end{aligned} \quad (1)$$

where $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $C(\cdot,\cdot)$, and $D(\cdot,\cdot)$ are matrix-valued functions that are known *a priori*. The variable k is time, and $\delta(k) := (\delta_1(k), \ldots, \delta_r(k)\,)$ is a vector of real scalar parameters. Such models differ from the standard LPV systems found in the literature in that the state-space matrices have *explicit dependence on time* in addition to the parameters; henceforth, standard LPV models having explicit dependence on the parameters only will be referred to as stationary LPV, or SLPV, systems. Models of the form in (1) are called nonstationary LPV (NSLPV) models, and some work on these systems formulated in an LFT framework can be found in [1], [2]. Clearly, in the context of control of nonlinear systems along *pre-specified* trajectories, NSLPV models arise naturally as a means to capture the nonlinear dynamics while maintaining a model that is amenable to control synthesis. Furthermore, in such a context, an NSLPV model is potentially far less conservative than a corresponding stationary one since, with an NSLPV model, we do not have to parameterize time-varying terms in the system equations, which are associated with the pre-specified trajectory and hence known a priori. Note that hybrid LPV systems are directly linked to nonstationary LPV ones, as will be evident in Section 4. In the case of hybrid systems, reference trajectories do not have to be pre-specified as long as the reference states and controls are within the covered state-space region.

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Unlike the work in [2], we will employ a parameter-dependent Lyapunov approach for the control problem in question, assuming of course that the rates of variation of the parameters are bounded. Clearly, a parameter-dependent Lyapunov approach is potentially less conservative than a corresponding parameter-independent one, but the trade-off is additional computational complexity. While in some cases, the attained improvement in performance may not justify the added computational complexity, there are certain control problems where the use of parameter-dependent Lyapunov functions is necessary irrespective of the performance betterment. One such problem is trajectory regulation in the presence of obstacles [3]. Here, the penalty weights on the tracking errors can be viewed as scheduling parameters. Then, the position of the vehicle in the obstacle environment will dictate the penalty weights on the control errors, and accordingly the control strategy will prominently change in order to prioritize the regulation of certain outputs over others. Hence, the key issue in this case is to make sure that the difference in control strategy is prominent when the vehicle is in proximity to obstacles. As a result, the use of a parameter-independent Lyapunov function is unfavorable because then all scheduled controllers will be inclined for worst-case-scenario behavior. Instead, a parameter-dependent Lyapunov function should be used, and furthermore, the rates of variation of the parameters should be of relatively small magnitude by design.

The outline of the paper is as follows: in Section 2, we formulate the control problem; in Section 3, we give analysis and synthesis conditions in terms of parameterized linear matrix inequalities (PLMIs) for eventually periodic NSLPV systems; and in Section 4, we give a result on hybrid LPV control. As for the notation, it is quite standard. We denote the set of non-negative integers by $\mathbb{N}_0$ and that of real $n \times m$ matrices by $\mathbb{R}^{n\times m}$. Also, we denote the space of continuous functions by $\mathcal{C}^0$. The adjoint of an operator X is written X^*, and we use $X \prec 0$ to mean it is negative definite. The normed space of square summable vector-valued sequences is denoted by ℓ_2. It consists of elements $x = (x_0, x_1, x_2, \ldots)$, with each $x_k \in \mathbb{R}^{n_k}$ for some n_k, having a finite 2-norm $\|x\|$ defined by $\|x\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2 < \infty$, where $\|x_k\|^2 = x_k^* x_k$.

II. NSLPV PLANT AND CONTROLLER

Let G be an NSLPV model defined by the following state-space equation: $\begin{bmatrix} x(k+1)^T & z(k)^T & y(k)^T \end{bmatrix}^T =$

$$\begin{bmatrix} A(\delta(k),k) & B_1(\delta(k),k) & B_2(\delta(k),k) \\ C_1(\delta(k),k) & D_{11}(\delta(k),k) & D_{12}(\delta(k),k) \\ C_2(\delta(k),k) & D_{21}(\delta(k),k) & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \quad (2)$$

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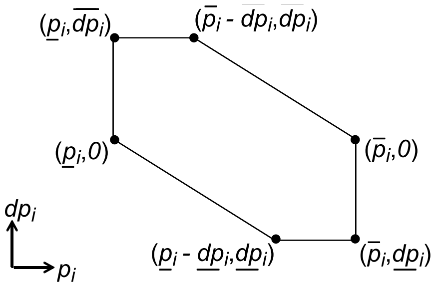

