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A convergence result for multiagent systems subject to noise

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***Abstract*— We present conditions under which a general class of multiagent systems subject to noise can reach agreement in expected value with probability one. The noise can be induced by the fact that each agent takes erroneous measurements of neighbors' positions. The class of systems considered may be nonlinear and requires that the diameter of the agents be bounded for all possible error measurements. The convergence result is related to previous work on the robustness of the rendezvous algorithm and the stability of multiagent systems with periodic connectivity. We illustrate the results in terms of a modified discrete-time Kuramoto system, which is amended to guarantee the system requirements.**

I. Introduction

The last years are witnessing an intense research activity in the area of cooperative control of multiagent systems and its applications to multi-vehicle sensor networks; see e.g. [1]. A main driving theme is the characterization of the system stability and robustness properties under different metrics.

For example, recent work has been devoted to the analysis of multiagent systems under switching graphs [2], [3], [4], [5], [6], asynchrony and delays [7], [8]. Robustness to noise, and how multiagent behavior is affected by the network size is also the subject of recent work; e.g. see [9], [10], [4], [11] on the input-to-state stability properties of consensus algorithms, and the degradation of formation control systems subject to noise.

Motivated by this, we look for general conditions that guarantee a class of discrete-time multiagent systems converge to an agreement state in expected value with probability one. For example, the systems can be subject to noise due to agents taking erroneous measurements of neighbors' positions. We assume that the possible disturbances belong to a compact space, and that the expected value of the measurements correspond to true position values. The class of systems considered here may be non-linear and extend the class of multiagent systems considered previously. A restriction that we impose to ensure a technical requirement is that the diameter of agents be upper bounded for any possible error measurement. We illustrate the results in terms of a discrete-time Kuramoto system introduced in [12]. This system is further modified in order to satisfy the assumptions of the main result of the paper. Simulations show that convergence occurs as predicted independently of this restriction.

The paper is organized as follows. In Section II we present some notation, preliminary concepts on graphs, and on multiagent systems modeled through set-valued maps. In particular, we revisit a discrete-time version of a Kuramoto oscillator from [12]; which is briefly analyzed using the tools described. In Section III, we present the class of multiagent systems subject to noise on which we focus and introduce a modified version of the Kuramoto oscillator system as an example. Section IV contains the main results of the paper which are then used to analyze the modified Kuramoto system. Section V presents simulations results and Section VI includes some concluding remarks.

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II. Notation and preliminary definitions

A. Preliminaries on geometric notions and graphs

In this subsection, we introduce some notation and preliminary concepts employed throughout the paper, see [13] for a more information on these.

In the sequel, X will either represent a (convex) subset of $\mathbb{R}^d$, for some $d \geq 0$. Consider a set of points $p_1, \dots, p_n \in X$, its convex hull is defined as $\operatorname{co}(p_1, \dots, p_n) = \{\lambda_1 p_1 + \dots + \lambda_n p_n | \ \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$. We will use tuples $P = (p_1, \dots, p_n) \in X^n$ to refer to the positions of the multiagent system in space. The algorithms we will consider are synchronous, implemented in discrete time over a time schedule $m = 0, 1, 2, 3, \dots$, and give rise to point sequences $\{P_m = (p_{1,m}, \dots, p_{n,m}) \in X^n\}_{m \geq 0}$.

A *(directed) graph* over a finite set of nodes V is a pair $G = (V, E)$, with $E = \{(i,j)| \ i,j \in V, \ i \neq j\} \subseteq V \times V \setminus \operatorname{diag}(V \times V)$. The graph is *undirected* when $(i,j) \in E$ if and only if $(j,i) \in E$. A *proximity graph function* $\mathcal{G}(P)$ associates to a point set $V_P = \{p_1, \dots, p_n\} \subset X$ an undirected graph with vertex set V_P and edge set $\mathcal{E}_{\mathcal{G}}(V_P) \subseteq V_P \times V_P \setminus \operatorname{diag}(V_P \times V_P)$. In other words, the edge set of a proximity graph may depend on the location of its vertices. We will also consider proximity graphs subject to link failures, $\mathcal{G}_{\mathcal{F}}(P)$. These are graphs on P with an edge set that may also be dependent on the location of the vertices. However, given (p_i, p_j) in $\mathcal{G}_{\mathcal{F}}(P)$, the reversed edge (p_j, p_i) may not be in $\mathcal{G}_{\mathcal{F}}(P)$. In other words, $\mathcal{G}_{\mathcal{F}}(P)$ is a directed graph. We will use these graphs to capture sensing or communication failures. When elements in the set P are indexed by $i \in \{1, \dots, n\} = V$, the graph $\mathcal{G}_{\mathcal{F}}(P)$ can be associated with a graph G over V in a natural way. With a slight abuse of notation, we will sometimes identify these two objects. We denote the set of neighbors of agent p_i in $\mathcal{G}_{\mathcal{F}}(P)$ by:

$$\mathcal{N}_i(\mathcal{G}_{\mathcal{F}}) = \{j \in \{1, \dots, n\} | \ (p_i, p_j) \in \mathcal{E}_{\mathcal{G}_{\mathcal{F}}}(P)\},$$

and the cardinality of $\mathcal{N}_i(\mathcal{G}_{\mathcal{F}})$ will be denoted as $n_i = |\mathcal{N}_i(\mathcal{G}_{\mathcal{F}})|$. Given a sequence of graphs $\{\mathcal{G}_{\mathcal{F},m}\}_{m \geq 0}$, we will use the shorthand notation $\mathcal{N}_{i,m} \equiv \mathcal{N}_i(\mathcal{G}_{\mathcal{F},m})$. We will denote the set of proximity graphs with failures as $\mathcal{G}_{\text{PF}}$. With

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a slight abuse of notation, the set $\mathcal{G}_{\rm PF}(P)$ will represent the set of directed graphs over n vertices.

