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Refining LaSalle's invariance principle

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Abstract— **This work deals with the problem of locating the omega-limit set of a bounded solution of a given autonomous vector field on a Riemannian manifold. The derived results extend LaSalle's invariance principle in such a way that the newly obtained conditions provide in certain situations a more refined statement about the location of the omega-limit set when the invariance principle is inconclusive. The derived conditions are for example useful for gradient-type vector fields and cascaded systems.**

I. Introduction

LaSalle's invariance principle [4] is one of the very useful theorems in dynamical systems and control theory with plenty of applications [3]. In recent years several generalizations covering non-classical frameworks have been investigated, ranging from switched systems [2], [8], to non-autonomous systems [6], to infinite dimensional systems [12] and to hybrid systems[9]. Instead the purpose of our contribution is to tackle some classical situations (autonomous finite dimensional continuous dynamical systems) in which the LaSalle's invariance principle can not be applied directly and in which it appears inconclusive. For example, consider a planar system of the form

$$\begin{aligned} \dot{x} &= a(x) + b(x,y) \\ \dot{y} &= c(y) + d(x,y), \end{aligned} \tag{1}$$

where a, b, c, d are appropriately defined functions with $c(0) = d(x,0) = a(0) = b(x,0) = 0$. Notice that for $d(x,y) = 0$, we have a cascaded system structure. Assuming that a proper Lyapunov function $V = V(x,y)$ is known such that $\dot{V}(x,y) \leq 0$ and $\dot{V}(x,y) = 0$ if and only if $y = 0$, one can conclude that the solutions $y(t, x(0), y(0))$ converge to zero. Hence the asymptotically residual dynamics is $\dot{x} = a(x)$. A natural question about the residual dynamics is the following: If another function $W = W(x)$ is known such that $\dot{W}(x) \leq 0$, is it possible to conclude that a bounded solution of (1) $x(t, x(0), y(0))$ converges to the set E defined by $\dot{W}(x) = 0$? In general, the answer is no and the application of LaSalle's invariance principle is not very helpful in such a situation, because the set of points $(x, 0)$ is invariant with respect to (1).

There are results available in the literature in which one can actually conclude that $x(t, x(0), y(0))$ converges to the set E, for example when this set is asymptotically stable ([7], [10], [11]). In a situation where the set E is not necessarily stable, not many results are seems to be known. Indeed, the authors are not aware of any result of this type.

In this paper, we address such problems where the set E is not necessarily stable and might be even disconnected. We derive sufficient conditions under which, for example for the scenario sketched above, it can be concluded that $x(t, x(0), y(0))$ approaches E. In particular, our results include the case when the set E consists of a set of points with at most a finite number of accumulation points or when the residual dynamics $\dot{x} = a(x) = \nabla h(x)$ is a gradient flow.

The outline of the paper is as follows: In Section II, the problem set-up is established and preliminary results are derived. In Section III, the main results are proved together with some of their consequences. Conclusions and a summary are provided in Section IV.

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II. Set-up and preliminary results

Main Assumptions. The set-up of our investigation is the following. On a Riemannian manifold $\mathcal{M}$ of class C^2 with metric g a locally Lipschitz continuous vector field

$$\dot{x} = f(x) \tag{2}$$

is given. The initial value $x(0)$ is such that the corresponding solution $x(t, x(0))$ is bounded; this is always the case if $\mathcal{M}$ is compact. It is not restrictive to assume that the omega-limit set $\Omega(x(0))$ which is a compact and connected (see Lemma 1) is contained in an embedded submanifold $\mathcal{S} \subset \mathcal{M}$, possibly equal to $\mathcal{M}$. Moreover, without loss of generality we can assume that the solution $x(t, x(0))$ is approaching $\mathcal{S}$. Since the omega-limit set $\Omega(x(0))$ is compact by Lemma 1, it is not restrictive to assume there exists a compact neighborhood K, such that the $O := \mathrm{Int}(K)$ is an open neighborhood of $\Omega(x(0))$. We assume that there exists a real valued C^1 function $W : O \to \mathbb{R}$ and such that $\dot{W}(x) \geq 0$ on $\mathcal{S} \cap O$, where $\dot{W}(x)$ is the derivative of $W(x)$ along the flow (Lie derivative). More specifically we assume that $\dot{W}(x) = 0$ on a subset $E \subset (\mathcal{S} \cap O)$, and $\dot{W}(x) > 0$ on $(\mathcal{S} \cap O) \setminus E$. Therefore $E = \{x \in \mathcal{S} : \dot{W}(x) = 0\}$ and in some applications it will be exactly the set of equilibria of the vector field (2), $E = \{x \in \mathcal{S} : f(x) = 0\}$, but our results hold without the latter assumption.

