2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Capabilities of Extended State Observer for Estimating Uncertainties

Xiaoxia Yang and Yi Huang

Abstract— **The capabilities of extended state observer (ESO) for estimating uncertainties is discussed in this paper. The scope of the disturbances that can be observed by LESO with bounded observing errors is given. The observing errors for several typical disturbances–constant disturbance, sine disturbance, ramp disturbance, and square wave disturbance are further analyzed. It is demonstrated that ESO can deal with a large class of disturbances. Finally, the results are tested by simulations.**

I. INTRODUCTION

Control design for the systems with uncertainties is a longstanding fundamental issue in automatic control. The uncertainties, which are universal in practice, usually stem from two sources: internal (parameter or structure) uncertainty and external (disturbance) uncertainty. Lots of control methods have been proposed centering on this issue, such as the widely used PID control [1], adaptive control [2], robust control [3] and disturbance-accommodation control [4] etc. What's more, many disturbance estimating techniques appeared, such as unknown input observer (UIO) [5], perturbation observer (POB) [6], the disturbance observer (DOB) [7], etc. Owing to the less dependence on model information, strong capabilities for disturbance rejection and simple control structure, the active disturbance rejection control (ADRC)[8][9][10][11] attracted many researchers' attention. The key in ADRC is the use of extended state observer (ESO) for on-line estimating the total uncertainties, which lumps the internal nonlinear and uncertain dynamics and the external disturbance. Hence the uncertainties of the system can be compensated actively.

The idea of ESO can be demonstrated in the following single-input and single-output system:

$$\begin{cases} x^{(n)} = f(x^{(n-1)}(t), x^{(n-2)}(t), \cdots, x(t), \omega(t), t) + bu(t), \\ y = x(t) \end{cases} \tag{1}$$

where n is the order of the plant, y is the output, u is the input, b is a constant, $\omega(t)$ is the external disturbance, $f(\cdot)$ is an unknown function which can be viewed as the total uncertainties or disturbances of the system, both internal and external. Introduce $h = df/dt$. If the function f is non-smooth, h denotes the generalized derivative of $f(\cdot)$. Treat the uncertainty f as an extended state of the system (1), the equation (1) can be written in the state form as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \vdots \\ \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = x_{n+1}(t) + bu, \\ \dot{x}_{n+1}(t) = h(\cdot), \end{cases} \tag{2}$$

where $X = [x_1, \cdots, x_{n+1}]^\tau \in \mathbb{R}^{n+1}$ represents the state of the system. The ESO for estimating both the states and the extended state for the uncertain system (1) can be given as follows[9][12]

$$\begin{cases} \dot{z}_1 = z_2 - \beta_1 fal(e_1, \alpha_1, \delta), \\ \vdots \\ \dot{z}_{n-1} = z_n - \beta_{n-1} fal(e_1, \alpha_{n-1}, \delta), \\ \dot{z}_n = z_{n+1} - \beta_n fal(e_1, \alpha_n, \delta) + bu, \\ \dot{z}_{n+1} = -\beta_{n+1} fal(e_1, \alpha_{n+1}, \delta), \end{cases} \tag{3}$$

where $Z = [z_1, \cdots, z_{n+1}]^\tau \in \mathbb{R}^{n+1}, e_1 = z_1 - x_1 and \ \ \beta_i(i \in \underline{n+1})$ are the state of ESO, the observing error and the observer gains, respectively,

$$fal(e, \alpha, \delta) = \begin{cases} |e|^\alpha sgn(e), |e| > \delta \\ e/\delta^{1-\alpha}, otherwise \end{cases} \quad 0 \le \alpha \le 1, \delta > 0.$$

ESO (3) is designed to have the property:

$$z_i(t) \to x_i(t)(i \in \underline{n+1}).$$

It should be noted that (3) takes the form of the classical Luenberger Observer, when $\alpha_i = 1(i \in \underline{n+1})$.On the other hand, (3) is consistent with the sliding mode observer, when $\alpha_i = 0(i \in \underline{n+1})$.

The idea of ESO has been used to deal with various kinds of engineering problems, such as flight control, web tension control, chemical process control etc [13][14][15]. At the same time, many researchers are of interest in the stability analysis of ADRC and ESO [13][16][17][18].

