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Singular Perturbation Analysis of Discrete-Time Output Feedback Sliding Mode Control with Disturbance Attenuation

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Abstract— **We address the output feedback sliding mode control problem for a sampled data linear system with disturbances. By taking into account the disturbance compensation, a deadbeat high gain output feedback control strategy with additional dynamics is able to attenuate the disturbances. It is shown that the closed loop system exhibits both singularly perturbed and weakly coupled characteristics. A numerical example of an aircraft attitude output feedback control is provided to demonstrate the effectiveness of the proposed approach.**

I. INTRODUCTION

The problem of output feedback sliding mode control with disturbances has been extensively studied for years [1], [2], [3], [4], [5], [6]. EL-Khazali and DeCarlo [3] provided a general framework for constructing a static sliding surface in the output space. A switching type of continuous-time variable structure control law was used to achieve the sliding mode. Zak and Hui [2] considered a generally accepted case when the number of output variables is greater than or equal than that of the input variables. It was pointed out that the problem of choosing the desired poles of the sliding mode dynamics can be approached by using the classical "squared-down" techniques [7]. In order to attain a global attraction to the sliding surface, Heck *et. al.* established numerical methods that adjust the switching gain to compensate for the unknown state and disturbance variables [4]. Edwards and Spurgeon [5] proposed a procedure to construct a sliding surface based on the output information by taking advantage of the fact that the invariant zeros of a system appear in the dynamics of the sliding motion. The remaining eigenvalues of the sliding mode dynamics can be chosen appropriately in the framework of a static output feedback pole placement problem for a subsystem [2], [5].

In this paper, we consider the output feedback sliding mode control for sampled-data linear systems. It is well-known that the exact continual sliding motion cannot be achieved in the discrete-time case due to the sample/hold effect [8]. Specifically, the system trajectory only travels in a neighborhood of the sliding surface forming a boundary layer [9]. Several approaches were proposed to address the problem of discrete-time output feedback sliding mode control [10], [11], [12], [13]. Some of them are devoted to sliding mode control of sampled-data systems. In [12], Xu and Adibi employed a disturbance observer and a state observer with an integral sliding surface to address the output tracking problem for sampled-data systems. Under their proposed control approach, the stability of the closed-loop system is guaranteed and the effect of external disturbances is reduced. An application of their method was realized for the problem of discrete-time output feedback control for a piezomotor-driven linear motion stage [13].

The deadbeat control strategy proposed in [16] is able to decouple external disturbances with an $O(\epsilon)$ accuracy, where ϵ is the sampling period. In [9], an one-step delayed disturbance approximation approach has been shown to be effective in dealing with disturbances that exhibit certain continuity characteristics. We shall exploit the continuity attribute of the state variables and the disturbances by using a similar approach to deal with the similar estimation problem encountered above.

A dynamic output feedback discrete-time control approach that takes into account the disturbance compensation as in [9] is proposed. It is pointed out that with high gain feedback control, the system exhibits the two-time scale behavior [14], [15], [16], [17]. By using singular perturbation analysis, we show that the sliding surface will be remained in an $O(\epsilon^2)$ accuracy when disturbances affect the system. Since we do not employ observers, our proposed approach is simpler than the one in [12], while obtaining the same characteristics such as the accuracy of sliding motion and the ultimate bound of state variables.

The structure of the paper is organized as follows. Section II presents the formulation of the problem, followed by Section III where the sampled-data system is transformed into a convenient form. A procedure to construct output feedback control with stability and accuracy analysis is shown in Section IV. Section V presents a numerical example to illustrate the effectiveness of the proposed method. Some conclusions are given in the final section.