We recall some concepts from Functional Analysis next. Let $\mathcal{F} = \{f : Y \to Z\}$ be a family of functions defined over metric spaces (Y, d_Y) and (Z, d_Z). Let $\|\cdot\|_Y$ (resp. $\|\cdot\|_Z$) denote the associated norm on Y (resp. on Z). A sequence $\{f_n\}_{n\geq 0} \subseteq \mathcal{F}$ converges uniformly to $f \in \mathcal{F}$ if for all $\epsilon > 0$ there is a $n_\epsilon > 0$ (independently of $y \in Y$) such that $\forall\, n \geq n_\epsilon$ we have that $\|f_n(y) - f(y)\|_Z \leq \epsilon$. The family $\mathcal{F}$ is uniformly bounded in Y if there is a constant $K \geq 0$ such that $\sup_{y\in Y} \|f(y)\|_Z \leq K$ for all $f \in \mathcal{F}$. The family $\mathcal{F}$ is equicontinuous if for all $y_0 \in Y$ and $\epsilon > 0$, there exists a $\delta_{y_0} > 0$ such that $\|f(y_0) - f(y)\|_Z < \epsilon$ for all $f \in \mathcal{F}$ and $y \in Y$ with $\|y_0 - y\|_Y < \delta$. The theorem of Ascoli-Arzela is the Bolzano-Weierstrass analogue for sequences of functions: every equicontinuous family $\{f_m\}_{m\geq 0}$ that is also uniformly bounded, has a uniformly convergent subsequence $\{f_{m_k}\} \subseteq \{f_m\}$.

B. Multiagent dynamics and Kuramoto oscillators

Here we present models of multiagent systems using set-valued maps and an associated convergence result. As an example, we describe a discrete-time Kuramoto multioscillator taken from [12]. We also point out how to partially prove a conjecture from [14] regarding the system convergence.

Given $P_0 \in X^d$, a proximity graph $\mathcal{G}$, and a map $f : X^n \to X^n$, we define a multiagent system evolution as:

$$P_m = f(P_{m-1}, \mathcal{G}(P_{m-1})) = (f_1(P_{m-1}, \mathcal{N}_1(\mathcal{G}(P_{m-1}))), \dots, f_n(P_{m-1}, \mathcal{N}_n(\mathcal{G}(P_{m-1})))) .$$

Observe that local multiagent interactions are introduced by the fact that $f_i(P, G) \equiv f_i(p_i, p_{j_1}, \dots, p_{j_{n_i}})$ where p_{j_l}, $l \in \{1, \dots, n_i\}$ are the neighbors of agent i in the graph $\mathcal{G}(P)$. To capture the possible failures in communication or sensing, we can employ a set-valued map $T_f : X^n \rightrightarrows X^n$ such that

$$P_m \in T_f(P_{m-1}) = \{f(P_{m-1}, G)\,|\,, G \subseteq \mathcal{G}(P_{m-1})\} .$$

A set-valued map T_f is closed if for any sequences P_k, Q_k such that $Q_k \in T_f(P_k)$ and $Q_k \to Q$, $P_k \to P$, we have that $Q \in T_f(P)$. It is easy to see that, provided that $f(P, G)$ is continuous in P for every fixed G, the set-valued map T_f is closed. A LaSalle invariance principle is available for multiagent systems as follows; see [13] for more information.

Lemma 1 ([13]): Let $T : X^n \rightrightarrows X^n$ be a set-valued map defining a discrete-time multiagent dynamical system. Assume that:

(i) there is a set $W \subseteq X^n$ that is strongly positively invariant under T;
(ii) there exists a function $V : X^n \to \mathbb{R}$ that is non-increasing along T on W.
(iii) all evolutions of the dynamical system with initial conditions in W are bounded; and
(iv) T is nonempty and closed at W and V is continuous on W.

Let M denote the largest weakly positively invariant set contained in $\{p \in \overline{W}\,|\,\exists\, p' \in T(p)$ such that $V(p') = V(p)\}$. Then there exists a $c \in \mathbb{R}$ such that all evolutions with initial conditions in W approach the set $M \cap V^{-1}(c)$.