Let us remark that most of the following results hold also in the slightly more general case of a Finsler manifold, but for the sake of readability we will not enforce this hypothesis, since in any case any manifold can be endowed with a Riemannian metric.

The following lemma about invariant sets, despite being well-known is important and it is the base of our investigation (see e.g. [4], Chapter 2, Theorem 5.2).

Lemma 1: Let $\Omega(x(0))$ denote the positive limit set (ω-limit set) of a bounded solution $x = x(t, x(0))$ of (2) above. Then $\Omega(x(0))$ is nonempty, compact, connected, positively and negatively invariant with respect to (2), and it is the *smallest* closed set that $x = x(t, x(0))$ approaches as $t \to \infty$, i.e., if $x = x(t, x(0))$ converges to a closed set which properly contains $\Omega(x(0))$, then $x = x(t, x(0))$ converges to $\Omega(x(0))$.
Since we will need to view manifolds as metric spaces, we use the Riemannian metric g on $\mathcal{M}$ to generate a corresponding distance function d. In the simplest case in which $\mathcal{M} = \mathbb{R}^n$ one can choose as d the norm associated to the Euclidean scalar product.

III. Main Results

All the Main Assumptions above are assumed to hold also in this section. Our goal is to find sufficient conditions that force $\Omega(x(0))$ to be contained in one of the connected components of E.
The following lemma describes some basic properties of the ω-limit set in relation to E.
Lemma 2: Assume that the Main Assumptions above are satisfied. Then the omega limit set $\Omega(x(0))$ and E have nonempty intersection, namely $\Omega(x(0)) \cap \{x \in \mathcal{S} : \dot{W}(x) = 0\} \neq \emptyset$. Moreover if $\Omega(x(0))$ does not reduce to a single point, then $\Omega(x(0)) \cap E$ does not contain any stable equilibrium point.

Proof: If $\Omega(x(0)) \cap E = \emptyset$, since $\Omega(x(0))$ and E are both closed, the distance between $\Omega(x(0))$ and E is strictly positive. Moreover, since the restriction of $\dot{W}(x)$ on $\mathcal{S} \cap O$, i.e $\dot{W}_{|\mathcal{S}\cap O}$ is continuous and zero only on E by assumption, this implies that there exists a $\delta > 0$ such that $\dot{W}_{|\mathcal{S}\cap O} \geq \delta > 0$ in $\Omega(x(0))$. On the other hand $\Omega(x(0))$ is compact, therefore W has a global maximum on $\Omega(x(0))$. Moreover since $\Omega(x(0))$ is invariant, all trajectories starting there will remain confined in $\Omega(x(0))$. Therefore we reach a contradiction, because $\int_0^{+\infty} \dot{W}_{|\mathcal{S}\cap O} dt$ is divergent along a solution starting in $\Omega(x(0))$, but this impossible since $\Omega(x(0))$ is compact and W has a global maximum in $\Omega(x(0))$. Thus $\Omega(x(0)) \cap E \neq \emptyset$. Assume now that $\Omega(x(0))$ is not a single point. Let $A_s \in \Omega(x(0)) \cap E$ be a stable equilibrium point and let $B \in \Omega(x(0))$ with $d(B, A_s) > \delta$. Then, for any $\epsilon > 0$ there exists a $t_1 > 0$, depending on ϵ such that $d(x(t_1), A_s) \leq \epsilon$, where $x(t, x(0))$ is the solution of (2) starting at $x(0)$, due to the fact that $A_s \in \Omega(x(0))$. Now, if one choses ϵ small enough, then it follows from stability that $d(x(t), A_s) \leq \delta$ for all $t > t_1$, hence the solution $x(t)$ cannot become $\delta/2$-close to B anymore, and consequently B is not a point in $\Omega(A(0))$, contrary to the initial assumption. ∎

