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Rank Tests for the Observability of Discrete-Time Jump Linear Systems with Inputs

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Abstract— **We analyze the observability of the continuous and discrete states of discrete-time jump linear systems (JLSs) with deterministic inputs. We consider several definitions of observability for JLSs depending on whether some or all inputs are considered. Unfortunately, checking these definitions can involve an exponential number of rank tests on the parameters of the JLS when the discrete state sequence is arbitrary. Our key contribution is to demonstrate that when there is a minimum separation between consecutive switches, one can verify observability by checking a number of rank tests that is only quadratic in the number of discrete states. The observability conditions we derive are natural generalizations of the well known rank test for linear systems. Moreover, they can be related to the Markov parameters of the individual linear systems.**

I. Introduction

A critical aspect in observer design and identification algorithms for hybrid systems is *observability*, i.e., the study of the conditions under which the continuous and discrete states of a system can be computed uniquely from input-output measurements. Existing works on observability of hybrid systems can be divided in two main categories. The first class proposes computationally simple observability conditions, often in the form of rank tests, but requires strong assumptions on the hybrid system, such as measured discrete states, autonomous systems, minimum dwell time, etc. The second class addresses the observability of more general classes of hybrid systems, such as switched or piecewise affine systems with inputs, but requires computationally expensive algorithms for checking observability, such as an exponential but decidable number of rank tests or mixed integer and quadratic programming.

Paper contributions and outline. The goal of this paper is to develop several notions of observability for discrete-time jump-linear systems (JLSs) with deterministic inputs, as well as computationally efficient tests for verifying when a JLS is observable. In §II we present several definitions of observability for JLSs. The first two definitions are *mode observability* [1], [2] and *strong mode observability*, which refer to the ability of uniquely reconstructing the discrete state for *some* or for *all* inputs, respectively. The other two definitions are *observability* and *strong observability*, which refer to the ability of uniquely reconstructing both continuous and discrete states for *some* or for *all* inputs, respectively.

Unfortunately, checking the observability of a JLS with N discrete states for all possible discrete state sequences of length T can involve $O(N^T)$ rank tests. To deal with this exponential complexity, we assume that the switching times are separated by a minimum dwell time that depends on the order of the JLS. Under this assumption, we show that the computational complexity reduces from $O(N^T)$ to $O(N^2)$. The proof is done in three steps.

In §III we consider the simple case of a JLS whose discrete state sequence is constant, i.e., the sequence has no switch. We show that the observability of the discrete and continuous states before the first switch can be verified by checking $O(N^2)$ rank tests on the parameters of the JLS. We also show that some of these rank tests are equivalent to certain conditions on the Markov parameters of the linear systems.

In §IV we consider a JLS whose discrete state sequence contains only one switch and study conditions under which the switching time can be detected after one or more steps. We show that detectability of a switch can be verified with $O(N^2)$ rank tests on the JLS model parameters.

In §V we consider a JLS whose discrete state sequence has multiple switches separated by a minimum dwell time. We show that such a JLS is observable when it is observable before the first switch and the switching times are detectable after one step. As a consequence, verifying observability requires only $O(N^2)$ rank tests, as claimed. §VI provides some numerical examples that show the simplicity and effectiveness of the proposed observability conditions.

It is worthwhile mentioning that the observability conditions for JLSs that we propose are only sufficient, but not necessary. While other works have proposed weaker definitions of observability or derived observability conditions under weaker assumptions, verifying such weaker notions is often computationally very complex. Therefore, our work offers a balance between the generality of the observability definitions and conditions, and the computational complexity of the tests for verifying such definitions.

Related work. Among the existing works on observability of discrete-time JLSs, the ones that are more closely related to our approach are [1], [3], [4], [2]. In comparison to the work of [3], the main contribution of this paper is that the switching sequence is viewed as an unknown parameter which needs to be estimated, rather than as a measured input. With respect to the work of [4] on observability of autonomous JSLs, the main contribution of our work is to consider more general definitions of observability for JLSs with inputs. In fact, our results can be seen as a natural generalization of those in [4] to non-autonomous systems.

With respect to the work of [1], [2], which also considered JLSs with inputs, there are two fundamental differences. First, we consider several different concepts of observability. Second, the conditions we obtain are simpler to check, thanks to the additional assumption of a minimum dwell time.

Prior work. For continuous-state linear systems, it is well known that the observability problem can be reduced to analyzing the rank of the so-called *observability matrix* [5].

For discrete-event systems, a definition of current-location observability was proposed in [6] as the ability to estimate the location of the system after a finite number of steps. A similar definition was given in [7] together with a polynomial test for observability, the so-called current-location tree, which depends on properties of the nodes of a finite state machine associated with the discrete-event system.

For hybrid systems, one of the first attempts to characterize observability can be found in [8], though the condition given in the paper is somewhat tautological. [9] addresses the observability and controllability of switched linear systems with known and periodic transitions. [10] gives conditions for the observability of a particular class of linear time-varying systems where the system matrix is a linear combination of a basis with respect to time-varying coefficients. [3] gives a condition for the observability of switched linear systems where the switching is assumed to be a measured input. [11] gives conditions for observability of bilinear and linear hybrid systems in continuous time, where the discrete states change according to a finite automaton and the discrete events are measured external inputs. [12] proposes the notion of incremental observability for piecewise affine systems, which can be tested by solving a mixed-integer linear program. [13] derives different rank tests for the weak observability of jump-Markov linear systems. [4] derives conditions for the observability of autonomous discrete-time JLSs that can be tested using simple rank tests on the structural parameters of the model. Such conditions are natural generalizations of well-known results for linear systems and can be extended to continuous-time JLSs [14] and to piecewise affine hybrid systems [15]. [16] gives observability conditions for stochastic linear hybrid systems in terms of the covariances of the outputs. [17] shows that the observability notions based on state indistinguishability do not imply state reconstructability and proposes new definitions of observability, and a weaker notion of detectability, based on the possibility of reconstructing the system state for discrete-time switching systems. [1] proposes various concepts of observability for autonomous and non-autonomous JLSs without imposing constraints on time separation between switches. [18] studies the observability of autonomous and non-autonomous continuous-time switched linear systems and shows that the observability of the discrete mode is equivalent to controlled-discernibility of pairs of different modes.

II. Problem Formulation

A. Jump Linear Systems

A discrete-time jump linear system (JLS) with deterministic inputs is a system whose evolution is determined by a collection of linear models with *continuous state* $x_t \in \mathbb{R}^n$ connected by switching among a number of *discrete states* or *modes* $q_t \in Q \triangleq \{1, \dots, N\}$, $N > 1$. The evolution of the continuous state x_t is described by the linear system

$$\Sigma : \begin{cases} x_{t+1} &= A(q_t)x_t + B(q_t)u_t \\ y_t &= C(q_t)x_t, \end{cases} \tag{1}$$

where $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times n_u}$ and $C(k) \in \mathbb{R}^{n_y\times n}$, for $k \in Q$, are the system parameters, x_{t_0} is the deterministic initial continuous state, u_t is a deterministic input, and $q_t \in Q$ is an unknown, deterministic and finite-valued input which is called the *discrete state (or mode)*.