Fig. 1. Parameter space in (p_i, dp_i)-plane

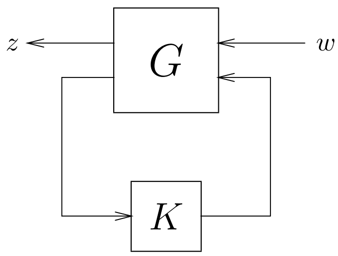

Fig. 2. Closed-loop system

$x(0) = 0$, for $w \in \ell_2$. The signals $w(k)$ and $z(k)$ denote the exogenous disturbances and errors, respectively, whereas $u(k)$ denotes the applied control and $y(k)$ the measurements. The vectors $x(k)$, $z(k)$, $w(k)$, $y(k)$, and $u(k)$ are real and have time-varying dimensions denoted by $n(k)$, $n_z(k)$, $n_w(k)$, $n_y(k)$, and $n_u(k)$ respectively. Like in [4], we assume the parameters $\delta(k) = (\delta_1(k), \ldots, \delta_r(k))$ and parameter increments $d\delta(k) = \delta(k+1) - \delta(k)$ such that $(\delta(k), d\delta(k)) \in \Gamma$ for all $k \in \mathbb{N}_0$, where Γ is a polytope defined as

$$\Gamma := \{(p, dp) \in \mathbb{R}^r \times \mathbb{R}^r \mid f_{i,j}(p_i, dp_i) \geq 0 \quad \text{for all } i = 1, \ldots, r \text{ and } j = 1, 2, 3\}, \quad (3)$$

$$\text{with} \quad \begin{aligned} f_{i,1} &= (p_i - \underline{p}_i)(\overline{p}_i - p_i), \\ f_{i,2} &= (dp_i - \underline{dp}_i)(\overline{dp}_i - dp_i), \\ f_{i,3} &= (p_i + dp_i - \underline{p}_i)(\overline{p}_i - p_i - dp_i), \\ &\underline{p}_i, \overline{p}_i, \underline{dp}_i, \overline{dp}_i \in \mathbb{R},\ \underline{dp}_i \leq 0,\ \overline{dp}_i \geq 0. \end{aligned}$$

Notice that, for each $i = 1, \ldots, r$, the set of points satisfying $f_{i,j}(p_i, dp_i) \geq 0$ for $j = 1, 2, 3$ defines a polygon which constitutes the projection of polytope Γ on the (p_i, dp_i)-plane, as shown in Figure 1. Thus the allowable parameter-trajectories δ reside in the set

$$\mathbf{\Delta}_\Gamma := \{\delta : \mathbb{N}_0 \to \mathbb{R}^r \mid (\delta(k), d\delta(k)) \in \Gamma \ \forall\ k \in \mathbb{N}_0\}. \quad (4)$$

It is important at this point to properly characterize the state-space matrix-valued functions. As the following applies to each of the state-space operators, for simplicity we will focus the discussion on the A-matrix only. First, we assume that the state-space matrices have continuous dependence on the parameters and are uniformly bounded for all admissible values of time and parameters. Then, setting $\mathcal{A}_k(p) = A(p, k)$, the matrix-valued function $A(p, k)$ can be viewed as a family of continuous functions of the parameter vector p, denoted $\mathcal{A}$, namely

$$\mathcal{A} = \{\mathcal{A}_k \in \mathcal{C}^0(\mathbb{R}^r, \mathbb{R}^{n(k+1)\times n(k)}) : k \in \mathbb{N}_0\}.$$

Furthermore, $\mathcal{A}$ is uniformly bounded, meaning that there exists a positive scalar λ such that $\|\mathcal{A}_k(p)\| \leq \lambda$ for all $p_i \in [\underline{p}_i, \overline{p}_i]$ and $k \in \mathbb{N}_0$. Alternatively, setting $A_\delta(k) = A(\delta(k), k)$, the function $A(\delta(k), k)$ can be regarded as a set of sequences, where, for each $\delta \in \mathbf{\Delta}_\Gamma$, the corresponding bounded sequence $A_\delta(k)$ is reminiscent of the A-sequence of a standard LTV system. Hence, for each $\delta \in \mathbf{\Delta}_\Gamma$, the NSLPV model G reduces to a standard LTV system, say, G_δ; in other words, $G = \{G_\delta : \delta \in \mathbf{\Delta}_\Gamma\}$. The uniform boundedness here ensures that there exists a positive scalar λ such that $\|A_\delta(k)\| \leq \lambda$ for all $\delta \in \mathbf{\Delta}_\Gamma$ and $k \in \mathbb{N}_0$.

We say an NSLPV model G, as defined in the preceding, is ℓ_2-stable if, for each $\delta \in \mathbf{\Delta}_\Gamma$, the resulting LTV system is exponentially stable. In the sequel, we will assume that both $\delta(k)$ and $d\delta(k)$ are measurable at each time instant k. As for computing the parameter increment $d\delta(k)$ online, one practical approach is to design a continuously differentiable parameter function, and then, assuming a measurable derivative $\dot{\delta}(k)$ at each k and a sufficiently small sampling time T, the value of $d\delta(k)$ can be obtained from the Euler approximation $d\delta(k) \approx T\dot{\delta}(k)$.