Example 2 (Discrete-time Kuramoto oscillators): The Kuramoto system was proposed in [15] to model synchronization in a population of oscillators. Different conditions for the stability of the system under switching graphs have recently been provided [16], [17], [2], [18]. A motivation for the study of the Kuramoto oscillators has been the coordination of multiple underwater vehicles [19], [12]. Since the communication among vehicles occurs naturally at discrete instants of time $K\Delta T$, the following discrete-time version of the algorithm was proposed in [12]. Consider an initial condition $\theta_{1,0}, \dots, \theta_{n,0}$. For every $k \in \{1, \dots, n\}$ and $m \geq 0$,

$$\theta_{k,m+1} = \theta_{k,m} + \frac{K\Delta T}{|\mathcal{N}_{k,m}| + 1} \sum_{i\in\mathcal{N}_{k,m}} \sin(\theta_{i,m} - \theta_{j,m}) \mod 2\pi\,, \tag{1}$$

where a particular identification of $\mathbb{S}$ as subset of $\mathbb{R}$ is chosen; i.e. $\mathbb{S} \equiv [\theta_*, \theta_* + 2\pi)$, for some origin θ_*, and the sum is understood modulo 2π. Observe that, if the initial conditions for the oscillators satisfy $-\pi \leq \theta_{i,0} - \theta_{j,0} \leq \pi$ for all $i, j \in \{1, \dots, n\}$, then $\sin(\theta_{i,0} - \theta_{j,0}) = \lambda_{ij}(0)(\theta_{i,0} - \theta_{j,0})$, with $0 \leq \lambda_{ij}(0) \leq 1$. Thus, for any choice of origin θ_*, we have that the system (1) satisfies:

$$\begin{aligned}
\theta_{k,1} &= \theta_{k,0} + \frac{K\Delta T}{|\mathcal{N}_{k,0}| + 1} \sum_{i\in\mathcal{N}_{k,0}} \lambda_{ik}(0)(\theta_{i,0} - \theta_{k,0}) \\
&= \theta_{k,0}\left(1 - \frac{K\Delta T}{|\mathcal{N}_{k,0}| + 1} \sum_{i\in\mathcal{N}_{k,0}} \lambda_{ik}(0)\right) \\
&\qquad + \frac{K\Delta T}{|\mathcal{N}_{k,0}| + 1} \sum_{i\in\mathcal{N}_{k,0}} \lambda_{ik}(0)\theta_{i,0}\,.
\end{aligned}$$

The above linear combination is a convex combination if

$$\begin{aligned}
&0 \leq \frac{K\Delta T}{|\mathcal{N}_{k,0}| + 1}\lambda_{ik}(0) \leq 1, \qquad \forall\, i \in \mathcal{N}_{k,0} \\
&0 \leq 1 - \frac{K\Delta T}{|\mathcal{N}_{k,0}| + 1} \sum_{i\in\mathcal{N}_{k,0}} \lambda_{ik}(0) \leq 1\,.
\end{aligned}$$

This is true provided that $K\Delta T \in [0, 2]$. In other words, $\theta_{i,1} \in \text{co}(\theta_{1,0}, \dots, \theta_{n,0}) \subseteq [\theta_* - \pi, \theta_* + \pi)$ and thus $-\pi \leq \theta_{i,1} - \theta_{j,1} \leq \pi$. In this way, the oscillator states remain in the set $[\theta_* - \pi, \theta_* + \pi)$ for all t, the system (1) is well defined as a system in $\mathbb{R}$ and it is not necessary to consider the modulo operation. Observe that if different agents choose different coordinates (determined by a different origin θ'_*), in order to implement equation (1), then it holds that $\theta'_{i,m} = \theta_{i,m} - \theta_* + \theta'_*$ for all $m \geq 0$. That is, the evolutions are the same except for the origin translation. Without lose of generality, we will assume that agents make use of a common origin in $\mathbb{S}$ to implement (1).

This system was analyzed in [12] and, under the assumption of all-to-all communication, it is seen that all agents' states get aligned for $K\Delta T \in [0, 2]$ under bounded delays for almost all initial conditions. The following result shows an application of Lemma 1 and proves a partial conjecture stated in [12].

Proposition 3: Consider the discrete-time Kuramoto system (1), with $K\Delta T \in [0,2]$. Let $\theta_{1,0},\ldots,\theta_{n,0}$ be an initial condition such that $\pi < \theta_{i,0}-\theta_{j,0} < \pi$ for all $i,j \in \{1,\ldots,n\}$. Let $\{G_m\}_{m\geq 0}$, be the sequence of proximity graphs with failures used to obtain the evolution of $\{\theta_{i,m}\}_{m\geq 0}$, $i \in \{1,\ldots,n\}$. If there exists $M>0$ such that G_{kM} is strongly connected for every $k \geq 1$, then $\lim_{m\to\infty}\theta_{k,m}=\theta^*$.