We introduce the following definition in order to state our main result:
Definition 3: Let $\{E_i\}_{i\in I}$ be the connected components of $E \cap \Omega(x(0))$. Given a function W as in the Main Assumptions, we say that the components $\{E_i\}_{i\in I}$ *are contained* in W if each E_i lies in a level set of W, and the subset $\{W(E_i)\}_{i\in I} \subset \mathbb{R}$ has at most a finite number of accumulation points in $\mathbb{R}$.
Observe that Definition 3 does not exclude the case in which two or more connected components of E lie in the *same* level set of W. Let us remark that when the function W is globally defined on $\mathcal{M}$ or in a tubular neighborhood of $\mathcal{S}$ it can be easier to check directly that all connected components of E are contained in W, rather than checking the condition for the components of E in $\Omega(x(0))$, since in general the ω-limit set is not known. For instance, this is what is done in the application of the results of this paper in [1]. Due to space limitations, other examples and applications will appear elsewhere.
The first main result is the following:
Theorem 4: Assume the Main Assumptions of the previous section hold. If the components $\{E_i\}_{i\in I}$ are contained in W, then $\Omega(x(0)) \subset \{E_i\}$ for a unique $i \in I$.

Proof: We analyze first the case in which $\Omega(x(0))$ is contained in a level set of W. Consider the level set LS of W where $\Omega(x(0))$ is contained. The invariant set $\Omega(x(0))$ can not be disjoint from the components of E sitting inside LS, because otherwise, reasoning as in Lemma 2 we would reach a contradiction. Call $\{E_{LS,k}\}_{k\in K}$ the connected component of E sitting inside LS and having nonempty intersection with $\Omega(x(0))$, where K is possibly an infinite set. We claim that necessarily $\Omega(x(0)) \subset E_{LS,j}$ for a unique j. Indeed if this is not the case, since $\Omega(x(0))$ is positive invariant and $\Omega(\Omega(x(0)) \subseteq \Omega(x(0))$ a solution of (2) starting in $\Omega(x(0)) \setminus \cup_{k\in K} E_{LS,k}$ will have $\dot{W}(x) > 0$, so the height function W is increasing, but on the other hand $\Omega(x(0))$ is invariant and contained in a level set of W. Contradiction. Therefore, the only possibility is that $\Omega(x(0)) \subset E_{LS,j}$ for some j. Since $\Omega(x(0))$ is connected by Lemma 1, then $\Omega(x(0)) \subset E_{LS,j}$ for a unique j.
Now consider the case in which $\Omega(x(0))$ is not contained in a level set of W. Since W is continuous and $\Omega(x(0))$ is compact, we have that $W(\Omega(x(0)) = [\min, \max]$ is a compact interval in the real line. Call now $\Omega_E := \{E_1, \ldots, E_k, \ldots\}$, $k \geq 1$, the sets in $\Omega(x(0))$ obtained by intersecting $\Omega(x(0))$ with the connected components of E. Ω_E is not empty by Lemma 2. Let $\mathcal{W} := \{w_1, \ldots, w_l\}$ be the corresponding values of the function W on $\{E_1, \ldots, E_k, \ldots\}$. These values are not necessarily distinct, in the sense that it can happen that for two different connected components E_i and E_j we have $w_i = w_j$. We will now distinguish three cases in the proof, even though in all these cases the idea of the proof is basically the same.
First case: the set Ω_E contains only one component, call it E_1 (see also Figure 1). Let $\mathcal{B}(\epsilon)$ be a closed neighborhood of $\Omega(x(0))$ in $\mathcal{M}$, $\mathcal{B}(\epsilon) \subset O$ such that $d(\mathcal{B}(\epsilon), \Omega(x(0))) \leq \epsilon$. Recall that O is the open neighborhood of $\Omega(x(0))$ with compact closure K where the function W is defined. Moreover, let $\mathcal{U}_1$ be an open neighborhood for E_1 in $\mathcal{M}$ and let $\mathcal{U}_1(\epsilon) = \mathcal{B}(\epsilon) \cap \mathcal{U}_1$. Denote with $\underline{b}_1 = \inf_{x\in\mathcal{U}_1} W(x)$, and with $\overline{b}_1 = \sup_{x\in\mathcal{U}_1} W(x)$. Since E_1 is closed in $\Omega(x(0))$ and by hypothesis $\Omega(x(0))$ is not contained in E_1, we can choose point $P \in \Omega(x(0)) \setminus E_1$ and choose a neighborhood $\mathcal{U}_P$ of P in $\mathcal{M}$, with $\mathcal{U}_P \subset O$. Call $\underline{b}_P = \inf_{x\in\mathcal{U}_P} W(x)$, and with $\overline{b}_P = \sup_{x\in\mathcal{U}_P} W(x)$ and $\mathcal{U}_P(\epsilon) = \mathcal{B}(\epsilon) \cap \mathcal{U}_P$. It is