Notice that the output of a JLS can be written explicitly in terms of the model parameters $\{A(\cdot), B(\cdot), C(\cdot)\}$, the initial continuous state x_{t_0}, the input u_t and the discrete q_t as

$$\begin{aligned} y_t = {} & C(q_t)A(q_{t-1})\cdots A(q_{t_0})x_{t_0} + \\ & C(q_t)A(q_{t-1})\cdots A(q_{t_0+1})B(q_{t_0})u_{t_0} + \cdots + \\ & C(q_t)A(q_{t-1})B(q_{t-2})u_{t-2} + C(q_t)B(q_{t-1})u_{t-1}. \end{aligned} \tag{2}$$

As in this paper we are interested in analyzing the input-output behavior of Σ on a finite time horizon $[t_0, t_0+T-1]$ of length T, we will restrict our attention to input sequences of length $T-1$, $u_{t_0}u_{t_0+1}\cdots u_{t_0+T-2}$, and output sequences of length T, $y_{t_0}y_{t_0+1}\cdots y_{t_0+T-1}$, and stack them into the vectors

$$\begin{aligned} \mathcal{U}_T &\triangleq [u_{t_0}^\top, u_{t_0+1}^\top, \cdots, u_{t_0+T-2}^\top]^\top, \\ \mathcal{Y}_T &\triangleq [y_{t_0}^\top, y_{t_0+1}^\top, \cdots, y_{t_0+T-1}^\top]^\top. \end{aligned} \tag{3}$$

Also, we will write a sequence of T discrete states as $w \triangleq q_{t_0}q_{t_0+1}\cdots q_{t_0+T-1}$, and denote the set of all mode sequences of length T as

$$Q_T = \{q_0q_1\cdots q_{T-1} \mid q_0, \dots, q_{T-1} \in Q\}. \tag{4}$$

With this notation, we can write the output of Σ in $[t_0, t_0+T-1]$ as

$$\mathcal{Y}_T \triangleq \mathcal{O}_T(w)x_{t_0} + \Gamma_T(w)\mathcal{U}_T, \tag{5}$$

where

$$\mathcal{O}_T(w) \triangleq \begin{bmatrix} C(q_{t_0}) \\ C(q_{t_0+1})A(q_{t_0}) \\ \vdots \\ C(q_{t_0+T-1})A(q_{t_0+T-2})\cdots A(q_{t_0}) \end{bmatrix} \tag{6}$$

will be called the *observability matrix* of the JLS and

$$\Gamma_T(w) \triangleq \begin{bmatrix} 0 & \cdots & 0 \\ C(q_{t_0+1})B(q_{t_0}) & \cdots & 0 \\ C(q_{t_0+2})A(q_{t_0+1})B(q_{t_0}) & \ddots & 0 \\ \vdots & & \vdots \\ & & C(q_{t_0+T-1})B(q_{t_0+T-2}) \end{bmatrix} \tag{7}$$

will be called the non-symmetric *Toeplitz matrix* of the JLS. Observe from (5) that the output vector $\mathcal{Y}_T$ can be thought of as the value of the input-output map $\mathcal{Y}(x_{t_0}, w, \mathcal{U}_T)$ induced by (x_{t_0}, w) for the input sequence $\mathcal{U}_T$. Notice that in the proper system-theoretic sense, the state of Σ is the pair (x_{t_0}, w), where $x_{t_0} \in \mathbb{R}^n$ is the initial continuous state and $w \in Q_T$ is the sequence of discrete modes.

Remark 1: Notice that whenever the mode sequence is constant, i.e., $w = q \cdots q \in Q_T$, the matrices $\mathcal{O}_T(w)$ and $\Gamma_T(w)$ reduce to the the well known extended observability matrix and non-symmetric Toeplitz matrix of the linear system $(A(q), B(q), C(q))$. With an abuse of notation, we will write these matrices simply as $\mathcal{O}_T(q)$ and $\Gamma_T(q)$.

B. W_T-Observability of JLSs

Given a JLS of the form (1) with known parameters $\{A(k), B(k), C(k);\ k \in Q\}$, we focus our attention on the study of the conditions under which the state of the system (x_{t_0}, w) can be uniquely determined from measurements of the input/output data $\mathcal{U}_T$ and $\mathcal{Y}_T$.

For linear systems, it is well known that the input has no effect on whether the continuous state x_{t_0} can be determined uniquely. Also for a JLS in which the mode sequence w is measured, the initial state x_{t_0} can be determined uniquely if and only if the matrix $\mathcal{O}_T(w)$ is full rank, as shown in [1]. However, notice that checking this condition is not straightforward, as it involves checking $O(N^T)$ rank tests.

In this paper we are interested in the more general situation in which w is not measured. Notice that in this case the input does have an effect on whether the continuous and discrete states can be uniquely determined. This is because the matrix $\Gamma_T(w)$ in equation (5) depends explicitly on the sequence of discrete states w. Therefore, there are several ways of defining observability for JLSs, depending on what assumptions are made about the input sequence.

In what follows, we define several possible notions of observability, which differ precisely on whether *all* or *some* inputs are considered. For reasons that will become apparent in the next subsection, we restrict the values of pairs of discrete state sequences via a relation $W_T \subseteq Q_T \times Q_T$ on the set of sequences of discrete states of length T, Q_T. This leads to the following notions of W_T-observability for JLSs.

Definition 1: We say that a JLS is *W_T-mode observable* on the interval $[t_0, t_0 + T - 1]$ if for any two different sequences of discrete states $w \neq \bar{w}$, $(w, \bar{w}) \in W_T$, there exists *some* input vector $\mathcal{U}_T$ such that for all pairs of initial continuous states $(x_{t_0}, \bar{x}_{t_0}) \neq 0$ the corresponding outputs $\mathcal{Y}_T = \mathcal{Y}(x_{t_0}, w, \mathcal{U}_T)$ and $\bar{\mathcal{Y}}_T = \mathcal{Y}(\bar{x}_{t_0}, \bar{w}, \mathcal{U}_T)$ are not equal, i.e.,

$$\forall (w, \bar{w}) \in W_T, w \neq \bar{w},\ \exists \mathcal{U}_T,\ \forall (x_{t_0}, \bar{x}_{t_0}) \neq 0 \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T.$$

Notice that our definition of mode observability differs from the definition of mode observability of [1], [2] in the way the quantifiers are placed.[1] There, it is required that there exists a universal input, i.e., an input independent of the switching sequences, such that for any two distinct switching sequences, the outputs corresponding to the inputs are different. In contrast, in Definition 1, the input $\mathcal{U}_T$ **may depend on the sequences** w **and** $\bar{w}$. Therefore, a system might be mode observable according to Definition 1, but not according to the definition of [1], [2]. Moreover, the sufficient conditions of [1], [2] are also sufficient conditions for mode observability in our setting.