Reference [16][17]discussed the capabilities of the linear ESO with$\alpha_i = 1(i \in \underline{n+1})$, which is given as:

$$\begin{cases} \dot{z}_1 = z_2 - \beta_1 e_1, \\ \vdots \\ \dot{z}_{n-1} = z_n - \beta_{n-1} e_1, \\ \dot{z}_n = z_{n+1} - \beta_n e_1, +bu, \\ \dot{z}_{n+1} = -\beta_{n+1} e_1, \end{cases} \tag{4}$$

The parameters are chosen in a special way as $s^{n+1} + \beta_1 s^n + \cdots + \beta_{n+1} = (s+\omega_0)^{n+1}$, where ω_0 denotes the bandwidth of the LESO (4). It was proved that if f is differentiable with respect to t and $h = \dot{f}$ is bounded, then the LESO (4) can estimate $f(t)$ with bounded error. In [18], the tracking errors

Xiaoxia Yang and Yi Huang are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences. Email:1984-yxx@163.com

of the LESO (4) were studied in the case $n=2$, under the assumptions that $f(t)$ is bounded, piecewise continuous and in each continuous subinterval $\dot{f}(t)$ is bounded. In conclusion, the existing results are all based on the assumption that $h=\dot{f}$ is bounded.

In this paper, the capabilities of LESO (4) will be analyzed for a wider scope of uncertain function $f(t)$. The main result is that it can estimate $f(t)$ with bounded error if either $h=df/dt$ is bounded or f is bounded. This enriches the existing theoretical analysis for ESO. For example, according to the result, the square wave disturbance, which usually exists in practice, can be dealt with by ESO with bounded errors since f is bounded while its derivative h is unbounded. Moreover, the observing errors for a set of uncertainties, which often appear in practice, are analyzed in this paper.

The paper is organized as follows. In Section II, the scope of the uncertainties that can be observed by LESO (4) with bounded observing errors is discussed. Then the observing errors for several typical forms of uncertainties are further analyzed in Section III. Simulation results are presented in section IV and the concluding remarks are given in section V.

II. MAIN RESULTS

Introducing the observing errors:

$$E=[e_1,\cdots,e_{n+1}]^T=Z-X$$

where $e_{n+1}=z_{n+1}-x_{n+1}=z_{n+1}-f$ is the observing error of the uncertainty, the error dynamics can be got from (2) and (4) :

$$\begin{cases}\dot{e}_1=e_2-\beta_1 e_1,\\ \vdots\\ \dot{e}_{n-1}=e_n-\beta_{n-1}e_1,\\ \dot{e}_n=e_{n+1}-\beta_n e_1,\\ \dot{e}_{n+1}=-\beta_{n+1}e_1-h,\end{cases}\tag{5}$$

which can be rewritten as

$$\dot{E}=AE+Bh\tag{6}$$

where

$$A=\begin{bmatrix}-\beta_1 & 1 & 0 & \cdots & 0\\ -\beta_2 & 0 & 1 & \cdots & 0\\ & \vdots & \vdots & & \\ -\beta_n & 0 & \cdots & 0 & 1\\ -\beta_{n+1} & 0 & 0 & \cdots & 0\end{bmatrix},B=\begin{bmatrix}0\\0\\ \vdots\\ -1\end{bmatrix}.$$

Solving (6), one can get

$$E(t)=e^{At}E(0)+e^{At}\int_0^t e^{-A\tau}Bh(\tau)d\tau\tag{7}$$

Next, the conditions for $f(t)$ or $h(t)$, under which $E(t)$ is bounded will be discussed.

Since the parameters $\beta_i(i\in\underline{n+1})$ can be chosen such that A is Hurwitz, the following discussions are all based on the fact that A is Hurwitz with real negative eigenvalues. Hence, the first part of the right side of (7) will converge to zero as long as $E(0)$ is bounded, that is

$$\lim_{t\to\infty}e^{At}E(0)=0\tag{8}$$

Then the analysis will be concentrated on the second part of (7), that is

$$G(t)=\int_0^t e^{A(t-\tau)}Bh(\tau)d\tau\tag{9}$$

Provided that A has r different real eigenvalues $\lambda_i<0(i\in\underline{r})$, where n_i is the multiplicity of λ_i,

$$\sum_{i=1}^{r}n_i=n+1,$$

then $e^{A(t-\tau)}$ can be expressed as:

$$e^{A(t-\tau)}=\sum_{i=1}^{r}e^{\lambda_i(t-\tau)}\sum_{j=0}^{n_i-1}S_{ij}\frac{(t-\tau)^j}{j!}\tag{10}$$

where $S_{ij}\in\mathbb{R}^{(n+1)\times(n+1)}$ is the constant matrix determined by A. It's obviously that A has $n+1$ different eigenvalues if $r=n+1$, and A has $n+1$ identical eigenvalues if $r=1$, which is just the situation as [16][17][18] discussed. Then G(t) can be unified in the following form

$$\begin{aligned}G(t) &= \int_0^t e^{A(t-\tau)}Bh(\tau)d\tau\\ &= \int_0^t\sum_{i=1}^{r}\sum_{j=0}^{n_i-1}S_{ij}B\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}h(\tau)d\tau\end{aligned}\tag{11}$$

Before discussing the property of $G(t)$, a lemma is given as follows.

Lemma 1. For $\forall\lambda<0$, the following formula holds

$$\lim_{t\to\infty}\int_0^t(t-\tau)^k e^{\lambda(t-\tau)}d\tau=\frac{k!}{(-\lambda)^{k+1}}\tag{12}$$

Proof Let $t-\tau=s$, then

$$\int_0^t(t-\tau)^k e^{\lambda(t-\tau)}d\tau=\int_t^0 s^k e^{\lambda s}d(t-s)=\int_0^t s^k e^{\lambda s}ds$$

Next the mathematical induction will be utilized to prove

$$\lim_{t\to\infty}\int_0^t s^k e^{\lambda s}ds=\frac{k!}{(-\lambda)^{k+1}}.\tag{13}$$

When $k=0$,

$$\lim_{t\to\infty}\int_0^t e^{\lambda s}ds=\lim_{t\to\infty}\frac{1}{\lambda}\left(e^{\lambda t}-1\right),$$

Since $\lambda<0$, it follows that $\lim\limits_{t\to\infty}e^{\lambda t}=0$. Hence,

$$\lim_{t\to\infty}\int_0^t e^{\lambda s}ds=-\frac{1}{\lambda},$$

(13) holds.

Assume that (13) holds when $k=m$, that is

$$\lim_{t\to\infty}\int_0^t s^m e^{\lambda s}ds=\frac{m!}{(-\lambda)^{m+1}},$$

Then when $k=m+1$,

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t s^{m+1}e^{\lambda s}ds\\
=\ &\lim_{t\to\infty}\frac{1}{\lambda}\int_0^t s^{m+1}de^{\lambda s}\\
=\ &\lim_{t\to\infty}\frac{1}{\lambda}\left(t^{m+1}e^{\lambda t}-(m+1)\int_0^t s^m e^{\lambda s}ds\right)
\end{aligned}$$

Since $\lambda<0$, it follows that

$$\lim_{t\to\infty}t^{m+1}e^{\lambda t}=0$$

So

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t s^{m+1}e^{\lambda s}ds &= \lim_{t\to\infty}-\frac{1}{\lambda}(m+1)\int_0^t s^m e^{\lambda s}ds\\
&= -\frac{1}{\lambda}(m+1)\frac{m!}{(-\lambda)^{m+1}}\\
&= \frac{(m+1)!}{(-\lambda)^{m+2}}
\end{aligned}$$

Therefore (13) holds when $k=m+1$. Q.E.D.

The following theorem declares the scope of disturbance f that can be estimated by the LESO (4) with bounded error E(t).

Theorem 1. $\lim\limits_{t\to\infty}E(t)$ is bounded if at least one of the following two conditions is satisfied:

a) $|h|\le M_1$ for a constant M_1 and all t.

b) $|f|\le M_2$ for a constant M_2 and all t.

Proof If property a) is satisfied, from (11), $|G(t)|$ has the upper bound as:

$$\begin{aligned}
|G(t)| &= \left|\int_0^t\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}h(\tau)d\tau\right|\\
&\le \sum_{i=1}^r\sum_{j=0}^{n_i-1}|S_{ij}B|M_1\int_0^t\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}d\tau
\end{aligned}$$

Since A is Hurwitz, $\lambda_i<0 (i\in\underline{r})$. According to lemma 1 we can get

$$\lim_{t\to\infty}|G(t)|\le\sum_{i=1}^r\sum_{j=0}^{n_i-1}|S_{ij}B|M_1\frac{1}{(-\lambda_i)^{j+1}}\tag{14}$$

Therefore, from (7) (8) and (14), $\lim\limits_{t\to\infty}E(t)$ is bounded.