not restrictive to assume that $W(P) < W(E_1)$ (if the other inequality is satisfied, simply switch the roles of P and E_1 in what follows). Since $W(P) < W(E_1)$ and W is continuous we can choose ϵ and neighborhoods $\mathcal{U}_P$ and $\mathcal{U}_1$ such that $\underline{b}_1 > \bar{b}_P$ and such that

$$\min_{x\in\mathcal{B}(\epsilon)\setminus(\mathcal{U}_1(\epsilon)\cup\mathcal{U}_P(\epsilon))} \dot{W}(x) \geq \frac{\delta}{2} > 0 \qquad (3)$$

holds for some $\delta > 0$. Moreover

$$\min_{x\in\Omega(x(0))\setminus(\mathcal{U}_1(0)\cup\mathcal{U}_P(0))} \dot{W}(x) \geq \delta > 0 \qquad (4)$$

can be always be satisfied, because $\Omega(x(0))\setminus(\mathcal{U}_1(0)\cup\mathcal{U}_P(0))$ can be contained in a compact set and $\dot{W}$ is strictly positive on $\Omega(x(0)) \setminus (\mathcal{U}_1(0) \cup \mathcal{U}_P(0))$. Hence, because W and $\dot{W}$ are continuous functions, (3) must be true for a sufficiently small ϵ. In other words, for ϵ sufficiently small, the set $\{x : \dot{W}(x) \leq 0, \ x \in \mathcal{B}(\epsilon)\}$ is contained in $\mathcal{U}_1(\epsilon) \cup \mathcal{U}_P(\epsilon)$. Since $\Omega(x(0))$ is the positive limit set, there exists a $t_1 > 0$ such that $x(t_1, x(0)) \in \mathcal{U}_1(\epsilon)$ and such that $x(t, x(0)) \in \mathcal{B}(\epsilon)$ for $t \geq t_1$, due to the fact that $\mathcal{B}(\epsilon)$ is closed, compact and contains $\Omega(x(0))$ (see Lemma 1). Moreover, there must exists a $t_2 > t_1$ such that $x(t_2, x(0)) \in \mathcal{U}_P(\epsilon)$, since P belongs to the ω-limit set. However, this is impossible because in order to reach $\mathcal{U}_P(\epsilon)$ we must have $W(x(t_2)) < \underline{b}_1$ for some $t_1 < t_2$. On the other hand $\dot{W} \geq \delta/2 > 0$ on $\mathcal{B}(\epsilon)\setminus(\mathcal{U}_1(\epsilon)\cup\mathcal{U}_P(\epsilon))$, so the value of W along $x(t)$ can not decrease. This leads to a contradiction with the assumption that $\Omega(x(0))$ is not contained in a level set of W or with the fact that $\Omega(x(0))$ is not contained in E_1. If the former is true then $\Omega(x(0))$ must be contained in a level set of W, from which the thesis follows by the first part of the proof. If the latter is true, the thesis follows immediately.