Proof: A sketch of the proof is given next. The system can be described with the help of a set-valued map $T_{\text{Kura}} = (T_1,\ldots,T_n) : X^n \rightrightarrows X^n$ as:

$$\theta_{i,m+1} = T_i(\theta_{1,m},\ldots,\theta_{n,m})$$
$$= \{\theta_{i,m} + \frac{K\Delta T}{|\mathcal{N}_k(G)|+1}\sum_{i\in\mathcal{N}_k(G)}\sin(\theta_{i,m}-\theta_{j,m}) \,|\, G \in \mathcal{G}_{\text{PF}}\}$$

In particular, the evolution of the oscillators under a particular choice of $\{G_m\}_{m\geq 0}$ is contained into the set of evolutions that is possible under the set-valued map. If $-\pi < \theta_{i,m}-\theta_{j,m} < \pi$, then it can be seen that $\theta_{i,m+1} \in \text{co}(\theta_{1,m},\ldots,\theta_{n,m})$, and $-\pi < \theta_{i,m+1}-\theta_{j,m+1} < \pi$ (see the above discussion for $m=0$). Thus the set $W = \text{co}(\theta_1(0),\ldots,\theta_n(0))^n$ is (strongly positively) invariant under the set-valued map T_{Kura}. Because we have that $\text{co}(\theta_{1,m+1},\ldots,\theta_{n,m+1}) \subseteq \text{co}(\theta_{1,m},\ldots,\theta_{n,m})$, one can see that the function diam : $X^n \to \mathbb{R}$, $\text{diam}(\theta_1,\ldots\theta_n) = \max_{i,j}(\theta_i-\theta_j)$ is decreasing along T_{Kura} on W. Note also that diam is continuous on W. Finally, since for every fixed graph the function $\theta_i + \frac{K\Delta T}{|\mathcal{N}_k(G)|+1}\sum_{i\in\mathcal{N}_k(G)}\sin(\theta_i-\theta_j)$ is continuous in $\theta_1,\ldots,\theta_n$, then T_{Kura} is closed on W. By Theorem 1, we have convergence to the largest invariant set M such that:

$$M \subseteq \{\theta \in \overline{W} \,|\, \exists\theta' \in T_{\text{Kura}}(\theta) \text{ s.t. } \text{diam}(\theta') = \text{diam}(\theta)\}.$$

It can be proved by contradiction that $M \subseteq \text{diag}([-\pi,\pi]^n)$. Otherwise, take $(\theta_1,\ldots,\theta_n) \in M$ such that $\text{diam}(\theta_1,\ldots,\theta_n) > 0$ and consider it as initial condition for T_{Kura}. It must be that the θ_i determining the diameter of $\text{co}(\theta_1,\ldots,\theta_n)$ remains stationary under T_{Kura}. Otherwise the diameter of $\text{co}(\theta_1,\ldots,\theta_n)$ will decrease strictly since when the θ_i move, they strictly move to the interior of the convex hull of neighbors. By hypothesis for all $Mk>0$, G_{Mk} is strongly connected. It can be argued that this implies that one of the agents determining $\text{diam}(\theta_1,\ldots,\theta_n)$, say i, will have a neighbor under G_{Mk}, for some $k\geq 0$, say j, with $\theta_{i,Mk} \neq \theta_{j,Mk}$. Since $|\theta_{i,m}-\theta_{j,m}| < \pi$ for all $m \geq 0$, then $\sin(\theta_{i,Mk}-\theta_{j,Mk}) \neq 0$ and necessarily $\theta_{i,Mk}$ will strictly move to the interior of $\text{co}(\theta_{i,0},\ldots,\theta_{n,0})$. Thus the diameter of the set will be strictly decreased, which is a contradiction with the fact that it is constant on M. Thus, it must be that $M \subseteq \text{diag}([-\pi,\pi]^n)$. Also, since $[-\pi,\pi]^n$ is a compact set, it must be that $\{\theta_{k,m}\}$ converge to a point in M. ■

III. A CLASS OF RANDOM MULTIAGENT SYSTEMS

In this section, we present the class of multiagent systems subject to noise that we consider in this paper. As an example, we introduce a modified version of the discrete-time Kuramoto system.

Consider a group of n agents, with states denoted by $P = (p_1,\ldots,p_n) \in X^n$, $i \in \{1,\ldots,n\}$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a compact probability space with $\Omega \subseteq \mathbb{R}^e$ for some integer e, and denote by $\mathcal{R} = \{R : \Omega \to X^n | \; R \text{ is a Random Variable}\}$ the set of random variables over Ω. We will denote the components of $R \in \mathcal{R}$ as $R = (R_1,\ldots,R_n)$. A sequence of random variables will be denoted by $\{R_m = (R_{1,m},\ldots,R_{n,m})\}_{m\geq 0}$.

Let us identify an initial multiagent configuration, $P_0 = (p_{1,0},\ldots,p_{n,0}) \in X^n$, with the constant random variable $R_0(\omega) = P_0$, $\omega \in \Omega$. Now, given a map $F : X^n \times \mathcal{G}_{\text{PF}} \to \mathcal{R}$, we can define a discrete-time Markov process as $R_{m+1} = F(R_m(\omega_m), \mathcal{G}_m(R_m(\omega_m)))$, $m \geq 1$, taking R_0 as initial condition[1]. In this way, a sequence of random outcomes, $\{\omega_m\}_{m\geq 0}$, graphs $\{G_m\}_{m\geq 0}$, and an initial condition, P_0, give rise to a particular sequence of multiagent states $\{R_m(\omega_m) = P_m = (p_{1,m},\ldots,p_{n,m})\}_{m\geq 0}$. To fix ideas, the outcome $\omega_m \in \Omega$ will correspond to arbitrary errors in e.g. measuring agents' positions. That is, $p_{i,m+1}$ will depend on the previous positions $p_{i,m}$ and the sensed position of neighbors as $p^i_{j,m} = s_i(p_{j,m},\omega)$, where $s_i : X \times \Omega \to X$ is the sensing function of agent i. For example, $s_i(p_{j,m},\omega) = p_{j,m} + \omega^i_j$, where $\omega = (\omega^i_j) \in \Omega$. In this way, depending on the error outcome ω, we will get different new positions $R_{m+1}(\omega)$. For every agent, we denote the set of neighbors' sensed positions at time m as $\mathcal{S}_{i,m} = \{p^i_{j,m} = s_i(p_{j,m},\omega_m) | \; j \in \mathcal{N}_{j,m}, \omega_m = (\omega^i_{j,m})\}$.