Usually, the parameters are used to replace time-varying and nonlinear terms in the system equations so that the resultant model would capture the nonlinear dynamics while still amenable to control synthesis. But these parameters can also be used for different purposes, for instance, the parameters can serve as penalty weights on the control errors, which are scheduled online appropriately. One specific application is in the control of vehicles about trajectories in the presence of obstacles, where the significance of the different control errors varies depending on the position of the vehicle in the obstacle environment. We refer to this as output scheduling.

Suppose that plant G is controlled by a controller K whose state-space equation is $\begin{bmatrix} x_K(k+1)^T & u(k)^T \end{bmatrix}^T =$

$$\begin{bmatrix} A_K(\delta(k), d\delta(k), k) & B_K(\delta(k), d\delta(k), k) \\ C_K(\delta(k), d\delta(k), k) & D_K(\delta(k), d\delta(k), k) \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \end{bmatrix}, \quad (5)$$

$x_K(0) = 0$, where $x_K(k) \in \mathbb{R}^{m(k)}$. The parameters $\delta_i(k)$ here are the same as those in the plant equations. It goes without saying that, when constructing the controller from the synthesis solutions, we will make sure that its system matrices are uniformly bounded functions, with continuous dependence on the parameters and their increments. The feedback interconnection of G and K is shown in Figure 2. We denote this closed-loop system by L and write its realization as $\begin{bmatrix} x_L(k+1)^T & z(k)^T \end{bmatrix}^T =$

$$\begin{bmatrix} A_L(\delta(k), d\delta(k), k) & B_L(\delta(k), d\delta(k), k) \\ C_L(\delta(k), d\delta(k), k) & D_L(\delta(k), d\delta(k), k) \end{bmatrix} \begin{bmatrix} x_L(k) \\ w(k) \end{bmatrix}, \quad (6)$$

where column vector $x_L(k) = (x(k), x_K(k)) \in \mathbb{R}^{n(k)+m(k)}$, and the closed-loop state-space matrices are given by

$$A_L = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix},\ B_L = \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix},$$

$$C_L = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix},\ D_L = D_{11} + D_{12} D_K D_{21}.$$

We now state the synthesis objective.

Definition 1: A controller K is a *γ-admissible synthesis* for NSLPV plant G if the closed-loop system in Figure 2 is

ℓ_2-stable and the performance inequality $\|w \mapsto z\|_{\ell_2 \to \ell_2} < \gamma$ is achieved for all $\delta \in \mathbf{\Delta}_\Gamma$.

Before concluding this section, we need to introduce the special class of eventually periodic NSLPV systems. In this case, the explicit time variation in the system realization is eventually periodic. Eventually periodic systems arise in two basic scenarios. The first is when parameterizing the nonlinear system equations about an eventually periodic trajectory, and the second is when the plant has an uncertain initial condition. An eventually periodic trajectory can be arbitrary for an initial amount of time, but then settles into a periodic orbit; a special case of this is when a system transitions between two operating points. Finite horizon and periodic systems are subclasses of eventually periodic systems. We now define an eventually periodic NSLPV system.

Definition 2: An NSLPV system G is (h, q)-eventually periodic for some integers $h \geq 0$, $q \geq 1$ if each of its state-space matrix-valued functions is (h, q)-eventually periodic with respect to the explicit time dependence; for instance, $A(\delta, k)$ would be of the form

$$\underbrace{A(\delta,0), A(\delta,1), \ldots, A(\delta,h-1)}_{h \text{ terms}}, \underbrace{A(\delta,h), \ldots, A(\delta,h+q-1)}_{q \text{ terms}}, \underbrace{A(\delta,h), \ldots, A(\delta,h+q-1)}_{q \text{ terms}}, \ldots$$

III. Analysis and Synthesis Results

Solving the control problem via a parameter-independent Lyapunov function is clearly conservative; whereas, if we allow the Lyapunov function to be parameter-dependent, this conservatism is likely to diminish. Also, the type of parameter dependence, be it linear, polynomial or rational, can be a factor. Of course, the more complicated the Lyapunov function is allowed to be, the more intensive the computational problem becomes. So, there is a trade-off in general between conservatism and computational complexity. In some cases, especially when the parameters are used to replace system dynamics only, the parameter-independent approach may yield satisfactory results; see for instance the hovercraft example given in [2]. However, this is not the case when the focus is output scheduling because then, in addition to ensuring stability and adequate performance, the controller must have the capacity to change strategy prominently so as to favor the regulation of certain outputs over others when necessary. This controller feature is a requisite as the critical outputs to track will generally outnumber the control inputs, especially in rotorcraft applications of interest.

We now state the following analysis and synthesis results.