Second case: the set $\mathcal{W}$ is finite, and there are possibly infinitely many components of E in the level sets of $W^{-1}(\mathcal{W}) \cap \Omega(x(0))$. Then $\mathcal{W}$ is contained in the closed interval $W(\Omega(x(0)) = [\min, \max]$. Let us call $\{w_1, \dots, w_k\}$ the *distinct* elements of $\mathcal{W}$, for some $k \geq 1$ and assume without loss of generality that they are ordered in such a way that $w_1 > w_2 > \dots > w_k$. Construct open intervals Z_i of w_i in $[\min, \max]$ such that $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Consider as before a compact neighborhood $\mathcal{B}(\epsilon)$ of $\Omega(x(0))$ in $\mathcal{M}$, $\mathcal{B}(\epsilon) \subset O$ such that $d(\mathcal{B}(\epsilon), \Omega(x(0))) \leq \epsilon$. By the fact that $\dot{W} > 0$ on $(\mathcal{S} \cap O) \setminus E$, and that $\dot{W}$ and W are continuous functions, it is possible to choose ϵ and the pairwise disjoint open neighborhoods $\{Z_i\}_{i=1,\dots,k}$ in $[\min, \max]$ in such a way that on the closed set $C = \mathcal{B}(\epsilon) \setminus (\mathcal{B}(\epsilon) \cap (W^{-1}(\cup_{i=1}^k Z_i))$, we have

$$\min_{x\in C} \dot{W}(x) \geq \frac{\delta}{2} > 0 \qquad (5)$$

for some $\delta > 0$. Observe also that, shrinking ϵ and Z_i if necessary, the open sets $\mathcal{U}_i := \mathcal{B}(\epsilon) \cap W^{-1}(Z_i)$ are such that $\underline{b}_i > \bar{b}_{i+1}$, where $\underline{b}_i := \inf_{x\in\mathcal{U}_i} W(x)$ and $\bar{b}_i := \sup_{x\in\mathcal{U}_i} W(x)$. Since $\Omega(x(0))$ is the ω-limit set, there must exist a time t_1 such that the solution of (2) $x(t_1, x(0)) \in \mathcal{U}_i$, and moreover, for any $t \geq t_1$, $x(t, x(0)) \in \mathcal{B}(\epsilon)$. On the other hand there must exists a $t_2 > t_1$ such that the solution $x(t_2, x(0)) \in \mathcal{U}_{i+1}$, which means in particular that $W(x(t_2, x(0))) < \underline{b}_i$, but this is impossible since on C $\dot{W}(x) \geq \delta/2 > 0$. So we reach a contradiction and $\Omega(x(0))$ must lie in a level set of W, and we conclude as in the first part.

As a third case, let us consider what happens when there are finitely many accumulation points for $\mathcal{W}$ in $[\min, \max]$. Let us call $\{P_1, \dots, P_k\}$ the finite set of accumulation points of $W(\Omega_E) = \mathcal{W}$. If we consider open neighborhoods $\{Z_i\}_{i=1,\dots k} \subset \mathbb{R}$ for each P_i, then the entire $\mathcal{W}$ is covered by these open neighborhoods except for possibly a *finite* residual set of points, call them $\{R_1, \dots, R_l\}$. Choose again open neighborhoods $\{X_i\}_{i=1,\dots l}$ for these residual points, and possibly shrink Z_i and X_i such that $Z_i \cap Z_j = \emptyset$ for any pair of distinct indices, $Z_i \cap X_j = \emptyset$ for any pair of indices and $X_i \cap X_j = \emptyset$ for any pair of distinct indices. By continuity of W and $\dot{W}$ and the fact that $\dot{W} > 0$ on $(\mathcal{S} \cap O) \setminus E$ we can reason as in the second case to reach a contradiction with the assumption that $\Omega(x(0))$ is not contained in a level set of W and it is at the same time the ω-limit set.