If property b) is satisfied, from (11), $|G(t)|$ has the upper bound as:

$$\begin{aligned}
|G(t)| &=\left|\int_0^t\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}h(\tau)d\tau\right|\\
&=\left|\int_0^t\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}df(\tau)\right|\\
&\le\left|\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\frac{(t-\tau)^j}{j!}e^{\lambda_i(t-\tau)}f(\tau)\Big|_0^t\right|\\
&\quad+\Big|\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\int_0^t\Big(\frac{\lambda_i(t-\tau)^j}{j!}\\
&\quad+\frac{(t-\tau)^{j-1}}{(j-1)!}\Big)e^{\lambda_i(t-\tau)}f(\tau)d\tau\Big|\\
&=\left|\sum_{i=1}^r S_{i0}Bf(t)-\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\frac{t^j}{j!}e^{\lambda_i t}f(0)\right|\\
&\quad+\Big|\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\int_0^t\Big(\frac{\lambda_i(t-\tau)^j}{j!}\\
&\quad+\frac{(t-\tau)^{j-1}}{(j-1)!}\Big)e^{\lambda_i(t-\tau)}f(\tau)d\tau\Big|
\end{aligned}$$

under the supposition that $(t-\tau)^{j-1}=0$ when $j-1<0$. Since $|f|\le M_2$, similarly, by applying lemma 1, it can be proved that

$$\begin{aligned}
\lim_{t\to\infty}\ &\Big|\sum_{i=1}^r\sum_{j=0}^{n_i-1}S_{ij}B\int_0^t\Big(\frac{\lambda_i(t-\tau)^j}{j!}\\
&+\frac{(t-\tau)^{j-1}}{(j-1)!}\Big)e^{\lambda_i(t-\tau)}f(\tau)d\tau\Big|\\
&\le\sum_{i=1}^r\sum_{j=0}^{n_i-1}|S_{ij}B|M_2\frac{2}{(-\lambda_i)^j}
\end{aligned}$$

So

$$\lim_{t\to\infty}|G(t)|\le\sum_{i=1}^r|S_{i0}B|M_2+\sum_{i=1}^r\sum_{j=0}^{n_i-1}|S_{ij}B|M_2\frac{2}{(-\lambda_i)^j}\tag{15}$$

Then, from (7) (8) and (15), $\lim\limits_{t\to\infty}E(t)$ is bounded. Q.E.D.

Conditions a) and b) covers a wide scope of uncertainties in the engineering practice, such as constant disturbance, square wave disturbance etc. The uncertainties described in the stability analysis in [13][16][17][18] all satisfy condition a). However, the square wave disturbance satisfies b) but not a).

Both the upper bounds in (14) and (15) can provide some guide for ESO's parameter design in practice. To further illustrate this, the special case when $r=1$ is studied for simplicity. In this case,

$$|S_{1j}B|=\begin{bmatrix}0\\ \vdots\\ 1\\ C_j^{j-1}(-\lambda)\\ \vdots\\ C_j^1(-\lambda)^{j-1}\\ (-\lambda)^j\end{bmatrix},\ j=0,\cdots,n\tag{16}$$

where $C_j^k=\frac{j\cdots(j-k+1)}{k!}$ $(k\le j)$, if the condition a) is satisfied, it follows that

$$\lim_{t\to\infty}|G(t)\le\sum_{j=0}^n M_1\begin{bmatrix}0\\ \vdots\\ \frac{1}{(-\lambda)^{j+1}}\\ \frac{C_j^{j-1}}{(-\lambda)^j}\\ \vdots\\ \frac{1}{-\lambda}\end{bmatrix},\tag{17}$$

and if the condition b) is satisfied, there is

$$\lim_{t\to\infty}|G(t)\leq \begin{bmatrix} 0\\ \vdots \\ 0 \\ 1\end{bmatrix} M_2+\sum_{j=0}^{n}2M_2\begin{bmatrix} 0\\ \vdots \\ \frac{1}{(-\lambda)^j} \\ \frac{C_j^{j-1}}{(-\lambda)^{j-1}} \\ \vdots \\ 1\end{bmatrix}. \tag{18}$$

According to (17), the larger $-\lambda$ is, the smaller the bound will be. What's more, the decay rate of the $i^{th}(i=1,\cdots,n+1)$ component is $O\Big(\frac{1}{\lambda^{n+2-i}}\Big)$. (18) shows that, the first n components of the bound will decrease as $-\lambda$ grows with the decay rate $O\Big(\frac{1}{\lambda^{n+1-i}}\Big)$ $i=1,\cdots,n$. The $(n+1)^{th}$ component in (18), which is the bound for estimating the uncertainty, is a constant.