Theorem 3: Closed-loop system L, defined in (6), is ℓ_2-stable and $\|w \mapsto z\|_{\ell_2 \to \ell_2} < \gamma$ for all $\delta \in \mathbf{\Delta}_\Gamma$, as defined in (4), if there exists a uniformly bounded matrix-valued function $X(p, k) \succ 0$, continuous in p, such that

$$\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}^* \begin{bmatrix} X(p+dp, k+1) & 0 \\ 0 & \frac{1}{\gamma^2} I \end{bmatrix} \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} - \begin{bmatrix} X(p,k) & 0 \\ 0 & I \end{bmatrix} \prec -\beta I, \quad (7)$$

for all $(p, dp) \in \Gamma$ as defined in (3), $k \in \mathbb{N}_0$, and some positive scalar β, where the dependence of the closed-loop system matrices on (p, dp, k) is suppressed for simplicity. A special case of this analysis result for stationary LPV systems is given in [4].

Proof: Consider any $\delta \in \mathbf{\Delta}_\Gamma$. Then, given the time-varying parameter-trajectory δ, NSLPV system L reduces to a standard discrete-time LTV system, say, L_δ. Suppose that inequality (7) holds for all $(p, dp) \in \Gamma$ and $k \in \mathbb{N}_0$. Then, given $\delta \in \mathbf{\Delta}_\Gamma$, the following inequality is valid:

$$F_L^*(\delta(k), d\delta(k), k) \begin{bmatrix} X(\delta(k) + d\delta(k), k+1) & 0 \\ 0 & \frac{1}{\gamma^2} I \end{bmatrix} \times F_L(\delta(k), d\delta(k), k) - \begin{bmatrix} X(\delta(k), k) & 0 \\ 0 & I \end{bmatrix} \prec -\beta I,$$

with $F_L = \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}$, for all $k \in \mathbb{N}_0$ and some positive scalar β; this immediately follows from the definition of $\mathbf{\Delta}_\Gamma$, which ensures that $(\delta(k), d\delta(k)) \in \Gamma$ for all $k \in \mathbb{N}_0$. Then, as $\delta(k+1) = \delta(k) + d\delta(k)$, we obtain that the sequence $X(\delta(k), k) \succ 0$, bounded above and below, satisfies the Kalman-Yakubovich-Popov (KYP) Lemma condition, with ℓ_2-gain performance level γ, for the LTV system L_δ, which, by [5, Corollary 12], implies that L_δ is stable and $\|L_\delta\|_{\ell_2 \to \ell_2} < \gamma$. ■

Theorem 4: Given NSLPV plant G defined in (2) with $\delta \in \mathbf{\Delta}_\Gamma$, suppose that

(A1) the matrices $\begin{bmatrix} B_2^*(\delta(k), k) & D_{12}^*(\delta(k), k) \end{bmatrix}$ and $\begin{bmatrix} C_2(\delta(k), k) & D_{21}(\delta(k), k) \end{bmatrix}$ have full-row rank uniformly for all $k \in \mathbb{N}_0$ and $\delta \in \mathbf{\Delta}_\Gamma$.

Then there exists a γ-admissible NSLPV synthesis K to G à la Definition 1 for some scalar γ if there exist uniformly bounded matrix-valued functions $R(p, k) \succ 0$, $S(p, k) \succ 0$, continuous in p, and some positive scalar σ such that

$$\begin{bmatrix} ARA^* - R^+ & ARC_1^* & B_1 \\ C_1RA^* & -\gamma I + C_1RC_1^* & D_{11} \\ B_1^* & D_{11}^* & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^* \prec -\beta I$$

$$\begin{bmatrix} A^*S^+A - S & A^*S^+B_1 & C_1^* \\ B_1^*S^+A & -\gamma I + B_1^*S^+B_1 & D_{11}^* \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} C_2^* \\ D_{21}^* \\ 0 \end{bmatrix} \begin{bmatrix} C_2^* \\ D_{21}^* \\ 0 \end{bmatrix}^* \prec -\beta I \quad (8)$$

$$\begin{bmatrix} R(p,k) & I \\ I & S(p,k) \end{bmatrix} \succeq 0$$

for all $(p, dp) \in \Gamma$, $k \in \mathbb{N}_0$, and some positive scalar β, where the dependence of R, S, and the state-space matrices on (p, k) is suppressed for simplicity, and

$$R^+ = R(p + dp, k+1) \qquad S^+ = S(p + dp, k+1).$$

Proof: As in the proof of Theorem 3, for each trajectory $\delta \in \mathbf{\Delta}_\Gamma$, NSLPV system G reduces to a standard discrete-time LTV system. Then, invoking [5, Theorem 19] along with applications of Finsler's lemma and a similar argument to that in the proof of Theorem 5.2 (iii) in [8] complete the proof. ∎

It will be convenient to write the synthesis conditions in (8) as

$$\begin{aligned} \mathcal{F}_1\left(R(p,k), R(p+dp,k+1), \sigma, p, k\right) &\prec -\beta I, \\ \mathcal{F}_2\left(S(p,k), S(p+dp,k+1), \sigma, p, k\right) &\prec -\beta I, \\ \mathcal{F}_3\left(R(p,k), S(p,k)\right) &\succeq 0, \end{aligned} \tag{9}$$

respectively, where $\mathcal{F}_i$ are defined in the obvious way.