Definition 2: We say that a JLS is *W_T-strong mode observable* on the interval $[t_0, t_0{+}T{-}1]$ if for any two different sequences of discrete states $w \neq \bar{w}$, $(w, \bar{w}) \in W_T$, for *all* input vectors $\mathcal{U}_T$ and for all pairs of initial continuous states $(x_{t_0}, \bar{x}_{t_0}) \neq 0$, the corresponding outputs are not equal, i.e.,

$$\forall (w, \bar{w}) \in W_T, w \neq \bar{w},\ \forall \mathcal{U}_T,\ \forall (x_{t_0}, \bar{x}_{t_0}) \neq 0 \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T.$$

Again, our definition of strong mode observability is different from the definition of mode observability of [1], [2]. The main difference is that strong mode observability in the sense of Definition 2 requires that all distinct sequences of discrete modes yield different outputs for all inputs, while the concept of [1], [2] requires only that there exists an input such that all distinct sequences of discrete modes yield different outputs for **that particular input**.

Definition 3: We say that a JLS is *W_T-observable* on the interval $[t_0, t_0 + T - 1]$ if for all states $(x_{t_0}, w) \neq (\bar{x}_{t_0}, \bar{w})$ with $(w, \bar{w}) \in W_T$ and $(x_{t_0}, \bar{x}_{t_0}) \neq 0$, there is *some* input vector $\mathcal{U}_T$ such that the corresponding outputs are not equal, i.e.,

$$\forall (x_{t_0}, w) \neq (\bar{x}_{t_0}, \bar{w}),\ (w, \bar{w}) \in W_T,\ (x_{t_0}, \bar{x}_{t_0}) \neq 0, \\ \exists \mathcal{U}_T \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T.$$

The definition of observability from Definition 3 says that there are no indistinguishable states, i.e., for any two distinct pairs of continuous states and sequences of discrete modes, there exists an input, possibly dependent on the initial continuous states and the sequences of discrete modes, such that the outputs corresponding to that particular input are different.

Definition 4: We say that a JLS is *W_T-strong observable* on the interval $[t_0, t_0 + T - 1]$ if for all states $(x_{t_0}, w) \neq (\bar{x}_{t_0}, \bar{w})$, with $(w, \bar{w}) \in W_T$ and $(x_{t_0}, \bar{x}_{t_0}) \neq 0$, and for *all* input vectors $\mathcal{U}_T$, the corresponding outputs are not equal, i.e.,

$$\forall (x_{t_0}, w) \neq (\bar{x}_{t_0}, \bar{w}),\ (w, \bar{w}) \in W_T,\ (x_{t_0}, \bar{x}_{t_0}) \neq 0, \\ \forall \mathcal{U}_T \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T.$$

Strong observability requires that any two distinct pairs of continuous states and sequences of discrete modes yield different outputs for every input sequence. Strong observability implies observability, but not vice-versa.

C. Minimum Dwell Time Observability of JLSs

If we let $W_T = Q_T \times Q_T$ in Definitions 1-4, we obtain definitions of observability for arbitrary discrete state sequence. Unfortunately, verifying observability for all possible

[1]More precisely, [1] used the expression strong mode observability, while [2] uses the expression mode observability. For the sake of simplicity, we will refer to the observability concept of both [1] and [2] as mode observability.

discrete state sequences is not computationally straightforward, because the number of sequences in Q_T is N^T. This motivates us to analyze definitions of observability for a subset of the discrete mode sequences of length T.

An important subset of Q_T is the set of discrete state sequences that are separated by a minimum dwell time. More precisely, let us denote by t_i the ith switching time and by τ_i the time spent in $[t_i, t_{i+1}-1]$, i.e., $\tau_i = t_{i+1} - t_i$. Also, let $\nu > 0$ be an integer indicating the minimum dwell time. The set of discrete state sequences of length T with switches separated by a minimum dwell time $\nu \le T$ is defined as

$$\begin{aligned} Q_T^\nu = &\big\{ q_1^{\tau_1} \cdots q_\ell^{\tau_\ell} \mid q_1, \ldots, q_\ell \in Q, \ell \ge 1, \\ &\sum_{j=1}^{\ell} \tau_j = T,\ \nu \le \tau_j \le T, \text{ for } j = 1, 2, \ldots, \ell \big\}, \end{aligned} \tag{8}$$

where q^τ stands for $qq\cdots q$ (τ-times). After letting

$$W_T^\nu = Q_T^\nu \times Q_T^\nu, \tag{9}$$

Definitions 1-4 can be specialized for JLSs with a minimum dwell time $\nu > 0$ as follows.

Definition 5 (Minimum dwell time observability): A JLS is called [strong] [mode] observable with a minimum dwell time ν if it is W_T^ν-[strong] [mode] observable on the time interval $[t_0, t_0 + T - 1]$.

Notice that restricting our definitions of observability from $Q_T \times Q_T$ to W_T^ν does not necessarily eliminate the issue of computational complexity, because the number of sequences in Q_T^ν is still $O(N^{T/\nu})$. This is precisely the key contribution of this paper: to find sufficient conditions for observability with minimum dwell time ν that involve checking only $O(N^2)$ rank tests.

The proof of our main result will be done in three main steps. First, we will consider the problem of uniquely determining the continuous and discrete states of a JLS before the first switch occurs. In §III, we will show that the observability of the initial continuous and discrete states before the first switch can be verified by checking $O(N^2)$ rank tests on the parameters of the constituent linear systems. The precise definitions of [strong] mode observability and [strong] observability before the first switch involve the set

$$W_T^T = \{(q^T, \bar{q}^T) \mid q, \bar{q} \in Q\}, \tag{10}$$

and can be stated as follows.

Definition 6: We will refer to W_T^T-[strong] mode observability and W_T^T-[strong] observability on the time interval $[t_0, t_0 + T - 1]$ as *[strong] mode observability before the first switch* on $[t_0, t_0 + T - 1]$ and *[strong] observability before the first switch* on $[t_0, t_0 + T - 1]$, respectively.

Second, we will consider the problem of uniquely determining the time instant at which the first switch occurs. For this purpose, let ν be the minimum dwell time and let $\eta \ge 1$ be the number of time-steps needed to detect a switch in a sequence of discrete modes. Consider also the set

$$W_T^{\nu,\eta} = \{(q^{\nu+\eta}, q^\nu \bar{q}^\eta) \mid q, \bar{q} \in Q, \nu + \eta = T\}. \tag{11}$$

We can now define the notions of [strong] mode detectability and [strong] detectability of the first switching time.