In order to consider arbitrary proximity graphs with failures, we will make use of a set-valued map $T_F : X^n \rightrightarrows \mathcal{R}$ such that:

$$R_{m+1} \in T_F(R_m(\omega_m)) = \{F(R_m(\omega_m)), G) | \; G \in G_{\text{PF}}\}\,. \tag{2}$$

The map $F : X^n \times \mathcal{G}_{\text{PF}} \to \mathcal{R}$ and a fixed graph G give rise to another map $F_G : X^n \times \Omega \to X^n$ such that $F_G(P,\omega) = F(P,G)(\omega)$. We will use the notation $F_G(P,\omega) = (F_{G,1}(P,\omega),\ldots,F_{G,n}(P,\omega))$.

The type of multiagent algorithms that we consider will satisfy the following properties:

Assumption 1: (i) $F_{G,i}(P,\omega)$, $i \in \{1,\ldots,n\}$, is continuous in (P,ω), for every G,

(ii) $F_G(P,\omega)$ is invariant modulo points in the diagonal of X. That is, $F_G(P+(q,\ldots,q),\omega) = F_G(P,\omega)+(q,\ldots,q)$ for all $(q,\ldots,q) \in \text{diag}(X^n)$.

(iii) the sequence $\{R_m\}_{m\geq 0}$, obtained through T_F and a given P_0 will satisfy $\|R_{i,m}(\omega)-R_{j,m}(\omega)\| \leq C_{P_0}$ for some positive constant C_{P_0},

(iv) for every $m \geq 0$, we have that $p_{i,m+1} \in \text{co}(\mathcal{S}_{i,m} \cup \{p_{i,m}\})$. More precisely, when there is $j \in \mathcal{N}_{i,m}$ such that $p_{j,m} \neq p_{i,m}$, and a certain constraint set $C_i(p_{i,m},\mathcal{S}_{i,m}) \neq \emptyset$, then $p_{i,m+1} \in \text{co}(\mathcal{S}_{i,m} \cup$

[1]Since we are dealing with random variables, we can understand the equality $R_{m+1} = F(R_m(\omega_m), \mathcal{G}_m(R_m(\omega_m)))$ in the almost sure sense. That is, $R_{m+1}(\omega) = F(R_m(\omega_m), \mathcal{G}_m(R_m(\omega_m)))(\omega)$ for all ω except for possibly for a set of probability (or measure) zero.

$\{p_{i,m}\}) \setminus \mathcal{S}_{i,m} \cup \{p_{i,m}\}$. When $C_i(p_{i,m}, \mathcal{S}_{i,m}) = \emptyset$, then $p_{i,m+1} = p_{i,m}$.

Given a map $F : X^n \times \mathcal{G}_{\text{PF}} \to \mathcal{R}$, we consider an associated map $\overline{F} : X^n \times \mathcal{G}_{\text{PF}} \to \mathcal{R}$ such that

$$\overline{F}(P, \mathcal{G}(P)) = F(P, \mathcal{G}(P)) - (\pi_1(F(P, \mathcal{G}(P))), \ldots, \pi_1(F(P, \mathcal{G}(P)))),$$

where $\pi_1 : X^n \to X$ is the natural projection. An alternative way of describing the algorithm defined through map F that satisfies Assumption 1 (ii) is the following:

$$\overline{R}_{m+1} \in \overline{T}_{\overline{F}}(\overline{R}_m(\omega_m)), \qquad m \geq 0, \tag{3}$$

where $\overline{T}_{\overline{F}} : X^n \rightrightarrows \mathcal{R}$ is given by

$$\overline{T}_{\overline{F}}(\overline{P}) = \{\overline{F}(\overline{P}, G) \,|\, G \subseteq \mathcal{G}(\overline{P})\},$$

To see this, suppose that $\overline{P}_0 = P_0$, and the specific sequences of graphs $\{G_m\}_{m\geq 0}$, and events $\{\omega_m\}_{m\geq 0}$ chosen to obtain $\{\overline{R}_m\}$ and $\{R_m\}$ are the same. Then we have that $\overline{R}_m = R_m - (\pi_1(R_m), \ldots, \pi_1(R_m))$, for all $m \geq 1$. We will make use of this relationship to prove our convergence results in the next section. Alternatively, using the notation of $\overline{F}_G(P, \omega) \equiv \overline{F}(P, G)(\omega)$, we have that $\overline{R}_{m+1}(\omega) = \overline{F}_{G_m}(\overline{R}_m(\omega_m), \omega)$, $m \geq 1$.