Throughout the paper, $\lambda(A)$ denotes the spectrum of matrix A, while I_m stands for an identity matrix of order m. A vector function $f(t,\epsilon) \in R^n$ is said to be $O(\epsilon)$ over an interval $[t_1, t_2]$ [18] if there exists positive constants k and ϵ^* such that

$$\|f(t,\epsilon)\| \leq k\epsilon \quad \forall \epsilon \in [0, \epsilon^*], \quad \forall t \in [t_1, t_2]$$

where $\|.\|$ is the Euclidean norm. Moreover, it is said to be

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$O(1)$ over $[t_1, t_2]$ if

$$\|f(t,\epsilon)\| \leq k, \quad \forall t \in [t_1, t_2].$$

II. PROBLEM FORMULATION

We consider a linear system described by

$$\begin{aligned} \dot{x}_0(t) &= A_0 x_0(t) + B_0 u(t) + D_0 f(t) \\ y(t) &= C_0 x_0(t) \end{aligned} \tag{1}$$

where $x_0(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $y(t) \in R^p$ is the output, $f(t) \in R^r$ is the unknown but bounded exogenous disturbance, with $m \leq p < n$. The system matrices A_0, B_0, C_0, D_0 are constant of appropriate dimensions with magnitudes $O(1)$. The following assumptions are made:

1) Matrices B_0 and C_0 have full rank.
2) (A_0, B_0, C_0) is controllable and observable [3].
3) The invariant zeros of system (1) are stable.

In addition, D_0 satisfies the matching condition [19]

$$\text{rank}([B_0|D_0]) = \text{rank}(B_0) \tag{2}$$

In other words, there exists a matrix K such that

$$D_0 = B_0 K. \tag{3}$$

The sliding surface under consideration is

$$s(t) = Hy(t) = HC_0 x_0(t) = 0, \tag{4}$$

where H is a full rank $m \times p$ matrix, designed to render stable sliding dynamics. It is shown that the eigenvalues of the sliding mode dynamics include the invariant zeros of the system (1) [6]. One can place the remaining eigenvalues of the zero dynamics of the sliding surface (4) if the Davison-Kimura condition [20] is satisfied [6]. In the case the Davison-Kimura condition is not satisfied, a dynamic compensator is constructed to produce a tractable structure for the sliding surface design [6]. Refer to [3], [2], and [6] for design of H. Note that HC_0B_0 is nonsingular. Our objective is to construct a discrete-time sliding mode controller given an output sliding surface $s(t) = 0$.

III. DISCRETE-TIME REGULAR FORM

In this section, we will use several similarity transformations to facilitate system design and analysis. Since $\text{rank}(B_0) = m$, there exists a coordinate transformation T_0 such that

$$B = T_0 B_0 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}.$$

where B_2 is a nonsingular square matrix of dimension m. The new variables are defined as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T_0 x_0.$$

The similarity transformation T_0 brings the original system (1) into the regular form [21]

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2 u(t) + D_2 f(t), \end{aligned} \tag{5}$$

where $D_2 = B_2 K$. The sliding surface (4) is now described as

$$\begin{aligned} s(t) &= HC_0 x_0(t) = HC_0 T_0^{-1} x(t) \\ &= Cx(t) = C_1 x_1(t) + C_2 x_2(t) = 0, \end{aligned} \tag{6}$$

where

$$HC_0 T_0^{-1} = C.$$

The zero dynamics of the sliding mode is represented by the eigenvalues of matrix $A_c = A_{11} - A_{12} C_2^{-1} C_1$. Note that H has been chosen in (4) to render a sliding surface coefficient matric C such that C_2 is invertible and A_c is stable [21].

Sampling the continuous-time system (5) with the sampling period ϵ results in the following discrete-time model

$$x(k+1) = \Phi x(k) + \Gamma u(k) + d(k) \tag{7}$$

where

$$\begin{aligned} \Phi &= e^{A\epsilon}, \quad \Gamma = \int_0^\epsilon e^{A\tau} d\tau B, \\ d(k) &= \int_0^\epsilon e^{A\tau} BKf((k+1)\epsilon - \tau) d\tau. \end{aligned} \tag{8}$$