Theorem 1 proves that the observing errors of LESO (4) are bounded when the uncertainty satisfies the condition a) or b). For some kinds of typical uncertainties, more accurate bounds of the observing errors can be obtained. Next, the bounds for a set of typical disturbances which often happen in engineering applications will be discussed.

III. OBSERVING ERRORS OF SEVERAL TYPICAL UNCERTAINTIES

A. f is a constant disturbance

Constant uncertain disturbance is the most simple and common case in practice. The widely used PID controller can treat constant disturbance very well because the use of integrator. Next we will show that the observing errors of LESO can converge to zero if f is an uncertain constant.

Assume $f=C$, where C is a uncertain constant. Then $h=\dot{f}=0$. From (9), there is $G(t)=0$, and $\lim\limits_{t\to\infty}E(t)=0$, which means $\lim\limits_{t\to\infty}z_{n+1}(t)=C$. Therefore the constant uncertainty can be precisely estimated by ESO.

B. f is the sinuous function

The sinuous disturbance can be viewed as the continuous and periodic uncertain dynamics exist in the system.

Assume $f=Msin\omega t$, M and ω are the amplitude and frequency of the sinuous function respectively. Obviously it satisfies property a) in Theorem 1 because $|h|\leq M\omega$. From (9), we can obtain

$$G(t)=M\omega\int_0^t e^{A(t-\tau)}Bcos\omega\tau d\tau \tag{19}$$

Integrating (19) by parts yields

$$G(t)=MBsin\omega t-\frac{MA}{\omega}Bcos\omega t+\frac{MA}{\omega}e^{At}B-\frac{A^2}{\omega^2}G(t)$$

Since matrix $I+\frac{A^2}{\omega^2}$ is positive definite, it is invertible. Then

$$G(t)=\Big(I+\frac{A^2}{\omega^2}\Big)^{-1}\Big(MBsin\omega t-\frac{MA}{\omega}Bcos\omega t+\frac{MA}{\omega}e^{At}B\Big)$$

So

$$\lim_{t\to\infty}|E(t)|\leq\Big|(I+\frac{A^2}{\omega^2})^{-1}B\Big|M+\Big|\Big(I+\frac{A^2}{\omega^2}\Big)^{-1}AB\Big|\frac{M}{\omega} \tag{20}$$

Equation (20) shows that the bounds of the observing errors are related to the frequency ω, the amplitude M and the Hurwitz matrix A.

C. f is a ramp function

The ramp function can be utilized to simulate the increasing disturbance in the system.

Assume $f=ct$, where c is an uncertain constant. Then $h=c$. It satisfies property a) in Theorem 1. It follows that

$$G(t)=c\int_0^t e^{A(t-\tau)}Bd\tau=cA^{-1}e^{At}B-cA^{-1}B.$$

So

$$\lim_{t\to\infty}E(t)=-cA^{-1}B \tag{21}$$

In most existing results for control methods of uncertain systems, the external disturbance is often assumed to be bounded. However, equation (21) reveals that LESO can observe ramp uncertainty with constant error although the ramp function will becoming infinite as the time going to infinite.

D. f is the square wave function

The square wave function represents a typical kind of uncertainties, for example, load change, which exists in many engineering systems. Actually it is a typical example, which only satisfies condition b) but not a) in theorem 1. Next, a more accurate bound for the observing errors will be presented for the square wave uncertainties, compared to that in (18).

Assume f is a square wave function as follows

$$f(t)=\begin{cases} L & t\in[(2k-2)T,(2k-1)T) \\ -L & t\in[(2k-1)T,2kT)\end{cases},k\in\mathbb{Z}^+ \tag{22}$$

where L and $2T$ are the amplitude and period of f respectively. Since $h(kT)$ is infinite, the condition a) is not satisfied. However, since $|f(t)|\leq L$ for all t, it satisfies condition b). Next the bound of $G(t)$ will be analyzed by two steps.