We assume henceforth that the state-space matrices have *polynomial* dependence on the parameters. Moreover, we will only seek solutions with polynomial parameter dependence for the synthesis inequalities. Specifically, we define the family of functions $\mathcal{X}$ to consist of all the uniformly bounded matrix-valued functions $X(p,k) \succ 0$, for all $p_i \in [\underline{p}_i, \overline{p}_i]$ and $k \in \mathbb{N}_0$, with polynomial dependence on the parameters, namely, with $v = (v_1, v_2, \ldots, v_r)$ and $J_\tau := \{(v_1, v_2, \ldots, v_r) \mid v_i \in \mathbb{N}_0 \text{ and } \sum_{i=1}^{r} v_i \leq \tau\}$, we have

$$X(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \ldots p_r^{v_r} X_v(k),$$

for some $\tau \in \mathbb{N}_0$, where, for each $v \in J_\tau$, the sequence $X_v(k)$ is bounded above and below. Next in this section, we will be focussing on eventually periodic plants which are presented in Definition 2. Thus, it will be convenient to specify (N,q)-eventually periodic matrix-valued functions in $\mathcal{X}$. Namely, given $X \in \mathcal{X}$ as aforementioned, we say X is (N,q)-eventually periodic if, for each $v \in J_\tau$, the sequence $X_v(k)$ is (N,q)-eventually periodic, i.e. $X_v(k)$ is of the form

$$\underbrace{X_v(0), X_v(1), \ldots, X_v(N{-}1)}_{N \text{ terms}}, \underbrace{X_v(N), \ldots, X_v(N{+}q{-}1)}_{q \text{ terms}}, \underbrace{X_v(N), \ldots, X_v(N{+}q{-}1)}_{q \text{ terms}}, \ldots$$

The synthesis conditions in (8) are convex but infinite dimensional both in time and parameters. However, if the NSLPV plant G is (h,q)-eventually periodic as in Definition 2, then the infinite dimensionality with respect to the explicit time dependence can be avoided as shown in the next results.

Proposition 5: Given NSLPV plant G defined in (2) with $\delta \in \mathbf{\Delta}_\Gamma$, suppose that G is q-periodic (i.e. $(0,q)$-eventually periodic), with assumptions **(A1)** and

(A2) the state-space matrices of G have *polynomial* dependence on the parameters $\delta \in \mathbf{\Delta}_\Gamma$.

Then there exist solutions in $\mathcal{X}$ to synthesis conditions (8) if and only if there exist q-periodic solutions in $\mathcal{X}$.

The following proof is inspired by that of a similar result for standard periodic systems in [5]. Also, a similar averaging technique is used in [9] in the context of time-varying control analysis.

Proof: The proof of the "if" direction is immediate.

As for the "only if" direction, suppose there exist solutions in $\mathcal{X}$ to the synthesis conditions in (8). Focussing on the first of these conditions, it is not difficult to show that this condition is equivalent to the existence of $R(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \ldots p_r^{v_r} R_v(k)$ in $\mathcal{X}$ for some $\tau \in \mathbb{N}_0$ such that

$$\begin{aligned} &M(p,k) \begin{bmatrix} R(p,k) & \\ & I \end{bmatrix} M^*(p,k) \\ &\quad - \begin{bmatrix} R(p+dp,k+1) & \\ & \gamma^2 I \end{bmatrix} - \sigma H(p,k) \prec -\beta I, \end{aligned} \tag{10}$$

$$\text{with } M(p,k) = \begin{bmatrix} A(p,k) & B_1(p,k) \\ C_1(p,k) & D_{11}(p,k) \end{bmatrix}$$

$$\text{and } H(p,k) = \begin{bmatrix} B_2(p,k) \\ D_{12}(p,k) \end{bmatrix} \begin{bmatrix} B_2(p,k) \\ D_{12}(p,k) \end{bmatrix}^*,$$

for all $(p,dp) \in \Gamma$, $k \in \mathbb{N}_0$, and some positive scalar β. As $M(p,k)$ and $H(p,k)$ are q-periodic, we have $M(p,k{+}iq) = M(p,k)$ and $H(p,k+iq) = H(p,k)$ for all $i \in \mathbb{N}_0$.