Definition 7: Consider a JLS with minimum dwell time ν. We say that the *first switching time is [strong] mode detectable after η steps with a minimum dwell time ν* on the time interval $[t_0, t_0 + T - 1]$, if the JLS is $W_T^{\nu,\eta}$-[strong] mode observable on the interval $[t_0, t_0 + T - 1]$. We say that the *first switching time is [strong] detectable after η steps with a minimum dwell time ν* on the time interval $[t_0, t_0 + T - 1]$, if the JLS is $W_T^{\nu,\eta}$-[strong] observable on the interval $[t_0, t_0 + T - 1]$.

Intuitively, [strong] [mode] detectability of the first switch after η steps means that if a change in a sequence of discrete modes occurs, then we can detect it in at most η steps after its occurrence, provided that the switching sequence has a minimum dwell time of ν. In §IV, we will derive conditions to uniquely recover the first switching time. As before, our conditions will involve checking $O(N^2)$ rank tests.

Third, we will show in §V that the combination of observability before the first switch and detectability of the first switch gives a sufficient condition for minimum dwell time observability of a JLS. As a consequence, minimum dwell time observability can be checked with $O(N^2)$ rank tests.

III. Observability Before The First Switch

In this section we study the observability of states before a switch occurs. More precisely, we are interested in W_T^T-[strong] mode observability and W_T^T-[strong] observability on $[t_0, t_0+T]$, where W_T^T is the set of pairs of discrete mode sequences with constant discrete states as defined in (10).

For the sake of simplicity, throughout the section we will omit the observability interval $[t_0, t_0 + T - 1]$ when speaking of [strong] [mode] observability before the first switch. Also, for notational convenience, we will use q_{t_0} to refer to the sequence of discrete modes $w = q_{t_0} \cdots q_{t_1 - 1}$, because $q_{t_0} = q_{t_0+1} = \cdots = q_{t_1-1}$. Thus, (x_{t_0}, q_{t_0}) and $\mathcal{O}_T(q_{t_0})$ will refer to the state (x_{t_0}, w) and the matrix $\mathcal{O}_T(w)$ with $w = q_{t_0} q_{t_0} \cdots q_{t_0}$, respectively. Also we will write $\mathcal{Y}_T = \mathcal{Y}_T(x_{t_0}, q_{t_0}, \mathcal{U}_T)$ and $\bar{\mathcal{Y}}_T = \mathcal{Y}_T(\bar{x}_{t_0}, \bar{q}_{t_0}, \mathcal{U}_T)$. Thus, according to (5), we can write

$$\begin{aligned} \mathcal{Y}_T &= \mathcal{O}_T(q_{t_0}) x_{t_0} + \Gamma_T(q_{t_0}) \mathcal{U}_T \\ \bar{\mathcal{Y}}_T &= \mathcal{O}_T(\bar{q}_{t_0}) \bar{x}_{t_0} + \Gamma_T(\bar{q}_{t_0}) \mathcal{U}_T. \end{aligned} \tag{12}$$

A. Case 1: W_T^T-Mode Observability

We first seek conditions under which a JLS is W_T^T-mode observable.

Lemma 1: Let $T \ge 2n$. A JLS is mode observable before the first switch if and only if for all $q_{t_0} \ne \bar{q}_{t_0} \in Q$

$$\operatorname{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]) = 2n, \quad \text{or} \tag{13}$$

$$\begin{aligned} &\operatorname{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0}), \Gamma_T(\bar{q}_{t_0}) - \Gamma_T(q_{t_0})]) \\ &\qquad > \operatorname{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]). \end{aligned} \tag{14}$$

Proof: We start by showing that condition (13) or (14) is sufficient. We do this by contradiction. Assume that (13) or (14) holds, but the JLS is not W_T^T-mode observable. Hence, there exist two sequences $w = q_{t_0} \cdots q_{t_0}, \bar{w} = \bar{q}_{t_0} \cdots \bar{q}_{t_0} \in W_T^T$ such that

$$\forall \mathcal{U}_T \quad \exists (x_{t_0}, \bar{x}_{t_0}) \neq 0 \quad : \quad \mathcal{Y}_T = \bar{\mathcal{Y}}_T. \tag{15}$$

After substituting $\mathcal{Y}_T$ and $\bar{\mathcal{Y}}_T$ from (12) into (15) we obtain that for every $\mathcal{U}_T$ there exists $(x_{t_0}, \bar{x}_{t_0}) \neq 0$ such that

$$\mathcal{O}_T(q_{t_0})x_{t_0} - \mathcal{O}_T(\bar{q}_{t_0})\bar{x}_{t_0} = (\Gamma_T(\bar{q}_{t_0}) - \Gamma_T(q_{t_0}))\mathcal{U}_T. \tag{16}$$

As (16) must hold for all $\mathcal{U}_T$, in particular it must hold for $\mathcal{U}_T$ in the kernel of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$. Hence we must have

$$\text{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]) < 2n, \tag{17}$$

so that equation (16) has a nonzero solution for the initial continuous states $(x_{t_0}, \bar{x}_{t_0})$. Also in order for (16) to hold for every $\mathcal{U}_T$ not in the kernel of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$, the range space of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$ must be contained in the range space of $[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]$, i.e.,

$$\begin{aligned} &\text{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0}), \Gamma_T(\bar{q}_{t_0}) - \Gamma_T(q_{t_0})]) \\ &\qquad = \text{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]). \end{aligned} \tag{18}$$

The two conditions (17) and (18), which must be simultaneously satisfied, are obviously in contradiction with (13) or (14). This completes the proof of sufficiency.

We now show that if the JLS is W_T^T-mode observable, at least one of the conditions in (13) or (14) must be met. Again we do this by contradiction. We assume that there exist $q_{t_0} \neq \bar{q}_{t_0}$ for which neither condition (13) nor (14) is satisfied, so (17) and (18) simultaneously hold. Thus, for every input vector $\mathcal{U}_T$, equation (16) admits nonzero solutions for the initial continuous states $(x_{t_0}, \bar{x}_{t_0})$, which clearly contradicts the definition of W_T^T-mode observability. ∎

Now, we show that the second condition of Lemma 1 can be translated into a relation among the Markov parameters of the constituent linear systems. To this end, we state the following basic result, whose proof is left to the reader.

Lemma 2: Given two linear systems (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ with extended observability matrices of order $\mu \geq 2n$, $\mathcal{O}_\mu = [C^\top, (CA)^\top, \ldots, (CA^{\mu-1})^\top]^\top$ and $\bar{\mathcal{O}}_\mu = [\bar{C}^\top, (\bar{C}\bar{A})^\top, \ldots, (\bar{C}\bar{A}^{\mu-1})^\top]^\top$, the kernel of $[\mathcal{O}_\mu, \bar{\mathcal{O}}_\mu]$ is M-invariant, where $M \triangleq \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}$.

Lemma 3: Let $T > 2n$. A JLS is mode observable before the first switch if and only if for all $q_{t_0} \neq \bar{q}_{t_0} \in Q$,

1) $\text{rank}([\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]) = 2n$, or
2) $\Gamma_T(q_{t_0}) \neq \Gamma_T(\bar{q}_{t_0})$, i.e., $\exists\, k \in \{0, 1, \ldots, 2n-1\}$ such that $C(q_{t_0})A^k(q_{t_0})B(q_{t_0}) \neq C(\bar{q}_{t_0})A^k(\bar{q}_{t_0})B(\bar{q}_{t_0})$.