The system matrices, Φ and Γ, of the sampled-data system (7) can be reformulated by taking the Taylor series expansion as

$$\Phi = e^{A\epsilon} = I + \epsilon A + \frac{\epsilon^2}{2!}A^2 + \cdots = I + \epsilon(A + \epsilon\Delta A) = O(1) \tag{9}$$

and

$$\Gamma = \int_0^\epsilon e^{A\tau} d\tau B = \epsilon(B + \epsilon\Delta B) = O(\epsilon), \tag{10}$$

where

$$\Delta A = \frac{1}{2!}A^2 + \frac{\epsilon}{3!}A^3 + \cdots = O(1) \tag{11}$$

and

$$\Delta B = \frac{1}{2!}AB + \frac{\epsilon}{3!}A^2 B + \cdots = \begin{bmatrix} \Delta B_1 \\ \Delta B_2 \end{bmatrix} = O(1), \tag{12}$$

where the dimensions of ΔB_1 and ΔB_2 are $(n-m) \times m$ and $m \times m$, respectively. Furthermore, since $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, Γ can be written as

$$\Gamma = \begin{bmatrix} \epsilon^2 \Delta B_1 \\ \epsilon \bar{B}_2 \end{bmatrix}, \tag{13}$$

where

$$\bar{B}_2 = B_2 + \epsilon \Delta B_2. \tag{14}$$

Due to the sampling process, the disturbance in the sampled-data system (7) exhibits unmatched components, as demonstrated by the following lemma [22].

Lemma 1: If the disturbance $f(t)$ in (1) and its first derivative are bounded, then

$$\begin{aligned} d(k) &= \int_0^\epsilon e^{A\tau} BKf((k+1)\epsilon - \tau) d\tau \\ &= \Gamma K f(k) + \frac{\epsilon}{2}\Gamma K v(k) + \epsilon^3 \Delta d \\ &d(k) - d(k-1) = O(\epsilon^2) \\ &d(k) - 2d(k-1) + d(k-2) = O(\epsilon^3) \end{aligned} \tag{15}$$

where $v(t) = df(t)/dt$.

To facilitate discrete-time sliding mode control design, we put (7) into a discrete-time regular form, which is similar to (5). To this end, we employ the following transformation for (7)

$$T_1 = \begin{bmatrix} I_{n-m} & -\epsilon \Delta B_1 \bar{B}_2^{-1} \\ 0 & I_m \end{bmatrix} \tag{16}$$

with new variables

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{17}$$

The inverse transformation of T_1 is

$$T_1^{-1} = \begin{bmatrix} I_{n-m} & \epsilon \Delta B_1 \bar{B}_2^{-1} \\ 0 & I_m \end{bmatrix}. \tag{18}$$

The $m \times m$ matrix $\bar{B}_2$ is an ϵ perturbed version of the nonsingular matrix B_2 as seen in (14), hence it is nonsingular so that the transformation matrices T_1 and T_1^{-1} both exist.

Under the transformation T_1, the discrete-time system in its regular form is given by

$$z(k+1) = \bar{\Phi} z(k) + \bar{\Gamma} u(k) + d_1(k), \tag{19}$$

in which the new system matrix $\bar{\Phi}$ still keeps it original form as an ϵ-perturbed identity matrix as in (9)

$$\bar{\Phi} = T_1 \Phi T_1^{-1} = T_1(I + \epsilon A + \epsilon^2 \Delta A) T_1^{-1} = I + \epsilon \bar{A} \tag{20}$$

$$\bar{A} = T_1(A + \epsilon \Delta A) T_1^{-1} = \begin{bmatrix} A_{11} + \epsilon \Delta \bar{A}_{11} & A_{12} + \epsilon \Delta \bar{A}_{12} \\ A_{21} + \epsilon \Delta A_{21} & A_{22} + \Delta \bar{A}_{22} \end{bmatrix} \tag{21}$$

while the first $n-m$ rows of the new control coefficient matrix $\bar{\Gamma}$ are nullified

$$\bar{\Gamma} = T_1 \Gamma = \begin{bmatrix} 0 \\ \epsilon \bar{B}_2 \end{bmatrix}. \tag{22}$$