Step 1: When $t\in[(2k-1)T,2kT)$, that is $f(t)=-L$, it follows that

$$\begin{aligned} G(t) &= \int_0^t e^{A(t-\tau)}Bh(\tau)d\tau \\ &= \int_0^t e^{A(t-\tau)}Bdf(\tau) \\ &= Bf(t)-e^{At}Bf(0)-\sum_{i=0}^{k-1}\int_{2iT}^{(2i+1)T}L\ de^{A(t-\tau)}B \\ &+ \sum_{i=0}^{k-2}\int_{(2i+1)T}^{(2i+2)T}L\ de^{A(t-\tau)}B+\int_{(2k-1)T}^{t}L\ de^{A(t-\tau)}B \end{aligned} \tag{23}$$

First, consider the third part of (23) :

$$\begin{aligned}
&\sum_{i=0}^{k-1}\int_{2iT}^{(2i+1)T} L\ de^{A(t-\tau)}B \qquad (24)\\
=\ & \sum_{i=0}^{k-1}\left(e^{A(t-2iT-T)}-e^{A(t-2iT)}\right)BL\\
=\ & \left(e^{A(t-T)}-e^{At}\right)\sum_{i=0}^{k-1}e^{-2iAT}BL
\end{aligned}$$

Since

$$\left(\sum_{i=0}^{k-1}e^{-2iAT}\right)\left(I-e^{-2AT}\right)=\left(I-e^{-2kAT}\right)$$

and $\left(I-e^{-2AT}\right)$ is invertible, so

$$\left(\sum_{i=0}^{k-1}e^{-2iAT}\right)=\left(I-e^{-2AT}\right)^{-1}\left(I-e^{-2kAT}\right) \qquad (25)$$

Applying (25) into (24) gives

$$\begin{aligned}
&\sum_{i=0}^{k-1}\int_{2iT}^{(2i+1)T} L\ de^{A(t-\tau)}B \qquad (26)\\
=\ & \left(e^{A(t-T)}-e^{At}\right)\left(I-e^{-2AT}\right)^{-1}\left(I-e^{-2kAT}\right)BL\\
=\ & e^{At}\left(I-e^{-2AT}\right)^{-1}\\
& \left(e^{-AT}-I-e^{-A(T+2kT)}+e^{-2kAT}\right)BL
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{i=0}^{k-2}\int_{(2i+1)T}^{(2i+2)T} L\ de^{A(t-\tau)}B\\
=\ & e^{At}\left(I-e^{-2AT}\right)^{-1} \qquad (27)\\
& \left(e^{-2AT}-e^{-AT}-e^{-2kAT}+e^{A(-2kT+T)}\right)BL
\end{aligned}$$

Applying (26) (27) to (23), we can obtain

$$\begin{aligned}
G(t)\ =\ & -BL-e^{At}BL-e^{At}\left(I-e^{-2AT}\right)^{-1}\\
& \left(e^{-AT}-I-e^{-A(T+2kT)}+e^{-2kAT}\right)BL\\
& +e^{At}\left(I-e^{-2AT}\right)^{-1}\\
& \left(e^{-2AT}-e^{-AT}-e^{-2kAT}+e^{A(-2kT+T)}\right)BL\\
& +\left(I-e^{A(t-2kT+T)}\right)BL\\
=\ & -e^{At}\left(I+e^{A(-2kT+T)}\right)BL+e^{At}\left(I-e^{-2AT}\right)^{-1}\\
& \left(I-e^{-AT}\right)^2\left(I+e^{A(-2kT+T)}\right)BL
\end{aligned}$$

Introducing $t-(2k-1)T\triangleq\Delta t, 0\le\Delta t<T$, there is

$$G(t)=-2e^{-AT}\left(I+e^{-AT}\right)^{-1}\left(e^{At}+e^{A\Delta t}\right)BL \qquad (28)$$

Hence,

$$\lim_{t\to\infty}G(t)=-2e^{-AT}\left(I+e^{-AT}\right)^{-1}e^{A\Delta t}BL \qquad (29)$$

Step 2: When $t\in[2kT,(2k+1)T)$, that is $f(t)=L$, introducing $t-2kT\triangleq\Delta t$, it can be similarly deduced that

$$G(t)=2e^{-AT}\left(I+e^{-AT}\right)^{-1}\left(e^{A\Delta t}-e^{At}\right)BL \qquad (30)$$

Hence,

$$\lim_{t\to\infty}G(t)=2e^{-AT}\left(I+e^{-AT}\right)^{-1}e^{A\Delta t}BL. \qquad (31)$$

Equations (29) and (31) show that, the observing error G(t) has a close relationship with Δt, which is the distance between t and the step time (the time when f steps from L to $-L$ or from $-L$ to L). In each period, G(t) will decrease as Δt grows when Δt exceeds a short period of time.