Proof: The first condition is the same as that in equation (13) for W_T^T-mode observability. So, we only need to show that condition (14) in Lemma 1 holds if and only if at least one of the Markov parameters of each linear system is different from the Markov parameters of other linear systems. This is equivalent to proving that condition (18) holds if and only if the Markov parameters of any two linear systems are equal to each other. First assume that the first $2n$ Markov parameters of two different linear models corresponding to q_{t_0} and $\bar{q}_{t_0}$ are equal to each other. By using the Cayley-Hamilton Theorem [5], one can easily show that $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0}) = 0$ and we immediately get the condition in equation (18). Next, assume that equation (18) holds but at least one of the Markov parameters of the linear system q_{t_0} and $\bar{q}_{t_0}$ are different from each other. Let j be the first index such that $C(q_{t_0})A^j(q_{t_0})B(q_{t_0}) \neq C(\bar{q}_{t_0})A^j(\bar{q}_{t_0})B(\bar{q}_{t_0})$, and $C(q_{t_0})A^i(q_{t_0})B(q_{t_0}) = C(\bar{q}_{t_0})A^i(\bar{q}_{t_0})B(\bar{q}_{t_0})$ for $i = 0, \ldots, j-1$. Because of the rank condition in (18), each column of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$ can be written as a linear combination of the columns of $\mathcal{O}_T(q_{t_0})$ and $\mathcal{O}_T(\bar{q}_{t_0})$. Therefore, there exist Θ_j and $\bar{\Theta}_j$ such that:

$$\begin{bmatrix} C & \bar{C} \\ \vdots & \vdots \\ CA^{T-2} & \bar{C}\bar{A}^{T-2} \\ CA^{T-1} & \bar{C}\bar{A}^{T-1} \end{bmatrix} \begin{bmatrix} \Theta_j \\ -\bar{\Theta}_j \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^jB - \bar{C}\bar{A}^j\bar{B} \end{bmatrix}. \tag{19}$$

Here, for ease of notation, we used A to denote $A(q_{t_0})$ and $\bar{A}$ to denote $A(\bar{q}_{t_0})$ (similarly for other matrices). Since $\begin{bmatrix} \Theta_j \\ -\bar{\Theta}_j \end{bmatrix}$ is in the kernel of $[\mathcal{O}_{T-1}(q_{t_0}),\ \mathcal{O}_{T-1}(\bar{q}_{t_0})]$ and $T - 1 \geq 2n$, Lemma 2 results in $CA^{T-1}\Theta_j - \bar{C}\bar{A}^{T-1}\bar{\Theta}_j = 0$, so $CA^jB - \bar{C}\bar{A}^j\bar{B} = 0$, which is obviously a contradiction. Thus we conclude that under condition (14), when $T > 2n$, at least one of the Markov parameters of each linear model is different from the Markov parameters of all other linear systems. ∎

B. Case 2 : W_T^T-Strong Mode Observability

A JLS is strong mode observable before the first switch if

$$\forall q_{t_0} \neq \bar{q}_{t_0} \in Q,\ \forall \mathcal{U}_T,\ \forall (x_{t_0}, \bar{x}_{t_0}) \neq 0 \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T. \tag{20}$$

Since the above relation must hold for all input vectors, we have two different cases. When the input vector $\mathcal{U}_T$ is in the kernel of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$, we have that $\forall q_{t_0} \neq \bar{q}_{t_0} \in Q$,

$$\forall (x_{t_0}, \bar{x}_{t_0}) \neq 0,\ \mathcal{O}_T(q_{t_0})x_{t_0} - \mathcal{O}_T(\bar{q}_{t_0})\bar{x}_{t_0} \neq 0. \tag{21}$$

This requires that for all $q_{t_0} \neq \bar{q}_{t_0} \in Q$

$$\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n. \tag{22}$$

When the input vector $\mathcal{U}_T$ is not in the kernel of $\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})$, we must have that $\forall q_{t_0} \neq \bar{q}_{t_0} \in Q, \forall (x_{t_0}, \bar{x}_{t_0}) \neq 0$,

$$\mathcal{O}_T(q_{t_0})x_{t_0} - \mathcal{O}_T(\bar{q}_{t_0})\bar{x}_{t_0} \neq (\Gamma_T(\bar{q}_{t_0}) - \Gamma_T(q_{t_0}))\mathcal{U}_T. \tag{23}$$

This is equivalent to requiring the intersection of the range spaces of $[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})]$ and $(\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0}))$ to be trivial. Thus, we must have the following condition

$$\begin{aligned} &\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0}), \Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})] = \\ &\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] + \text{rank}(\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})). \end{aligned} \tag{24}$$

Therefore, in order for a JLS to be W_T^T-mode strong observable, the conditions of the following lemma must hold.

Lemma 4: Let $T \geq 2n$. A JLS is strong mode observable before the first switch if and only for all $q_{t_0} \neq \bar{q}_{t_0} \in Q$

1) $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n$, and
2) $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0}), \Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})] = 2n + \text{rank}(\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0}))$.

C. Case 3: W_T^T-Observability

Up to now, we have studied conditions that guarantee the observability of the mode sequence before the first switch. We now study conditions for recovering both the continuous and discrete states before the first switch.

For the sake of contradiction, assume that the JLS is not observable before the first switch. This means that there exist (x_{t_0}, q_{t_0}) and $(\bar{x}_{t_0}, \bar{q}_{t_0})$ not equal to each other and $(x_{t_0}, \bar{x}_{t_0}) \neq 0$ such that for all input vectors $\mathcal{U}_T$ their corresponding outputs are equal, i.e.,

$$\mathcal{O}_T(q_{t_0})x_{t_0} - \mathcal{O}_T(\bar{q}_{t_0})\bar{x}_{t_0} = (\Gamma_T(\bar{q}_{t_0}) - \Gamma_T(q_{t_0}))\mathcal{U}_T. \quad (25)$$

We have the following two cases:

1) If $q_{t_0} = \bar{q}_{t_0}$, then there exist two different initial conditions $x_{t_0} \neq \bar{x}_{t_0}$ such that $\mathcal{O}_T(q_{t_0})(x_{t_0} - \bar{x}_{t_0}) = 0$. This is equivalent to $\text{rank}(\mathcal{O}_T(q_{t_0})) < n$ for some $q_{t_0} \in Q$. As a consequence, if one of the linear systems is not observable, then the JLS is not W_T^T-observable.
2) If $q_{t_0} \neq \bar{q}_{t_0}$, then there exist $(x_{t_0}, \bar{x}_{t_0}) \neq 0$ such that the left hand side of (25) is equal to the right hand side for all $\mathcal{U}_T$. Since the left hand side does not depend on $\mathcal{U}_T$, in order for (25) to hold for all $\mathcal{U}_T$, both the left hand side and the right hand side must be identically zero. We thus have that for some $(x_{t_0}, \bar{x}_{t_0}) \neq 0$, $\mathcal{O}_T(q_{t_0})x_{t_0} - \mathcal{O}_T(\bar{q}_{t_0})\bar{x}_{t_0} = 0$, which implies that $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] < 2n$. In addition, we have that for all $\mathcal{U}_T$, $(\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0}))\mathcal{U}_T = 0$, hence we must have $\Gamma_T(q_{t_0}) = \Gamma_T(\bar{q}_{t_0})$. This means that the Markov parameters of the two linear systems are equal to each other.