Moreover, all the matched portion in the original disturbance vector $d(k)$ is now transferred to the bottom m row of the new disturbance vector $d_1(k)$, leaving only the $O(\epsilon^3)$ mismatched portion in the first $n-m$ rows

$$\begin{aligned} d_1(k) &= T_1 d(k) = T_1(\Gamma K f(k) + \frac{1}{2}\Gamma K v(k) + \epsilon^3 \Delta d) \\ &= \begin{bmatrix} 0 \\ \epsilon \bar{B}_2 \end{bmatrix} (K f(k) + \frac{\epsilon}{2} K v(k)) + \epsilon^3 T_1 \Delta d = \begin{bmatrix} d_{11}(k) \\ d_{12}(k) \end{bmatrix} \end{aligned} \tag{23}$$

where

$$d_{11}(k) = O(\epsilon^3) \tag{24}$$

$$d_{12}(k) = \epsilon \bar{B}_2 K(f(k) + \frac{\epsilon}{2} v(k)) + O(\epsilon^3) = O(\epsilon). \tag{25}$$

The sliding surface vector in the new coordinates is now described as

$$s(k) = C T_1^{-1} z(k) = C_1 z_1(k) + \bar{C}_2 z_2(k) \tag{26}$$

where the $m \times m$ matrix $\bar{C}_2$ is an ϵ perturbed version of the original nonsingular matrix in (6):

$$\bar{C}_2 = C_2 - \epsilon C_1 \Delta B_1 \bar{B}_2^{-1}. \tag{27}$$

Therefore, $\bar{C}_2$ is nonsingular if ϵ is small.

IV. Main results

A. Output Feedback Control Design

In this section, we develop a control strategy that forces the state to reach the sliding surface (4) in a finite time. Applying the transformation

$$T_2 = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & \bar{C}_2 \end{bmatrix} \tag{28}$$

with state variables

$$\begin{bmatrix} z_1(k) \\ s(k) \end{bmatrix} = T_2 \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{29}$$

recasts the sampled-data system (19) into

$$\begin{aligned} z_1(k+1) &= A_s z_1(k) + \epsilon \bar{A}_{12} \bar{C}_2^{-1} s(k) + d_{11}(k) \\ s(k+1) &= \epsilon \Omega_1 z_1(k) + (I_m + \epsilon \Omega_2) s(k) + \\ &\quad \epsilon \bar{C}_2 \bar{B}_2 u(k) + C_1 d_{11}(k) + \bar{C}_2 d_{12}(k), \end{aligned} \tag{30}$$

where

$$A_s = I_{n-m} + \epsilon(\bar{A}_{11} - \bar{A}_{12} \bar{C}_2^{-1} C_1) \tag{31}$$

$$\Omega_1 = (\bar{C}_1 \bar{A}_{11} + \bar{C}_2 \bar{A}_{21}) - (\bar{C}_1 \bar{A}_{12} + \bar{C}_2 \bar{A}_{22}) \bar{C}_2^{-1} \bar{C}_1 \tag{32}$$

$$\Omega_2 = (\bar{C}_1 \bar{A}_{12} + \bar{C}_2 \bar{A}_{22}) \bar{C}_2^{-1}. \tag{33}$$

In view of (27), the inverse of $\bar{C}_2$ can be written as

$$\bar{C}_2^{-1} = C_2^{-1} + \epsilon \Delta C_{2i}. \tag{34}$$

where ΔC_{2i} is a constant matrix. From (31) and (34), we have

$$A_s = I_{n-m} + \epsilon(A_c + \epsilon \Delta A_c). \tag{35}$$

where

$$A_c = A_{11} - A_{12} C_2^{-1} C_1 \tag{36}$$

and

$$\Delta A_c = \Delta \bar{A}_{11} - (\Delta \bar{A}_{12} \bar{C}_2^{-1} + A_{12} \Delta C_{2i}) C_1. \tag{37}$$