■

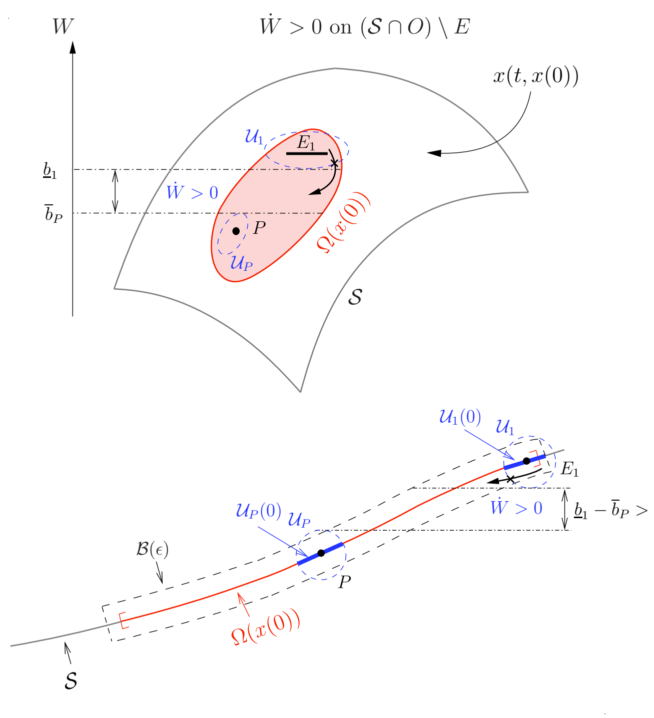

Fig. 1. Basic idea and illustration of the first case in the proof of Theorem 4.

Two special cases of the Theorem 4 appear frequently in applications and are worthwhile to mention as distinct results.

The first one deals with the case in which the set E is known to be a countable set of points. In this case we have

Theorem 5: Assume the Main Assumptions hold. Then if the set E is a countable set of points $\{P_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$ such that $W(E\cap\Omega(x(0)))$ has at most a finite number of accumulation

points, then $\Omega(x(0)) = P_i$ for a unique $i \in I$.
Proof: It is an immediate consequence of Theorem 4. Indeed under the current hypotheses, E satisfies automatically the requirement for being contained in W and Theorem 4 gives the desired result. ■

Another important special case is when the submanifold $\mathcal{S}$ is invariant for (2) and the vector field (2) becomes a gradient flow when restricted to $\mathcal{S}$. Let us recall that on a Riemannian manifold $(\mathcal{M}, g)$, the gradient of $h \in C^1(\mathcal{M}, \mathbb{R})$ is defined as the unique vector field $\nabla_g h$ on $\mathcal{M}$ such that $g(\nabla_g h, \cdot) = dh$, where dh is the differential of h. The same construction applied to $\mathcal{S}$ works when $\mathcal{S}$ is an embedded submanifold of $\mathcal{M}$.

Before stating the case of a gradient flow, we proceed with the following easy observation:

Lemma 6: Let
$$\dot{x} = \nabla_g h(x) \tag{6}$$
be a gradient flow on Riemannian submanifold (M, g) of dimension n. If h is of class at least C^n then the connected components $\{E_i\}_{i\in I}$ of the equilibria set E of (6) lie on level sets of h.

Proof: This is an immediate consequence of Sard's lemma [13]. Indeed by continuity $h(E_i)$ is a connected subset of $\mathbb{R}$ and by Sard's lemma it has to have zero Lebesgue measure, so it is a point and therefore it lies on a level set of h. ■

Remark 7: The condition about the regularity of h in Lemma 6 is in general the best possible. Indeed, it is possible to construct interesting counterexamples where h is of class C^{n-1} on a manifold of dimension n and a connected component of the locus where the gradient of h vanishes *is not contained* in a level set of h. The first counterexample, for a function h of class C^1, $h : \mathbb{R}^2 \to \mathbb{R}$ was given in [14].

Theorem 8: Assume that $\mathcal{M}$ is a smooth Riemannian manifold, $\mathcal{S}$ is a smooth closed embedded submanifold containing the ω-limit set $\Omega(x(0))$ of the smooth vector field f appearing in (2). Assume that the vector field f restricts to a vector field $f_{\mathcal{S}}$ on $\mathcal{S}$ and that $f_{\mathcal{S}}$ is a gradient vector field on $\mathcal{S}$, namely $f_{\mathcal{S}} = \nabla_{\tilde{g}} h$ for some smooth function $h \in C^\infty(\mathcal{S}, \mathbb{R})$, where $\tilde{g}$ is the induced metric on $\mathcal{S}$. Call $\{E_i\}_{i\in I}$ the connected components of the equilibria set E for $f_{\mathcal{S}}$. If the subset $\{h(E_i)\}_{i\in I}$ has at most a finite number of accumulation points in $\mathbb{R}$, then $\Omega(x(0)) \subset E_i$ for a unique $i \in I$.