In the following, we set k to be some *fixed* integer in $\mathbb{N}_0$. Then, due to the linearity of (10), the following inequality holds:

$$\begin{aligned} &M(p,k) \begin{bmatrix} Y_\lambda(p,k) & \\ & I \end{bmatrix} M^*(p,k) \\ &\quad - \begin{bmatrix} Y_\lambda(p+dp,k+1) & \\ & \gamma^2 I \end{bmatrix} - \sigma H(p,k) \prec -\beta I, \end{aligned} \tag{11}$$

for all $(p,dp) \in \Gamma$, where $Y_\lambda(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \ldots p_r^{v_r} Y_{v,\lambda}(k)$ and

$$Y_{v,\lambda}(k) = \frac{1}{\lambda} \sum_{i=0}^{\lambda-1} R_v(k+iq), \qquad \text{for } \lambda \geq 1.$$

With k fixed and for each $v \in J_\tau$, since the sequence $R_v(k+iq)$ for all $i \geq 0$ is bounded, then so is the sequence $Y_{v,\lambda}(k)$. Then, like in the proof of [5, Theorem 20], there exists a subsequence $Y_{v,\lambda_t}(k)$ which converges to some matrix $Y_v(k)$ in the weak operator topology; see [5] for a definition of such a convergence and [10] for more details on this property. Clearly, as $R(p,k) \succeq \alpha I$ for all $p_i \in [\underline{p}_i, \overline{p}_i]$ and some positive scalar α, then so is $Y(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \ldots p_r^{v_r} Y_v(k)$. Also, by the properties of weak convergence, $Y(p,k)$ solves the non-strict version of inequality (11) and, since

$$Y_{v,\lambda}(k{+}q) - Y_{v,\lambda}(k) = \frac{1}{\lambda}\left(R_v(k+\lambda q) - R_v(k)\right) \xrightarrow{\lambda \to \infty} 0,$$

the equality $Y_v(k+q) = Y_v(k)$ follows.

A similar argument can be used to show that the second synthesis condition admits a q-periodic solution as well. Last, given the way these q-periodic solutions are constructed, it is not difficult to see that they also satisfy the coupling condition in (8). ∎

Proposition 6: Suppose NSLPV plant G is (h,q)-eventually periodic, along with assumptions (A1–A2). Then there exist solutions in $\mathcal{X}$ to synthesis conditions (8) if and only if there exist (N,q)-eventually periodic solutions in $\mathcal{X}$ for some integer $N \geq h$.

Proof: The proof of the "if" direction is immediate.

We now prove "only if". By assumption, there exist solutions in $\mathcal{X}$ to (8). We can then equivalently rewrite the first condition in (8) as inequality (10), where $R(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \dots p_r^{v_r} R_v(k) \in \mathcal{X}$ for some $\tau \in \mathbb{N}_0$. As $M(p,k)$ and $H(p,k)$ in (10) are (h,q)-eventually periodic in this case, we have $M(p,k+h+iq) = M(p,k+h)$ and $H(p,k+h+iq) = H(p,k+h)$ for all $k,i \in \mathbb{N}_0$. Then, appealing to Proposition 5, there exists a q-periodic $Y(p,k) = \sum_{v \in J_\tau} p_1^{v_1} p_2^{v_2} \dots p_r^{v_r} Y_v(k) \in \mathcal{X}$ such that

$$M(p,k+h) \begin{bmatrix} Y(p,k) & \\ & I \end{bmatrix} M^*(p,k+h) - \begin{bmatrix} Y(p+dp,k+1) & \\ & \gamma^2 I \end{bmatrix} - \sigma H(p,k+h) \prec -\beta I,$$

for all $(p,dp) \in \Gamma$, $k \in \mathbb{N}_0$, and some positive scalar β. Based on the continuity and convexity properties of LMIs, and following a similar argument to that used in the proof of [6, Lemma 7], we can show that there exists a matrix-valued function $Q_\varepsilon(p,k) = \sum_{v \in J_\tau} p_1^{v_1} \dots p_r^{v_r} Q_{\varepsilon,v}(k) \in \mathcal{X}$ satisfying inequality (10), where

$$Q_{\varepsilon,v}(k) = \frac{R_v(k)+\varepsilon(k) Y_v(k-h)}{1+\varepsilon(k)},$$

$$\text{with} \quad \varepsilon(k) = \begin{cases} 0 & \text{for } k < h \\ \epsilon + \left(\text{floor}\left(\frac{k-h}{q}\right)\right)\xi & \text{for } k \geq h \end{cases}$$

for sufficiently small positive scalars ϵ and ξ. Clearly, as $k \to \infty$, $Q_{\varepsilon,v}(k) \to Y_v(k-h)$. Then, for a sufficiently large k, say $k = N$, and due to the continuity properties of LMIs, we can replace $Q_{\varepsilon,v}(N)$ by $Y_v(N-h)$ and the corresponding inequality at instant $k = N$ would remain valid for all $(p,dp) \in \Gamma$. It follows that the (N,q)-eventually periodic function $Q(p,k) \in \mathcal{X}$ solves (10) for all $(p,dp) \in \Gamma$ and $k \in \mathbb{N}_0$, where the time-varying coefficients $Q_v(k)$ are such that $Q_v(k) = Q_{\varepsilon,v}(k)$ for $k < N$ and $Q_v(k) = Y_v(k-h)$ for $k \geq N$, and so, as $Y_v(k)$ is q-periodic, then $Q_v(k)$ is (N,q)-eventually periodic.