Rewrite the dynamics of $s(k)$ in (30) as

$$s(k+1) = (I_m + \epsilon \Omega_2) s(k) + \epsilon \bar{C}_2 \bar{B}_2 u(k) + g(k), \tag{38}$$

where $g(k)$ contains the state variables in $z_1(k)$ and the portion of disturbances lying in the control range space

$$g(k) = \epsilon \Omega_1 z_1(k) + d_2(k) \tag{39}$$

with

$$d_2(k) = C_1 d_{11}(k) + \bar{C}_2 d_{12}(k) = O(\epsilon). \tag{40}$$

By solving $s(k+1) = 0$, we obtain the discrete-time equivalent control law [23] as follows

$$u^{eq}(k) = -\frac{1}{\epsilon}(\bar{C}_2 \bar{B}_2)^{-1}((I_m + \epsilon \Omega_2) s(k) + g(k)). \tag{41}$$

Since $g(k)$ contains the unmeasurable state variables in $z_1(k)$ and disturbances $d_2(k)$, it is unknown to the controller at time step k, hence the equivalent control law (41) is not realizable.

Nevertheless, equation (38) reveals that we are able to approximate $g(k)$ numerically by $g(k-1)$

$$g(k-1) = s(k) - (I_m + \epsilon \Omega_2) s(k-1) - \epsilon \bar{C}_2 \bar{B}_2 u(k-1). \tag{42}$$

Use of $g(k-1)$ in place of $g(k)$ in $u^{eq}(k)$ leads to a realizable control law with additional dynamics in $u(k)$

$$
\begin{aligned}
u(k) &= -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((I_m+\epsilon\Omega_2)s(k)+g(k-1)) \\
&= -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((2I_m+\epsilon\Omega_2)s(k)- \\
&\quad (I_m+\epsilon\Omega_2)s(k-1))+u(k-1).
\end{aligned} \tag{43}
$$

A similar technique was employed by Su *et. al.* [9] for state feedback discrete-time sliding mode control, where only the external disturbances are to be approximated. In this paper, however, the unknown term $g(k)$ under consideration includes both the external disturbances and the unmeasured state variables.

B. Stability Analysis

The discrete-time equivalent control law (41) is also known as a deadbeat control law that brings about a gigantic control action leading to a fast-time system behavior in the state vector $s(k)$ of (30). It was also pointed out in [16], [17], [24] that such a deadbeat high gain control will lead to a singular perturbation behavior for sampled-data systems. In this section, we will employ the singular perturbation methodology to analyze the stability of the closed-loop system driven by the proposed sliding mode control law (43).

The involvement of the past disturbance $g(k-1)$ in the control law (43) produces additional m order dynamics for the closed-loop system. Following a similar way as in [9], we study an augmented dynamic system of $z_1(k)$, $s(k)$, and $u(k)$. From (42) and (43), we have

$$
\begin{aligned}
&u(k+1) = -(\bar{C}_2\bar{B}_2)^{-1}(2I_m+\epsilon\Omega_2)\Omega_1 z_1(k) \\
&-\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}(I_m+\epsilon\Omega_2)^2 s(k)-\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}(I_m+\epsilon\Omega_2)\epsilon \\
&\bar{C}_2\bar{B}_0 u(k)-(\bar{C}_2\bar{B}_2)^{-1}(2I_m+\epsilon\Omega_2)d_2(k).
\end{aligned} \tag{44}
$$

Introduce a new variable

$$
\gamma(k) = \epsilon\bar{C}_2\bar{B}_2 u(k). \tag{45}
$$

Thus, the new augmented system of $z_1(k), \xi(k)$, and $v(k)$ is described by

$$
\begin{bmatrix} z_1(k+1) \\ s(k+1) \\ \gamma(k+1) \end{bmatrix} = A_{aug}\begin{bmatrix} z_1(k) \\ s(k) \\ \gamma(k) \end{bmatrix} + \begin{bmatrix} d_{11}(k) \\ d_2(k) \\ -(2I_m+\epsilon\Omega_2)d_2(k) \end{bmatrix}, \tag{46}
$$

where

$$
A_{aug} = \begin{bmatrix} A_s & \epsilon\bar{A}_{12}\bar{C}_2^{-1} & 0 \\ \epsilon\Omega_1 & (I_m+\epsilon\Omega_2) & I_m \\ -\epsilon(2I_m+\epsilon\Omega_2)\Omega_1 & -(I_m+\epsilon\Omega_2)^2 & -(I_m+\epsilon\Omega_2) \end{bmatrix}. \tag{47}
$$