Noticing that the condition $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n$ implies the condition of $\text{rank}(\mathcal{O}_T(q_{t_0})) = n$, we have the following lemma for the W_T^T-observability of a JLS.

Lemma 5: Let $T \geq 2n$. A JLS is observable before the first switch if and only if for all $q_{t_0} \neq \bar{q}_{t_0} \in Q$,

1) $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n$ or,
2) $\text{rank}(\mathcal{O}_T(q_{t_0})) = n$ and $\Gamma_T(q_{t_0}) \neq \Gamma_T(\bar{q}_{t_0})$.

D. Case 4 : W_T^T-Strong Observability

The analysis for this case is similar to what we did for strong mode observability before the first switch. The only difference is that, in addition, we have to consider the case where $q_{t_0} = \bar{q}_{t_0}$ with $x_{t_0} \neq \bar{x}_{t_0}$, which results in $\text{rank}(\mathcal{O}_T(q_{t_0})) = n$. One can easily see that this condition is satisfied whenever $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n$. Thus, the conditions in this case are the same as the conditions for strong mode observability before the first switch. That is:

Lemma 6: Let $T \geq 2n$. A JLS is strong observable before the first switch if and only if, for all $q_{t_0} \neq \bar{q}_{t_0} \in Q$,

1) $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0})] = 2n$, and
2) $\text{rank}[\mathcal{O}_T(q_{t_0}), \mathcal{O}_T(\bar{q}_{t_0}), \Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0})] = 2n + \text{rank}(\Gamma_T(q_{t_0}) - \Gamma_T(\bar{q}_{t_0}))$.

IV. Detectability of the First Switching Time

In the previous section, we derived conditions for recovering either the discrete state or both the continuous and discrete states before the first switch. Now, assuming that these conditions are satisfied, we derive additional conditions that enable us to recover the time instant at which the first switch occurs. Once the first switching time is recovered, we can repeat the procedure by finding the discrete (and continuous) state after the first switch, recovering the second switching time, and so on. We will discuss this case in §V.

Recall now the notion of [strong] [mode] detectability of the first switching time after η steps with a minimum dwell time ν in the interval $[t_0, t_0 + T - 1]$ from Definition 7. For the sake of simplicity, throughout the section we will omit the observability interval $[t_0, t_0 + T - 1]$ when referring to [strong] [mode] detectability of the first switching time. We have the following result.

Lemma 7: Consider a JLS with minimum dwell time $\nu \geq 2n$. Assume that the JLS is observable before the first switch on $[t_0, t_0 + \nu - 1]$. For any sequence of discrete states w of length T, let $\hat{\Gamma}_\eta(w)$ be the matrix obtained by taking only the last $\eta \times n_y$ rows of $\Gamma_T(w)$ defined in (7). The first switching time is detectable after η steps, if and only if for all $q_{t_0} \neq q_{t_1} \in Q$,

$$\text{rank}((\mathcal{O}_\eta(q_{t_1}) - \mathcal{O}_\eta(q_{t_0}))A^\nu(q_{t_0})) = n, \text{ or} \quad (26)$$

$$\hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta}) \neq \hat{\Gamma}_\eta(q_{t_0}^\nu q_{t_1}^\eta). \quad (27)$$

Proof: According to Definition 7, the first switching time of a JLS is detectable after η steps if

$$\forall (x_{t_0}, q_{t_0}^{\nu+\eta}) \neq (\bar{x}_{t_0}, q_{t_0}^\nu q_{t_1}^\eta), (q_{t_0}^{\nu+\eta}, q_{t_0}^\nu q_{t_1}^\eta) \in W_T^{\nu,\eta}, \\ (x_{t_0}, \bar{x}_{t_0}) \neq 0, \exists \mathcal{U}_T \Rightarrow \mathcal{Y}_T \neq \bar{\mathcal{Y}}_T. \quad (28)$$

When $x_{t_0} \neq \bar{x}_{t_0}$, then by observability of the JLS before the first switch, we have that $\mathcal{Y}_T \neq \bar{\mathcal{Y}}_T$.

When $x_{t_0} = \bar{x}_{t_0}$, we want to distinguish between the outputs of $(x_{t_0}, q_{t_0}^{\nu+\eta})$ and $(x_{t_0}, q_{t_0}^\nu q_{t_1}^\eta)$. In order to derive the conditions, we use the negation of (28) when $\bar{x}_{t_0} = x_{t_0}$,

$$\exists (x_{t_0}, q_{t_0}^{\nu+\eta}) \neq (x_{t_0}, q_{t_0}^\nu q_{t_1}^\eta), (q_{t_0}^{\nu+\eta}, q_{t_0}^\nu q_{t_1}^\eta) \in W_T^{\nu,\eta}, \\ x_{t_0} \neq 0, \forall \mathcal{U}_T \Rightarrow \mathcal{Y}_T = \bar{\mathcal{Y}}_T. \quad (29)$$

Since the outputs of the system associated with $(x_{t_0}, q_{t_0}^{\nu+\eta})$ and $(x_{t_0}, q_{t_0}^\nu q_{t_1}^\eta)$ are the same in the interval $[t_0, t_0 + \nu - 1]$, we have to search for conditions that yield equal outputs in the interval $[t_0 + \nu, t_0 + T - 1]$. To this end, we will denote the system matrices $(A(q_{t_i}), B(q_{t_i}), C(q_{t_i}))$ by (A_i, B_i, C_i) for $i = 0, 1$. With this notation, let $y_{t_0+\nu+k}$ and $\bar{y}_{t_0+\nu+k}$ denote the outputs of the system at $t_0 + \nu + k$ corresponding to $(x_{t_0}, q_{t_0}^{\nu+\eta})$ and $(x_{t_0}, q_{t_0}^\nu q_{t_1}^\eta)$, respectively. Then we have:

$$y_{t_0+\nu+k} = C_0 A_0^{k+\nu} x_{t_0} + C_0 A_0^{k-1+\nu} B_0 u_{t_0} + \\ \cdots + C_0 B_0 u_{t_0+\nu+k-1}, \quad (30)$$

$$\bar{y}_{t_0+\nu+k} = C_1 A_1^k A_0^\nu x_{t_0} + C_1 A_1^{k-1} A_0^\nu B_0 u_{t_0} + \\ \cdots + C_1 B_0 u_{t_0+\nu+k-1}. \quad (31)$$