A similar argument can be employed to show that the second synthesis condition in (8) admits an eventually periodic solution in $\mathcal{X}$. As for the coupling condition, it is always possible to construct eventually periodic solutions for the first and second conditions in (8) using the same $\varepsilon(k)$ sequence; in such a case, it is routine to show that the coupling condition holds. ■

The result states that, as far as eventually periodic NSLPV plants are concerned, a solution to the synthesis conditions, if existent, can always be chosen to be eventually periodic, having the same periodicity as the plant but probably exhibiting a longer finite horizon. From a practical perspective, the preceding means that, given an eventually periodic plant, it may be possible to improve the closed-loop performance by allowing for eventually periodic controllers with longer finite horizons than the plant. This is also the case for standard LTV systems as shown in [6], [7].

We conclude this subsection with the following corollary.

Corollary 7: Given NSLPV plant G defined in (2) with $\delta \in \mathbf{\Delta}_\Gamma$, suppose that G is (h,q)-eventually periodic, along with assumptions (A1–A2). Then, with $N \in \mathbb{N}_0$, there exists a γ-admissible (N,q)-eventually periodic LPV synthesis K to plant G for some scalar γ if there exist polynomial matrix-valued functions $R_0(p), \dots, R_{N+q-1}(p) \succ 0$, $S_0(p), \dots, S_{N+q-1}(p) \succ 0$, and some positive scalar σ such that

$$\mathcal{F}_1 \left(R_0(p), R_1(p+dp), \sigma, p, 0\right) \prec 0,$$
$$\mathcal{F}_1 \left(R_1(p), R_2(p+dp), \sigma, p, 1\right) \prec 0,$$
$$\vdots$$
$$\mathcal{F}_1 \left(R_{N+q-2}(p), R_{N+q-1}(p+dp), \sigma, p, N+q-2\right) \prec 0,$$
$$\mathcal{F}_1 \left(R_{N+q-1}(p), R_N(p+dp), \sigma, p, N+q-1\right) \prec 0,$$

and similarly, for $k = 0, 1, \dots, N+q-1$,

$$\mathcal{F}_2 \left(S_k(p), S_k(p+dp), \sigma, p, k\right) \prec 0, \ \mathcal{F}_3 \left(R_k(p), S_k(p)\right) \succeq 0$$

for all $(p,dp) \in \Gamma$, where $S_{N+q}(p) \equiv S_N(p)$.

In the event that G is q-periodic, i.e. $h = 0$, then N can be set equal to zero with no added conservatism.

Thus, when the explicit time dependence in the system equations is of an eventually periodic nature, the infinite dimensionality of the synthesis PLMIs with respect to time k can be bypassed, as evident from Corollary 7. This PLMI problem though remains infinitely constrained. Fortunately, several PLMI relaxation methods are available in the literature. The reader is referred to [11] for a survey on the hierarchies of semidefinite relaxations and their main properties, and to the tutorial paper [12] which focuses on PLMI problems with rational dependence on uncertainties and their important role in robust control. Also, see the latest YALMIP [13] features for solving PLMIs and PMIs (polynomial matrix inequalities). With this said, applying the aforesaid corollary to an NSLPV plant with numerous sampling points ($h, q \gg 0$) is in general a formidable computational problem, regardless of the PLMI relaxation method employed. The next section presents a way to reduce the computational complexity of such a problem.

IV. NSLPV CONTROL OF SWITCHED SYSTEMS

Clearly, in many scenarios where time-varying system parameters are known a priori, the use of nonstationary LPV models instead of stationary ones is quite advantageous as a means for less conservative representations. The trade-off however is an added computational complexity to the synthesis approach; this is by the same token that an LTV approach is computationally more expensive than an LTI one. The computational issue is even more severe when a parameter-dependent Lyapunov function is sought since then each PLMI in a stationary LPV problem would correspond to at least $N+q$ PLMIs in an (N,q)-eventually periodic NSLPV problem. It might be possible to avoid a list of PLMIs if the explicit time dependence can be approximated by polynomial functions, bearing in mind that the larger the polynomial degree is, the more computationally intensive the problem becomes. In general, obtaining polynomial approximations can be a very challenging task, and a practical solution to this computational predicament is to divide the state-space region into a number of divisions in which the explicit time variation becomes very small and the plant dynamics, as a result, can be fairly represented by a stationary LPV model. In other words, we propose to work with switched stationary LPV systems as an alternative to NSLPV models so as to reduce the computational complexity of the synthesis problem to a manageable level. We note that the approach here requires that each of the stationary LPV models of the switched system be strongly stabilizable, as defined next.