The above augmented system is both a weakly coupled system and a singularly perturbed discrete-time system [14], [15], [25], [26] where the slow dynamics is represented by $z_1(k)$, and fast dynamics is represented by $s(k)$ and $\gamma(k)$. The slow and fast components affect each other by a regular $O(\epsilon)$ perturbation (or by weak couplings). To see this, partition the augmented matrix A_{aug} as

$$
A_{aug} = \begin{bmatrix} A_s & O(\epsilon) \\ O(\epsilon) & A_f \end{bmatrix}
$$

and

$$
A_f = \begin{bmatrix} (I_m+\epsilon\Omega_2) & I_m \\ -(I_m+\epsilon\Omega_2)^2 & -(I_m+\epsilon\Omega_2) \end{bmatrix}.
$$

It is then natural to expect (46) to be separated into distinct slow and fast subsystems by using a decoupling transformation [14] provided that $I_{2m}-A_f$ is nonsingular. The following lemma shows that the existence of a decoupling transformation is satisfied [14].

Lemma 2: The $2m\times 2m$ matrix A_f possesses $2m$ zero eigenvalues.

Proof: The eigenvalues of A_f are solutions of the following equation

$$
\begin{aligned}
0 &= \det(\lambda I_{2m} - \begin{bmatrix} (I_m+\epsilon\Omega_2) & I_m \\ -(I_m+\epsilon\Omega_2)^2 & -(I_m+\epsilon\Omega_2) \end{bmatrix}) \\
&= \det(\begin{bmatrix} \lambda I_m-(I_m+\epsilon\Omega_2) & -I_m \\ (I_m+\epsilon\Omega_2)^2 & \lambda I_m+(I_m+\epsilon\Omega_2) \end{bmatrix}).
\end{aligned}
$$

Premultiplying the first m rows by $(I_m+\epsilon\Omega_2)$ and adding to the last m rows results in

$$
\begin{aligned}
0 &= \det(\begin{bmatrix} \lambda I_m-(I_m+\epsilon\Omega_2) & -I_m \\ \lambda(I_m+\epsilon\Omega_2) & \lambda I_m \end{bmatrix}) \\
&= \lambda^m \det(\begin{bmatrix} \lambda I_m-(I_m+\epsilon\Omega_2) & -I_m \\ -(I_m+\epsilon\Omega_2) & I_m \end{bmatrix}).
\end{aligned}
$$

Adding the last m rows to the first m rows yields

$$
0 = \lambda^m \det(\begin{bmatrix} \lambda I_m & 0 \\ (I_m+\epsilon\Omega_2) & I_m \end{bmatrix}) = \lambda^{2m} \tag{48}
$$

∎

The stability of system (46) is further investigated by using a decoupling transformation [14], which is stated in the following lemma.

Lemma 3: There exists a transformation P with new variables

$$
\begin{bmatrix} w(k) \\ \eta(k) \end{bmatrix} = P\begin{bmatrix} z_1(k) \\ s(k) \\ \gamma(k) \end{bmatrix} \tag{49}
$$

where $w(k)\in R^{n-m}$ and $\eta(k)\in R^{2m}$, such that system (46) is decomposed into reduced-order systems

$$
\begin{bmatrix} w(k+1) \\ \eta(k+1) \end{bmatrix} = \begin{bmatrix} A_s+O(\epsilon^2) & 0 \\ 0 & A_f+O(\epsilon^2) \end{bmatrix}\begin{bmatrix} w(k) \\ \eta(k) \end{bmatrix} + d_3(k) \tag{50}
$$

where

$$
d_3(k) = \begin{bmatrix} O(\epsilon^2) \\ d_2(k)+O(\epsilon^2) \\ -(2I_m+\epsilon\Omega_2)d_2(k)+O(\epsilon^2) \end{bmatrix}. \tag{51}
$$

Proof: Refer to [14]. ∎

Equation (50) implies that the stability of the closed-loop system under the control law (43) depends on the eigenvalues of A_s and A_f. It is pointed in Lemma 2 that the eigenvalues of A_f lie in the origin of the unit circle. The positions of the eigenvalues of A_s are justified in the following lemma.