Remark 9: The regularity on h can be dropped, as long as h is at least C^m where m is the dimension of $\mathcal{S}$, due to Lemma 6.

Proof: First of all, notice that in this case the function W used in Theorem 4 will be provided by an extension of the potential function h to a tubular neighborhood of $\mathcal{S}$. Indeed, by Lemma 6 applied to the Riemannian submanifold $\mathcal{S}$ the connected components $\{E_i\}_{i\in I}$ lie on the level sets of h. Besides, under the current hypotheses, the subset $\{h(E_i)\}_{i\in I}$ has at most finite many accumulation points. Moreover since $\dot{h} =< dh, f >= \tilde{g}(\nabla_{\tilde{g}} h, f) = \tilde{g}(\nabla_{\tilde{g}} h, \nabla_{\tilde{g}} h)$ on $\mathcal{S}$, we have $\dot{h} \geq 0$ on $\mathcal{S}$ and $\dot{h} = 0$ only on E. Now since $h \in C^\infty(\mathcal{S}, \mathbb{R})$, there exists a smooth function $\tilde{h}$ extending h to a tubular neighborhood O of $\mathcal{S}$ in $\mathcal{M}$ [5]. Moreover, since $f_{\mathcal{S}}$ is tangent to $\mathcal{S}$, the function $\tilde{h}$ can be chosen in such a way that the following equality is true:
$$(\dot{\tilde{h}})_{\mathcal{S}} = (\mathcal{L}_f \tilde{h})_{\mathcal{S}} = \mathcal{L}_{f_{\mathcal{S}}} h = \dot{h},$$
where $\mathcal{L}$ denotes Lie derivative. Therefore the extended potential $\tilde{h}$ satisfies all the hypothesis of the function W in Theorem 4 in a tubular neighborhood O of $\mathcal{S}$, therefore a fortiori in an open neighborhood of $\Omega(x(0))$. In particular the components $\{E_i\}_{i\in I}$ are contained in $\tilde{h}$. Thus applying Theorem 4 with $W = \tilde{h}$ we get immediately the claim. ■

Remark 10: Observe that in Theorem 8 we do not need to assume the existence of a function W like in the Main Assumptions, since the function W is automatically provided by a suitable extension of h on a tubular neighborhood of $\mathcal{S}$.

Sometimes in applications, one is working in a more rigid category, where some hypotheses of the previous results are automatically satisfied. This is for instance the case of non singular real algebraic variety, the simplest example of which is given by $\mathbb{R}^n$ or the set of zeros of polynomial functions in $\mathbb{R}^n$, when these sets are non singular.

We can restate Theorem 4 in this set up in a simpler way:

Theorem 11: Assume the Main Assumptions above hold. Assume moreover that M is a nonsingular real algebraic variety on which the vector field (2) is defined. If the function W is such that $\dot{W}(x)$ is algebraic, and such that each connected component E_i of the equilibria set E of (2) lies on a level set of W, then $\Omega(x(0)) \subset E_i$ for a unique i.

Proof: The only difference with respect to the hypotheses of Theorem 4 is the fact that we do not have to check if the set $\{W(E_i)\}_{i\in I}$ has a finite number of accumulation points or not. Indeed, the set $\{W(E_i)\}_{i\in I}$ can not have an accumulation point at all in this case. This is due to the fact that since $\mathcal{M}$ is algebraic and $\dot{W}$ is algebraic, the set E is a possibly singular real algebraic variety, and as such it can have only a finite number of components [15]. ■

A direct consequence of Theorem 11 is the following:

Corollary 12: Suppose the vector field (2) is defined over $\mathbb{R}^n$. If we are given a function W as above, such that $\dot{W}(x)$ is algebraic, and such that each connected component E_i of the equilibria set E of (2) lies on a level set of W, then $\Omega(x(0)) \subset E_i$ for a unique i.