IV. SIMULATION

Because second-order systems occupy an important place in the engineering and practice, the following second-order system with single-input u and single-output y is considered to test the capability of ESO for estimating uncertainties.

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=f+bu,\quad x_1(0)=0,x_2(0)=0,\\ y=x_1\end{cases} \qquad (32)$$

where $b=1$. The ADRC law is designed to regulate the output y to a constant $r=2$ when the following set of uncertainties exist in the system respectively:

$$\begin{cases}f &= 3;\\ f &= 2\sin(3t);\\ f &= t;\\ f &= sgn(\sin(\pi t)).\end{cases} \qquad (33)$$

The ADRC law is designed to be

$$u=-k_1(x_1-r)-k_2x_2-z_3, \qquad (34)$$

where $k_1=30,k_2=10$. The ESO is designed as

$$\begin{cases}\dot{z}_1=z_2-\beta_1(z_1-x_1),\\ \dot{z}_2=z_3-\beta_2(z_1-x_1)+u,\\ \dot{z}_3=-\beta_3(z_1-x_1),\end{cases} \qquad (35)$$

To implement the simulation, we use the Euler discrete method with the step size $\tau=0.01$. In order to guarantee the convergence of the digital calculation for ESO, β_1 is usually chosen as

$$\beta_1\approx\frac{1}{\tau}$$

The parameters in (35) are chosen as

$$\beta_1=120,\ \beta_2=4800,\ \beta_3=64000$$

Remark 1. In the simulation, the bandwidth of ESO (35) is designed to be around 6Hz, which is larger than the frequency of disturbances in (33). This will guarantee the convergence of ESO in a short time. Since the high frequency noise always exists in the practice, the bandwidth of ESO should be determined according to the practical bandwidth limit. One can consult [13][14][15] to see how ESO is used in practical engineering problems.

The simulation results are demonstrated in Fig.1 to Fig.3. Fig.1 is the output y when the different disturbances (33) are added. It shows that ADRC (34)(35) has strong ability for control uncertain systems. Its ability can be explained by Fig.2 and Fig.3. Fig.2 is the output z_3 of ESO (35), which is designed to estimate the uncertainty f. It can be seen that ESO (35) has an excellent capabilities to estimate a wide kind of uncertainties. Fig.3 is the static regulating error $r-y(t>1)$ and the observing error z_3-f. It can be seen that:

1) When $f=3$, the observing error z_3-f converges to zero, and the regulating error $r-y$ converges to zero.

2) When $f=2\sin(3t)$, both the observing error z_3-f and the regulating error $r-y$ are bounded.

3) When $f=t$, as fig. 2 shows, although the disturbance increases with the time t, z_3 successfully approaches the increasing disturbance and the observing error z_3-f is bounded. Hence, the regulating error $r-y$ is bounded.

4) When $f=sgn(\sin(\pi t))$, both the observing error z_3-f and the regulating error $r-y$ are bounded. Furthermore, although an observing error happens at the time when f steps, it decreased with t until the next step. The simulation results show that ESO has strong capabilities for estimating uncertainties. Because the design of ESO is completely independent of the disturbance, a ESO with the same structure and same parameters can be used to deal with a large class of disturbances.