Lemma 4: There exists a small enough ϵ such that the eigenvalues of A_s lie in the unit circle.

Proof: From (35), we have

$$\lambda(A_s) = 1 + \epsilon\lambda(A_c + \epsilon\Delta A_c). \tag{52}$$

Since the eigenvalues of A_c have negative real parts, it is strait forward to show that the eigenvalues of A_s lie in the unit circle provided there exists a sufficiently small ϵ. ∎

The decoupled system (50) shows that the eigenvalues of the closed-loop (augmented) system can be dissected into two groups. The eigenvalues of the first group lie in an $O(\epsilon^2)$ neighborhood of the eigenvalues of A_s. Note that the eigenvalues of A_s which represent transmission zeros of the sliding surface lie in the unit circle due to Lemma 4. With sufficiently small ϵ, the eigenvalues of the first group represent slow modes that are asymptotically stable. The eigenvalues of the second group lie in an $O(\epsilon^2)$ neighborhood of the zero eigenvalues of A_f. They represent fast modes with sufficiently small ϵ; thus the fast dynamics is asymptotically stable. Therefore, the closed-loop system (46) is asymptotically stable with sufficiently small ϵ.

We summarize the above discussion in the following theorem.

Theorem 1: The discrete-time output feedback sliding mode control law (43) renders the sampled-data system (7) asymptotic stability.

C. Accuracy Analysis

In this subsection, we deal with the accuracy issues of the sliding mode when the closed-loop system is under the influence of external disturbances. From (38) and (43), we have

$$s(k+1) = g(k) - g(k-1). \tag{53}$$

In view of (39) and (40), it follows that

$$s(k+1) = \epsilon\Omega_1(z_1(k) - z_1(k-1)) + d_2(k) - d_2(k-1) \tag{54}$$

From (30) and (31), we have

$$\begin{aligned} z_1(k) - z_1(k-1) = \epsilon(\bar{A}_{11} - \bar{A}_{12}\bar{C}_2^{-1}C_1)z_1(k-1) + \\ \epsilon\bar{A}_{12}\bar{C}_2^{-1}s(k) + d_{11}(k) = O(\epsilon). \end{aligned} \tag{55}$$

If the first derivative of the disturbance is bounded, then we have

$$\begin{aligned} d_{12}(k) - d_{12}(k-1) = \epsilon\bar{B}_2K\left(\int_{(k-1)\epsilon}^{k\epsilon} v(\tau)d\tau\right) \\ + \frac{\epsilon^2}{2}\bar{B}_2K(v(k) - v(k-1)) + O(\epsilon^3) = O(\epsilon^2). \end{aligned} \tag{56}$$

This implies

$$d_2(k) - d_2(k-1) = O(\epsilon^2). \tag{57}$$

From (54), (55) and (57), we obtain

$$s(k+1) = O(\epsilon^2). \tag{58}$$

Employing the same arguments and techniques as in [9], we conclude that the system state $x_0(t)$ will stay in an $O(\epsilon^2)$ boundary layer of the sliding surface or $s(t) = O(\epsilon^2)$. Our findings are summarized in the following theorem.

Theorem 2: Consider the linear dynamic system

$$\begin{aligned} \dot{x}_0(t) &= A_0x_0(t) + B_0u(t) + D_0f(t) \\ y(t) &= C_0x_0(t) \end{aligned}$$

with the associated sliding surface

$$s(t) = Hy(t) = HC_0x_0(t) = 0.$$

If the exogenous disturbance $f(t)$ and its derivative are bounded, then the sampled-data output feedback control

$$\begin{aligned} u(k) = -\frac{1}{\epsilon}(\bar{C}_2\bar{B}_2)^{-1}((2I_m + \epsilon\Omega_2)Hy(k) \\ - (I_m + \epsilon\Omega_2)Hy(k-1)) + u(k-1) \end{aligned}$$

produces sliding motion in an $O(\epsilon^2)$ boundary layer of the sliding surface.