A similar corollary of Theorem 11 can be obtained for gradient vector fields. We leave the details to the reader.

There is another situation which is not covered by Definition 3 and Theorem 4 where it is possible to obtain a weaker result. First we introduce the following definition:

Definition 13: Let $\{E_i\}_{i\in I}$ be the connected components of E. Given a function W as above, we say that W *strictly separates* the components $\{E_i\}_{i\in I}$ if for any pair of distinct indices i and j in I the distance on the real line between the closed subsets $W(E_i \cap \Omega(x(0)))$ and $W(E_j \cap \Omega(x(0)))$ is greater or equal to a positive constant c.

Observe that Definition 3 does not imply and is not implied by Definition 13.

Proposition 14: Assume the Main Assumptions hold. Assume moreover that W strictly separates the connected components $\{E_i\}_{i\in I}$. Then $\Omega(x(0)) \cap \{E_i\}_{i\in I}$ is not empty for a unique $i \in I$, call it m. Moreover if E_m is stable, then $\Omega(x(0)) \subset E_m$.

Proof: Arguing as in Theorem 4 it is easy to see that if the connected components are strictly separated by W, then only one of them has non empty intersection with $\Omega(x(0))$, say E_l. In particular, for any $\epsilon > 0$, there exists a time t_1 such that $d(x(t_1, x(0)), E_l) \leq \epsilon$, where d denotes the Hausdorff distance. By stability of E_l, choosing ϵ sufficiently small, the solution $x(t, x(0))$ will remain confined in a δ-neighborhood of E_l. In particular, for any $\delta > 0$, there is an $\epsilon > 0$ such that if the solution intersect the ϵ-neighborhood at a time t_1, then it will remain trapped in the δ-neighborhood for all times $t > t_1$. But this provides an immediate contradiction with the assumption that $\Omega(x(0))$ is not contained in E_l. ■

IV. Conclusions

In this paper, the following problem was studied: A bounded solution $x(t, x(0))$ of the vector field (2) on a Riemannian manifold $\mathcal{M}$ is given and it is known that this solution approaches a possibly compact submanifold $\mathcal{S}$, i.e. the omega-limit set $\Omega(x(0))$ is contained in $\mathcal{S}$. Moreover, a function W on $\mathcal{S} \cap O$ is known such that $\dot{W}(x) \geq 0$ on $\mathcal{S} \cap O$, but not necessarily in a neighborhood O. What can be said about the convergence behavior of $x(t, x(0))$? The results in this paper provide sufficient conditions under which it can be shown that $\Omega(x(0))$ lies in the set E where $\dot{W}$ vanishes. In particular, it is shown that if each connected component of E is contained in a level set of W, then $x(t, x(0))$ approaches a single connected component E_i, i.e. $\Omega(x(0)) \subset E_i$, provided a certain condition is satisfied on accumulation points. As shown, the established results are in particular useful when the set E consists of isolated points or when the flow on $\mathcal{S}$ is a gradient flows. In these cases, it can be concluded that $\Omega(x(0))$ lies in the set E. It is important to notice that, if for example $\mathcal{S}$ is an invariant set and if a Lyapunov function V is know such that $\dot{V}(x) \leq 0$ on M and $\dot{V}(x) = 0$ if and only if $x \in \mathcal{S}$, then LaSalle's invariance principle would not allow to conclude anything stronger than that any omega-limit set of a bounded solution lies in $\mathcal{S}$, since $\mathcal{S}$ itself is the largest invariant set. The results in this paper, however, allow to give a sharper statement on the location of omega-limit sets, assuming that a positive semidefinite function W on $\mathcal{S}$ (or in a neighborhood O of $\Omega(x(0))$ is known. Hence, the results in this paper can be considered as a refinement of LaSalle's invariance principle.

For future research, one open question is to ask how far one can extend the present results to the case in which each connected component E_i is not necessarily contained in a single level set of W. In general, one cannot expect that the present results generalize to this case. For a counterexample, see [4], p.67. Nevertheless, due to the many applications of LaSalle's invariance principle, further research in this direction is interesting and in particular necessary in situations where the invariance principle is not helpful.

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