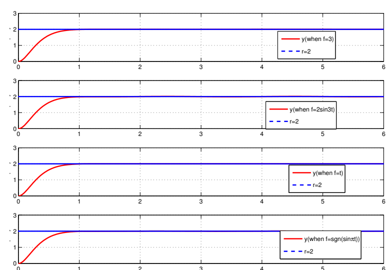

Fig. 1. The output y with the different disturbances in the system

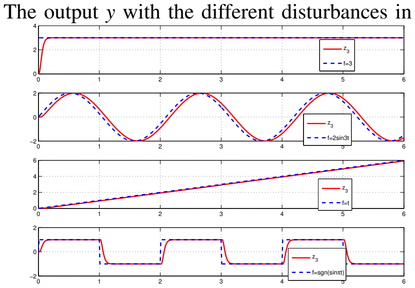

Fig. 2. The output z_3 of ESO and the disturbance f

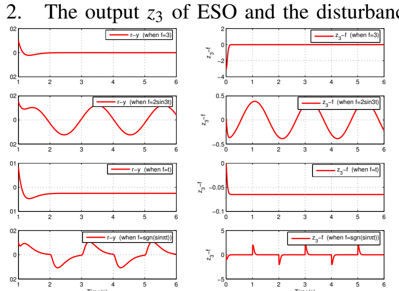

Fig. 3. The regulating errors and observing errors

V. CONCLUSIONS

In this paper, the capabilities of ESO for estimating uncertainties are analyzed. It demonstrates that the error of ESO is bounded if the uncertainty f is bounded or its derivative (or generalized derivative) is bounded. Furthermore, the observing errors are analyzed for several typical kinds of uncertainties and the analysis results are tested by simulations. The results enriched the existing ones from the theoretical perspective.

REFERENCES

[1] K. J.Astrom and T. Hagglund, *PID controllers: Theory, design and tuning.Research Triangle Park*, N.C. Instrument Society of America, 1995.

[2] K..J.Astrom and B. Wittenmark, *Adaptive Control*, 2nd ed. Reading, MA: Addison-Wesley, 1995.

[3] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal Control:* Prentice-Hall, 1996.

[4] C. D. Johnson, “Accommodation of external disturbances in linear regulator and servomechanism problem”, *IEEE Transactions on automatic control*, vol.2, Ac-16, no. 6, pp635-644, 1971.

[5] G. Basile and G. Marro, “On the observability of linear, time-invariant systems with unknown inputs”, *Journal of optimization theory and applications*, vol. 2, no 6,pp410-415, 1969.

[6] S. Kwon and W. K. Chung, “A discrete-time design and analysis of perturbation observer for motion control applications”,*IEEE Transactions on control systems technology*, vol. 11, no.3 , pp399-407,2003.

[7] E. Schrijver and J. van Dijk, “Disturbance observers for rigid mechanical systems: Equivalence, stability and design”,*ASME Journal of dynamics systems, measurement and control*, vol. 124, no. 4, pp.539-548, 2002.

[8] Jingqing Han, “Active disturbance rejection controller and its applications”, *Control and Decision*, Vol 13,No 1,pp19-23,1998.(in Chinese)

[9] Z.Gao, Y. Huang, and J. Han, “An alternative paradigm for control system design”, *Proc. of IEEE conference on decision and control*, pp.4578-4585,2001.

[10] Z. Gao, “Active disturbance rejection control: A paradigm shift in feedback control system design” *Proc. of the 2006 American Control Conference*, pp 2399-2405, 2006.

[11] Jingqing Han,“From PID to Active Disturbance Rejection Control”,*IEEE Transactions on Industrial Electronics*, Vol.56, No.3, pp.900-906,2009

[12] Jingqing Han, “A class of extended state observers for uncertain systems”,*Control and Decision*, Vol. 10 No.1, 1995. (in Chinese)

[13] Yi Huang, Kekang Xu, Jingqing Han, James Lam, “Flight control design using extended state observer and non-smooth feedback”,*Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida USA,pp223-228,2001

[14] Wankun Zhou and Zhiqiang Gao,“An active disturbance rejection approach to tension and velocity regulation in web processing lines”, *16th IEEE international conference on control and applications*, pp 842-848, 2007.

[15] Zhongzhou Chen, Qing Zheng, Zhiqiang Gao, “Active disturbance rejection control of chemical processe”,*16th IEEE international conference on control and applications*, pp 855-861, 2007.

[16] Qing Zheng, Linda Q. Gao, and Zhiqiang Gao, “On stability analysis of active disturbance rejection control for nonlinear time-varing plant with unknown dynamics” *Proceedings of the 46th IEEE Conference on Decision and Control*, pp3501-3506,2007.

[17] Qing Zheng, Zhiqiang Gao,“On estimation of plant dynamics and disturbance from input-output data in real tim”,*16th IEEE International Conference Control Applications*, pp.1167-1172, 2007

[18] D.Yoo, S.S-T.YAU, Z. Gao, “Optimal fast tracking observer bandwidth of the linear extended state observer”,*International Journal of Control*, vol80, No 1, pp102-111, January, 2007.