By taking every possible $k \in \{0, 1, \cdots, \eta - 1\}$ in equations (30) and (31), we obtain the following expressions for the outputs $\mathcal{Y}_\eta$ and $\bar{\mathcal{Y}}_\eta$ on the interval $[t_0+\nu, t_0+T-1]$ corresponding to (q_{t_0}, x_{t_0}) and $(\bar{q}_{t_0}, \bar{x}_{t_0})$:

$$\mathcal{Y}_\eta = \mathcal{O}_\eta(q_{t_0})A_0^\nu x_{t_0} + \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})\mathcal{U}_T, \quad (32)$$
$$\bar{\mathcal{Y}}_\eta = \mathcal{O}_\eta(q_{t_1})A_0^\nu x_{t_0} + \hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta})\mathcal{U}_T. \quad (33)$$

Thus, by equality of equations (32) and (33), we get

$$(\mathcal{O}_\eta(q_{t_1}) - \mathcal{O}_\eta(q_{t_0}))A_0^\nu x_{t_0} = (\hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta}))\mathcal{U}_T. \quad (34)$$

Since this equation must hold for all $\mathcal{U}_T$, in an analogous analysis to what we did in §III-C we get

$$\operatorname{rank}((\mathcal{O}_\eta(q_{t_1}) - \mathcal{O}_\eta(q_{t_0}))A^\nu(q_{t_0})) < n, \quad \text{and} \quad (35)$$
$$\hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta}) = \hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) \quad (36)$$

Thus, from the negation of the above conditions we get the proposed conditions (26) and (27) of the Lemma. ■

In an analogous fashion, one can derive the following conditions for strong detectability, mode detectability, and strong mode detectability of the first switching time.

Lemma 8: Consider a JLS with minimum dwell time $\nu \geq 2n$ and which is strong observable before the first switch on $[t_0, t_0+\nu-1]$. The first switching time is strong detectable after η steps, if and only if for all $q_{t_0} \neq q_{t_1} \in Q$,

$$\operatorname{rank}((\mathcal{O}_\eta(q_{t_1}) - \mathcal{O}_\eta(q_{t_0}))A^\nu(q_{t_0})) = n, \text{ and}$$
$$\operatorname{rank}([(\mathcal{O}_\eta(q_{t_1}) - \mathcal{O}_\eta(q_{t_0}))A^\nu(q_{t_0}), \hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})]) = n + \operatorname{rank}(\hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})).$$

Lemma 9: Consider a JLS with minimum dwell time $\nu \geq 2n$. Assume that for each discrete mode $q \in Q$ the matrix $A(q)$ is invertible and that the system $(C(q), A(q))$ is observable. The first switching time is mode detectable after η steps, if and only if for all $q_{t_0} \neq q_{t_1} \in Q$,

$$\operatorname{rank}[(\mathcal{O}_\eta(q_{t_0}) - \mathcal{O}_\eta(q_{t_1}))A^\nu(q_{t_0})] = n, \quad \text{or}$$
$$\operatorname{rank}[(\mathcal{O}_\eta(q_{t_0}) - \mathcal{O}_\eta(q_{t_1}))A^\nu(q_{t_0}), \hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})] > \operatorname{rank}[\mathcal{O}_\eta(q_{t_0}) - \mathcal{O}_\eta(q_{t_1})].$$

Lemma 10: Consider a JLS with minimum dwell time $\nu \geq 2n$. Assume that for each discrete mode $q \in Q$ the matrix $A(q)$ is invertible and the system $(A(q), C(q))$ is observable. The first switching time is strong mode detectable after η steps, if and only if for all $q_{t_0} \neq q_{t_1} \in Q$,

$$\operatorname{rank}[(\mathcal{O}_\eta(q_{t_0}) - \mathcal{O}_\eta(q_{t_1}))A^\nu(q_{t_0})] = n, \quad \text{and}$$
$$\operatorname{rank}[(\mathcal{O}_\eta(q_{t_0}) - \mathcal{O}_\eta(q_{t_1}))A^\nu(q_{t_0}), \hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})] = n + \operatorname{rank}(\hat{\Gamma}_\eta(q_{t_0}^{\nu} q_{t_1}^{\eta}) - \hat{\Gamma}_\eta(q_{t_0}^{\nu+\eta})).$$

V. Minimum Dwell Time Observability

In this section we show that [strong] [mode] observability before the first switch and [strong] [mode] detectability of the first switching time after one step are sufficient for [strong] [mode] observability with minimum dwell time. More specifically, we show the following (see the Appendix for the proof).

Theorem 1: Consider a JLS with minimum dwell time $\nu \geq 2n+1$. Assume that for each discrete mode $q \in Q$, the matrix $A(q)$ is invertible. Assume also that

- the JLS is [strong] [mode] observable before the first switch with minimum dwell time ν on the interval $[t, t+\nu-1]$ for any t, i.e., the JLS is W_ν^ν [strong] [mode] observable on the interval $[t, t+\nu-1]$ for any t; and
- the first switching time is [strong] [mode] detectable after one step with a minimum dwell time ν on the interval $[t, t+\nu]$ for any t, i.e., the JLS is $W_{\nu+1}^{\nu,1}$ [strong] [mode] observable on the interval $[t, t+\nu]$, $t \geq 0$.

Then the JLS is [strong] [mode] observable on $[t_0, t_0+T-1]$, i.e., it is W_T^ν [strong] [mode] observable on $[t_0, t_0+T-1]$.

Recall from §III-IV that both observability before the first switch and detectability of the first switching time can be characterized with $O(N^2)$ rank tests on the parameters of the JLS. Therefore, it follows from Theorem 1 that the following $O(N^2)$ rank conditions are sufficient for [strong] minimum dwell time observability on the interval $[t_0, t_0+T-1]$.

Corollary 1: (Minimum dwell time observability): A JLS is observable with minimum dwell time $\nu \geq 2n+1$ on $[t_0, t_0+T-1]$, if $A(q)$ is invertible for every $q \in Q$ and for any pair of distinct discrete states $q \neq \bar{q} \in Q$,

1) $\operatorname{rank}[\mathcal{O}_\nu(q), \mathcal{O}_\nu(\bar{q})] = 2n$, or
   $\operatorname{rank}(\mathcal{O}_\nu(q)) = n$ and $\Gamma_\nu(q) \neq \Gamma_\nu(\bar{q})$; and
2) $\operatorname{rank}((C(\bar{q}) - C(q))A^\nu(q)) = n$, or
   $\mathcal{N}(q, \bar{q}) \neq 0$.

where $\mathcal{N}(q, \bar{q}) = (\bar{C} - C)\left[A^{\nu-1}B, A^{\nu-2}B, \cdots, B\right]$ with $\bar{C} = C(\bar{q})$ and $(A, B, C) = (A(q), B(q), C(q))$.