Example 4 (Modified discrete-time Kuramoto system): Suppose that every agent $k \in \{1, \ldots, n\}$ can only take noisy measurements of neighbors i according to $\theta_i^k = \theta_i + \omega_i^k$, with ω_i^k e.g. uniformly distributed over $[-\sigma, \sigma]$, for some $\sigma > 0$.

As a consequence of the noise, the direct implementation of the discrete-time Kuramoto update law (1), could make agents' get out of the invariant region $-\pi < \theta_i - \theta_k < \pi$. Suppose that $\pi < \theta_{i,0} - \theta_{j,0} < \pi$ for all $i, j \in \{1, \ldots, n\}$. Then, the update law in (1) can be modified to guarantee $-\pi < \theta_{i,m} - \theta_{j,m} < \pi$ for all $m \geq 0$. In what follows we use the notation $\Phi_m(\omega) = (\phi_{1,m}(\omega), \ldots, \phi_{n,m}(\omega))$ to refer to the random variables obtained through the algorithm, and $\Theta_m = (\theta_{1,m}, \ldots, \theta_{n,m})$ to refer to the specific multiagent states the system evolves through. In this way, $\phi_{i,m}(\omega_m) = \theta_{i,m}$. Then, $\Phi_{m+1}(\omega) = (\phi_{1,m+1}(\omega), \ldots, \phi_{n,m+1}(\omega))$ is obtained as:

$$\phi_{k,m+1}(\omega) = F_k(\Phi_m(\omega_m), G_m)(\omega) = \theta_{k,m} + \frac{K\Delta T}{|\mathcal{N}_k(G_m)| + 1} \sum_{i \in \mathcal{N}_k(G_m)} \sin(\overline{\theta}_{i,m}^k - \theta_{k,m}),$$

where $\overline{\theta}_{i,m}^k = \theta_{i,m} + \omega_i^k$ if $-\pi < \theta_{i,m}^k - \theta_{k,m} < \pi$, otherwise $\overline{\theta}_{i,m}^k = \theta_{k,m}$. Observe that this condition on the definition of $\overline{\theta}_{i,m}^k$ defines a constraint set, $C_k(\theta_{k,m}, \mathcal{S}_k)$, for each agent. In other words, $C_k(\theta_{k,m}, \mathcal{S}_k) = \emptyset$ if $|\theta_{i,m} + \omega_i^k - \theta_{k,m}| \geq \pi$ for all $i \in \mathcal{N}_{k,m}$, otherwise $C_k(\theta_{i,m}, \mathcal{S}_k) = X$.

It is easy to see that all conditions of Assumption 1 (i) through (iii) are satisfied. To see that condition (iv) holds, we can rewrite the system as:

$$\phi_{k,m+1}(\omega) = b_{kk,m}\theta_{k,m} + \sum_{i \in \mathcal{N}_k(G_m)} b_{ik,m}\overline{\theta}_{i,m}^k,$$

for some constants $0 \leq b_{ij,m} \leq 1$ (see reasoning before Proposition 3). It can be seen that whenever $\overline{\theta}_{i,m}^k \neq \theta_{k,m}$, then $\theta_{k,m+1} \in \text{co}(\mathcal{S}_k, \theta_{k,m}) \setminus \mathcal{S}_k \cup \{\theta_{k,m}\}$. Because of the definition of $C_(k\theta_{k,m}, \mathcal{S}_k)$ taken and the algorithm definition, condition (iv) holds.

IV. Convergence Results

This section contains the main convergence results for a multiagent system satisfying Assumption 1 (i)–(iv). The proofs of the following theorems and results are contained in an extended version of this paper; see [20], where we also analyze related circumcenter algorithms subject to noisy measurements.

Proposition 5: Consider a multiagent system defined through a set-valued map as in (2) and (3), satisfying Assumption 1. Let $\{\overline{R}_m\}_{m\geq 0}$ be the sequence obtained through it from the initial condition $P_0 \in X^n$. Then, the family of functions $\{\overline{R}_m\}_{m\geq 0}$ is uniformly bounded in Ω and is equicontinuous.

Given a discrete-time Markov process, we define an omega-limit set for it as follows.

Definition 1: Consider a discrete-time Markov process $\{R_m(\omega)\}$, we define its *omega-limit* set as

$$\Omega(R_m) = \{R : \Omega \to X^n \text{ random variable} \,| \, \exists \{R_{m_k}\} \subseteq \{R_m\} \text{ such that } R_{m_k} \to R \text{ uniformly in } \Omega\}. \tag{4}$$

Observe that since a sequence $\{R_m\}$ satisfying Assumption 1 (iii) is uniformly bounded and equicontinuous, by the Ascoli-Arzela theorem, there is always a subsequence $\{R_{m_k}\}$ that is uniformly convergent $\{R_{m_k}\} \to R$. In this way, $R \in \Phi(R_{m_k}) \neq \emptyset$.