Definition 8: We say an SLPV system, with $\delta \in \mathbf{\Delta}_\Gamma$, is strongly stabilizable by a feedback operator $F(\delta(k), d\delta(k))$ for all $\delta \in \mathbf{\Delta}_\Gamma$ if there exists a bounded polynomial function $X(p) \succ 0$ such that, for all $(p, dp) \in \Gamma$, we have

$$\Big(A(p) + B(p)F(p, dp)\Big)\, X(p)\, \Big(A(p) + B(p)F(p, dp)\Big)^* - X(p + dp) \prec 0.$$

Consider a nonlinear system and a corresponding NSLPV model G which captures the nonlinear system dynamics over some state-space region $\mathcal{E}$. Suppose it is possible to divide $\mathcal{E}$ into N subregions $\mathcal{E}^{(i)}$ for $i = 1, 2, \ldots, N$ such that, over each subregion $\mathcal{E}^{(i)}$, the explicit time variation of the NSLPV model G is sufficiently small that the system dynamics can be satisfactorily represented by a strongly stabilizable SLPV model $G^{(i)}$. The resulting SLPV models constitute a switched system denoted by $G_s := \{G^{(1)}, \ldots, G^{(N)}\}$. We also denote the boundary between subregions $\mathcal{E}^{(i)}$ and $\mathcal{E}^{(j)}$ by $\mathcal{B}_{ij}$; a nonexistent boundary is set equal to the empty set.

It is obvious that switched systems are directly linked to NSLPV systems; so the results of the previous section are still usable here. As mentioned before, the aim here is to simplify the time-varying nature of the plant in order to render the associated computational problem practicable. This is indeed possible as long as each of the constituent SLPV models of the switched system is strongly stabilizable, as evident from the next result.

Theorem 9: Given a switched SLPV system $G_s = \{G^{(1)}, \ldots, G^{(N)}\}$ with $\delta \in \mathbf{\Delta}_\Gamma$, suppose that each of the constituent SLPV models is strongly stabilizable along with assumptions (A1–A2). Then there exists a γ-admissible switched SLPV synthesis K_s to plant G_s if, for $i, j = 1, 2, \ldots, N$, $i \neq j$, there exist polynomial matrix-valued functions $R_i(p) \succ 0$, $S_i(p) \succ 0$, and some positive scalar σ such that

$$\begin{aligned} &\mathcal{F}_1^{(i)}\left(R_i(p), R_i(p+dp), \sigma, p\right) \prec 0, \\ &\mathcal{F}_2^{(i)}\left(S_i(p), S_i(p+dp), \sigma, p\right) \prec 0, \\ &\mathcal{F}_3^{(i)}\left(R_i(p), S_i(p)\right) \succeq 0, \end{aligned} \tag{12}$$

and, across each *existent* boundary $\mathcal{B}_{ij}$,

$$\begin{aligned} &\mathcal{F}_1^{(i)}\left(R_i(p), R_j(p+dp), \sigma, p\right) \prec 0, \\ &\mathcal{F}_2^{(i)}\left(S_i(p), S_j(p+dp), \sigma, p\right) \prec 0, \end{aligned} \tag{13}$$

for all $(p, dp) \in \Gamma$, where the notation $\mathcal{F}_x^{(y)}$ is as defined in (9) with the superscript y indicating that the SLPV state-space data used correspond to subsystem $G^{(y)}$, and the explicit dependence on k is dropped.

Proof: Given a parameter-trajectory $\delta \in \mathbf{\Delta}_\Gamma$, say the state-space subregions covered are $\mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(N)}$, where the time-intervals in which the system stays in these subregions are $[0, k_1]$, $[k_1{+}1, k_2]$,. . ., $[k_{N-1}, \infty[$, respectively. Then, the matrix-valued functions $R(p,k) \equiv R_i(p)$, $S(p,k) \equiv S_i(p)$ for $i = 1, \ldots, N$, $k = k_{i-1} + 1, \ldots, k_i$, with $k_0 = -1$ and $k_N = \infty$, and the positive scalar σ solve the synthesis conditions for the NSLPV system G, whose A-matrix, for example, is defined as $A(p,k) \equiv A^{(i)}(p)$ for i, k as aforementioned. Invoking Corollary 7 completes the proof. ■

Remark 10: In the preceding result, the switching takes place over one discrete-time instant. It is not difficult to re-write the conditions so that the switching occurs over several time instants, bearing in mind that this would incur additional PLMIs and hence increase the computational complexity. Of course, more work needs to be done to further realize the switching logic and link it to what is currently available in the literature.

V. Conclusions

The paper deals with the control of eventually periodic and hybrid LPV systems. The use of parameter-dependent Lyapunov functions in the context of generalized LMI-based H_∞ control results in analysis and synthesis conditions in the form of parameterized linear matrix inequalities. As for online controller construction, we refer the reader to [3] for fast and easy-to-implement algorithms based on the results of [14] and [15].

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