Corollary 2: (Minimum dwell time strong observability): A JLS is strong observable with minimum dwell time $\nu \geq 2n+1$ on the interval $[t_0, t_0+T-1]$, if $A(q)$ is invertible for every $q \in Q$ and for any pair of distinct modes $q \neq \bar{q} \in Q$,

$$\operatorname{rank}[\mathcal{O}_\nu(q), \mathcal{O}_\nu(\bar{q})] = 2n,$$
$$\operatorname{rank}[\mathcal{O}_\nu(q), \mathcal{O}_\nu(\bar{q}), \Gamma_\nu(q) - \Gamma_\nu(\bar{q})] = 2n + \operatorname{rank}(\Gamma_\nu(q) - \Gamma_\nu(\bar{q})),$$
$$\operatorname{rank}((C(\bar{q}) - C(q))A^\nu(q)) = n, \quad \text{and}$$
$$\operatorname{rank}[(C(\bar{q}) - C(q))A^\nu(q), \mathcal{N}(q, \bar{q})] = n + \operatorname{rank}(\mathcal{N}(q, \bar{q})).$$

Analogous sufficient rank conditions can be formulated for [strong] mode observability with minimum dwell time by combining Theorem 1 with the rank conditions for [strong] mode observability before the first switch and [strong] mode detectability.

VI. Numerical Example

Consider a JLS of the form (1) with $n = N = 2$ and $\nu = 5$ where

$$A(1) = A(2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; C(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C(2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$B(1) = [1 \; 0]^\top, \; B(2) = [2 \; 1]^\top.$$

Since $C(1)B(1) \neq C(2)B(2)$, by Lemma 3 we have that the JLS is mode observable before the first switch and since

$\text{rank}(\mathcal{O}_5(1)) = \text{rank}(\mathcal{O}_5(2)) = 2$, according to Lemma 5, the JLS is also observable before the first switch. However, the JLS does not satisfy the conditions of Lemmas 4 and 6, so it is not strong [mode] observable before the first switch ($\text{rank}([\mathcal{O}_5(1), \mathcal{O}_5(2)]) = 2 \neq 4$). One can easily form

$$\hat{\Gamma}_1(1^{5+1}) = \begin{bmatrix} 1\,1\,1\,1\,1\,1 \\ 0\,0\,0\,0\,0\,0 \end{bmatrix}, \quad \hat{\Gamma}_1(1^5 2^1) = \begin{bmatrix} 2\,2\,2\,2\,2\,2 \\ 0\,0\,0\,0\,0\,0 \end{bmatrix}$$

and see that $\hat{\Gamma}_1(1^{5+1}) \neq \hat{\Gamma}_1(1^5 2^1)$ (similarly $\hat{\Gamma}_1(2^{5+1}) \neq \hat{\Gamma}_1(2^5 1^1)$). Thus the first switching time is mode detectable after one step. In addition, because of the fact that $\text{rank}((\mathcal{O}_1(2) - \mathcal{O}_1(1))A^5(1)) = 2$, the first switching time is detectable after one step according to Lemma 7. Since $A(1)$ and $A(2)$ are invertible, and the JLS is [mode] observable before the first switch and also the first switching time is [mode] detectable after one step, according to Theorem 1, the JLS is [mode] observable. However, the JLS does not satisfy the conditions of Lemmas 8 and 10, so the first switch is not strong [mode] detectable. Thus we cannot assert the strong [mode] observability of the JLS, because the sufficient conditions are not satisfied.

VII. Conclusions

We presented an analysis of the observability of the continuous and discrete states of discrete-time JLSs. We considered several definitions for observability of JLSs and derived rank conditions for each definition. Future work includes removing the assumption of minimum dwell time, as well as addressing the observability conditions for stochastic inputs.

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Appendix: Sketch of the proof of Theorem 1

To simplify the terminology, throughout the proof we will omit the expression: *with minimum dwell time ν on the time interval $[t, t+\nu]$ for any t.*

We have to prove the following implications.

(a) If a JLS is [strong] observable before the first switch and the first switching time is [strong] detectable after one step, then the JLS is [strong] observable.

(b) If a JLS is [strong] mode observable before the first switch and the first switching time is [strong] mode detectable, then the JLS is [strong] mode observable.

We only prove (a) since the proof of (b) is similar to (a).

Recall the definition of W_T^ν from (9) and consider two sequences of discrete modes $w, \bar{w} \in W_T^\nu$. Let $x_{t_0}, \bar{x}_{t_0} \in \mathbb{R}^n$ be two initial continuous states, $(x_{t_0}, \bar{x}_{t_0}) \neq 0$. We have to show that if the system is [strong] observable before the first switch and the first switching time is [strong] detectable after one step, then for some [all] inputs the outputs of (x_{t_0}, w) and $(\bar{x}_{t_0}, \bar{w})$ are different.

Two cases have to be distinguished: 1) $w = \bar{w}$ with $x_{t_0} \neq \bar{x}_{t_0}$, and 2) $w \neq \bar{w}$. Case 1 follows from [strong] observability before the first switch. Case 2 is more involved.

Notice that if $w \neq \bar{w}$ then w and $\bar{w}$ can be rewritten as follows; $w = vq^i s$ and $\bar{w} = v\bar{q}^j \bar{s}$ where s, $\bar{s}$ and v are sequences of discrete modes such that each discrete mode is repeated consecutively at least ν times, $i, j \geq \nu$, and the following holds: either $\bar{q} \neq q$ or $\bar{q} = q$ with $i \neq j$, say $i < j$. We will show that in both cases the outputs of (x_{t_0}, w) and $(\bar{x}_{t_0}, \bar{w})$ are different for some [all] inputs.

First assume that $\bar{q} \neq q$. Then using [strong] observability before the first switch we can show that (x_v, q^i) and $(\bar{x}_v, \bar{q}^j)$ generate different outputs for the some [all] inputs, where x_v and $\bar{x}_v$ denote the states reached from x_{t_0} and $\bar{x}_{t_0}$, respectively under the switching sequence v and constant zero input [the corresponding initial segment of the input $\mathcal{U}_T$]. Notice that by invertability of $A(i)$, $i \in Q$, we get that $(x_v, \bar{x}_v) \neq 0$, if $(x_{t_0}, \bar{x}_{t_0}) \neq 0$. In turn, this implies that (x_{t_0}, w) and $(\bar{x}_{t_0}, \bar{w})$ generate different outputs for some [all] inputs.

Now, assume that $q = \bar{q}$ and $i < j$. Let q_1 be the first letters of s, i.e., $s = q_1 s_1$ for some sequence s_1. Then for the pair (x_{t_0}, w) there is a switch from discrete mode q to q_1, while for $(\bar{x}_{t_0}, \bar{w})$ the system stays in q. Using [strong] detectability of the first switching time, we can then show that the outputs of (x_{t_0}, w) and $(\bar{x}_{t_0}, \bar{w})$ are different for some [all] inputs.