Suppose that $\{R_m\}_{m\geq 0}$ is determined from the set-valued map $\overline{T}_{\overline{F}}$, the initial condition P_0 and a sequence of outcomes $\{\omega_m\}_{m\geq 0}$. We say that $\Omega(R_m)$ is (weakly) invariant with respect to $\overline{T}_{\overline{F}}$ if for every $\overline{R} \in \Omega(R_m)$ there exist $\omega \in \Omega$, and $R' \in T(\overline{R}(\omega))$ such that $R' \in \Phi(R_m)$. The following result holds.

Theorem 6: Let $\overline{T}_{\overline{F}}$ be a set-valued map as in (3) associated with a map as in 2 satisfying Assumption 1. The omega-limit set of the discrete-time Markov process (3) defined in (4) is invariant with respect to $\overline{T}_{\overline{F}}$.

Theorem 7: Let $p_{1,0}, \ldots, p_{n,0} \in X$ be the initial positions of a multiagent system. Let $\{\overline{R}_m(\omega) = (\overline{R}_{1,m}(\omega), \ldots, \overline{R}_{n,m}(\omega))\}_{m\geq 0}$ denote the discrete-time Markov process obtained by applying a set-valued map algorithm as in (3), associated with a map F satisfying Assumption 1. Then $\{R_m\}$ converges to the largest (weakly positively) invariant set, $\mathcal{M}$, contained in

$$\{R \text{ random variable } | \exists T(R(\omega)) \text{ for some } \omega \in \Omega \text{ and } \operatorname{diam}(E[R']) = \operatorname{diam}(E[R])\}.$$

A. Analysis of the modified discrete-time Kuramoto system

From Theorem 7, we would like to conclude that in fact $\lim_{m\to\infty} \operatorname{diam}(E[R_m]) = c = 0$, so that we have multiagent agreement in expected values. Due to the fact that

$C_i(p_{i,m}, \mathcal{S}_{i,m})$ may be empty for all $i \in \{1, \dots, n\}$, agents may remain stationary and $\lim_{m\to\infty} \operatorname{diam}(E[\overline{R}_m]) > 0$. As long as one can prove that $C_i(p_{i,m}, \mathcal{S}_{i,m}) \neq \emptyset$ and the multi-agent system is connected infinitely often then agreement in expected value will follow. In the next subsection, we prove this holds for the modified Kuramoto system. For simplicity we use sequences of undirected graphs. The result can be extended to the case of directed graphs as in Proposition 3 and also for graphs that are jointly connected over a fixed time window.

Theorem 8: Consider the modified discrete-time Kuramoto system proposed in Example 4 and the associated Markov process $\{\Phi_m\}_{m\geq 0}$ obtained from an initial condition $\Theta_0 = (\theta_{1,0}, \dots, \theta_{n,0})$ such that $|\theta_{i,0} - \theta_{j,0}| < \pi$ for all $i, j \in \{1, \dots, n\}$. Let $\{G_m\}_{m\geq 0}$ be the sequence of graphs used to obtain $\{\Phi_m\}_{m\geq 0}$ and suppose that there is $M > 0$ such that G_{kM} is connected for all $m \geq 0$. Then we have that $\lim_{m\to\infty} E[\Phi_m] = 0$ with probability one.

Proof. By Theorem 7 we have that $\lim_{m\to\infty} E[\Phi_m] = c$ and that Φ_m converges almost surely to $\mathcal{M} \cap (\operatorname{diam} \circ E)^{-1}(c)$. Using a contradiction argument we see next that if $\overline{R} \in \mathcal{M}$ then $\operatorname{diam}(E[\overline{R}]) = 0$ with probability one.

Let $\Phi \in \mathcal{M}$ and consider the initial condition $\overline{\Theta} = E[\Phi] = (\overline{\theta}_1, \dots, \overline{\theta}_n)$ for the Kuramoto system. By the weak invariance property, it must be that we can always find a sequence such that $\operatorname{diam}(E[R_m]) = c$ for all $m \geq 0$. We will show that, with probability one, for all sequences chosen there is $m_0 > 0$ such that $\operatorname{diam}(E[R_{m_0}] < c$ unless $c = 0$.

Suppose that $\operatorname{diam}(\overline{\Theta}) > 0$. By the algorithm definition, this implies that all the agents i that determine the diameter of the set $\operatorname{co}(\theta_{1,m}, \dots, \theta_{n,m})$ must remain stationary. Otherwise, the diameter $\operatorname{diam}(E[\Phi_m])$ will decrease strictly. In particular, there exists an agent i such that $\overline{\theta}_i \in \partial(\operatorname{co}\{\overline{\theta}_1, \dots, \overline{\theta}_n\})$, with $\overline{\theta}_i$ determining the diameter of $\operatorname{co}\{\overline{\theta}_1, \dots, \overline{\theta}_n\}$, for which there is an agent j connected to i an infinite number of times. Recall that the algorithm update law makes $|\theta_{j,m} - \theta_{i,m}| < \pi$ hold for all $m \geq 0$.