V. NUMERICAL EXAMPLE

We use the L-1011 aircraft model as in [4] for sampled-data output feedback sliding mode control design where parameters are given by

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 0.0386 & -0.996 & -0.0003 & -2.117 & 0 & 0.02 & 0 \\ 0 & 0.5 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25 \end{bmatrix} \tag{59}$$

$$B_0 = D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix}^T \tag{60}$$

$$C_0 = \begin{bmatrix} 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{61}$$

The coefficient matrix of the sliding surface $s(t) = Hy(t) = 0$ was chosen as

$$H = \begin{bmatrix} -0.0067 & 0.0167 & 0.0033 & 0 \\ 0.0167 & -0.0333 & 0 & 0.0333 \end{bmatrix}. \tag{62}$$

Consider the external disturbance vector

$$f(t) = \begin{bmatrix} 1 + \sin(2t) \\ 1.5\sin(4t) \end{bmatrix}, \tag{63}$$

which satisfies the disturbance conditions in Lemma 1. The sampling period is $\epsilon = 0.01$ second. The initial condition is $x(0) = [1 \quad -2 \quad 2 \quad -4 \quad 3 \quad 4 \quad -1]^T$.

Employing the proposed control law (43) renders the evolution of the seven state variables, the two sliding variables as shown in Fig. 1–Fig. 2. It is observed in Fig. 1 that the state variables converge to the origin asymptotically. Fig. 2 depicts the particulars of the sliding motion, in which an accuracy of $O(\epsilon^2)$ can be seen. These numerical results agree with the accuracy analysis in the previous section.

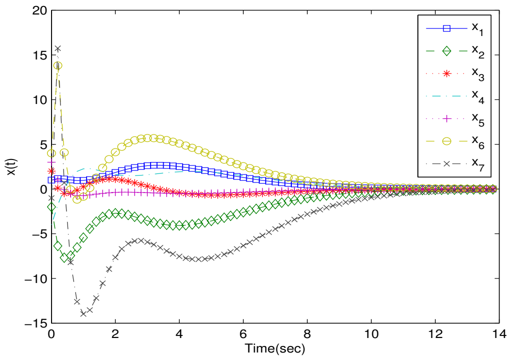

Fig. 1. The evolution of the state variables under the influence of external disturbances (63)

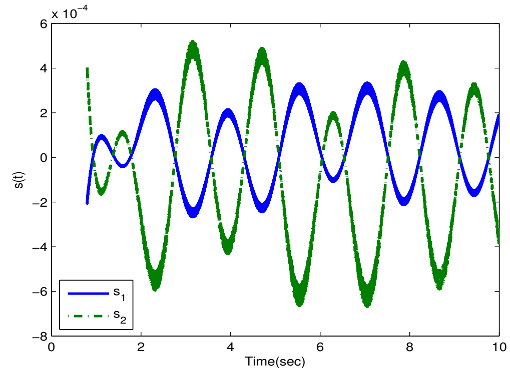

Fig. 2. The accuracy of sliding motion is $O(\epsilon^2)$

VI. CONCLUSION

We have investigated the output feedback sliding mode control problem of sampled-data systems with external disturbances. By some suitable linear transformations and changes of variables, the closed-loop system under high gain control law (43) is shaped into a two-time scale representative. This paves way to the framework of discrete-time singular perturbation analysis, by which the eigenvalues of the closed-loop system are clustered into two groups: the slow and fast eigenvalues. It is also pointed out that the slow eigenvalues are located in an $O(\epsilon^2)$ neighborhood of transmission zeros of the sliding surface, while the fast eigenvalues stay in an $O(\epsilon^2)$ vicinity of the origin. Therefore, the stability of the closed-loop system is guaranteed as no external disturbances occur. The idea of approximating disturbances by the past information equips the high gain control law (43) with an ability to maintain the system state in an $O(\epsilon^2)$ boundary layer of the sliding surface. In other words, the closed-loop exhibits good robustness against exogenous disturbances. As an illustration, a numerical example has been provided to show the efficiency of the proposed method.

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