The measurement model error assumes that $\theta^i_{j,m}$ is uniformly distributed over a disk of radius σ centered at $\theta_{j,m}$. Let $\alpha >$ and $0 < \sigma - \alpha < \pi$ then, with probability one, there is an infinite number of time instants for which $|\theta^i_{j,m} - \theta_{i,m}| \leq \pi - \sigma + \alpha < \pi$. To see this, observe that:

$$\mathbb{P}(|\theta^i_{j,m} - \theta_{i,m}| \leq \pi - \sigma + \alpha) = \begin{cases} 1, & \text{if } |\theta_{j,m} - \theta_{i,m}| \leq \pi - 2\sigma, \\ \frac{1}{\pi\sigma^2} \int_D d\,\mathbb{P} > 0, & \text{otherwise,} \end{cases}$$

where $D = [\theta_{j,m} - \sigma, \theta_{j,m} + \sigma] \cap [\theta_{i,m} - \pi + \sigma - \alpha, \theta_{i,m} + \pi - \sigma + \alpha] = [\theta_{j,m} - \sigma, \theta_{i,m} + \pi - \sigma + \alpha]$. Since $\theta_{i,m} - \pi + \sigma - \alpha < \theta_{j,m} + \sigma$ is equivalent to $\theta_{i,m} - \theta_{j,m} < \pi + \alpha$, then $D \neq \emptyset$. Denote by $\mathbb{P}(|\theta_{i,m} - \theta^i_{j,m}| \leq r - \sigma + \alpha) = a > 0$ and let us compute the probability of the event $A = \{\exists\, m > 0 |\ |\theta_{i,m} - \theta^i_{j,m}| \leq r - \sigma + \alpha\}$. In fact, we can write A as the disjoint union of events A_m, $m \geq 0$:

$$A = \cup_{m=0}^{\infty} A_m = \cup_{m=0}^{\infty} \{|\theta_{i,s} - \theta^i_{j,s}| > r - \sigma + \alpha, \ \forall s \leq m - 1, \text{ and } |\theta_{i,m} - \theta^i_{j,m}| \leq r - \sigma + \alpha\}.$$

In this way,

$$P(A) = \sum_{m=0}^{\infty} P(A_m) = \sum_{m=0}^{\infty} a(1-a)^m = \frac{a}{1-(1-a)} = 1.$$

Therefore, there exists a time $m > 0$ such that agent i makes a measurement of agent $j \in \mathcal{N}_{i,m}$ and $|\theta_{i,m} - \theta^i_{j,m}| < \pi$. Thus, we have that $C_i(\theta_{i,m}\mathcal{S}_{i,m}) \neq \emptyset$ and $\theta_{i,m+1} \in \operatorname{co}(\theta_{i,m}, \theta^i_{j,m} | j \in \mathcal{N}_{j,m}) \setminus \{\theta_{i,m}\} \cup \{\theta^i_{j,m} | j \in \mathcal{N}_{j,m}\}$. This implies that $\theta_{i,m}$ strictly moves inside the convex hull of the θ_k, with probability one.

In all, we have proven that w.p.1 $\operatorname{diam}(E[\overline{\Phi}_m]) < \operatorname{diam}(E[\overline{\Phi}])$ for some $m > 0$. Since $\operatorname{diam}(E[\Phi_m]) = \operatorname{diam}(E[\overline{\Phi}])$ for all $m \geq 0$ by the invariance property of the omega limit set, we obtain a contradiction. ■

V. Simulations

Figure 2 shows a run of the diameter of a multi-oscillator system that evolves under the modified Kuramoto dynamics. The proximity graph considered is the $\mathcal{G}_{\text{disk}}(r)$ proximity graph, for $r = 1$, which is subject to random failures every 4 time steps. The number of oscillators is $n = 15$, and $\sigma = 0.5$. As it can be seen in this figure the algorithm behaves as expected from the analysis. Oscillators converge to a practical stability ball that wanders in space. This behavior is representative of what we have seen in many repeated simulations with different initial conditions and relations $r/\sigma > 1$, as long as connectivity is guaranteed periodically.

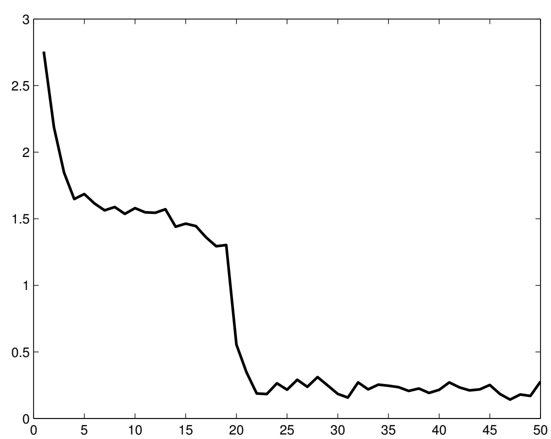

Fig. 1. Diameter evolution of a group of oscillators implementing a modified Kuramoto system with the $\mathcal{G}_{\text{disk}}(r)$.

The size of the stability ball is very much affected by the sparsity of the connectivity graph. Figure **??** presents a run of the diameter of a multi-oscillator system that is connected through the Delaunay graph in $\mathbb{R}$. In this particular simulation, $n = 15$ and $\sigma = 0.05$. We have observed in simulations that with this type of graph, the size of the stability ball is typically much larger and increases with the addition of more agents.

Fig. 2. Diameter evolution of a group of oscillators implementing a modified Kuramoto system with the Delaunay graph.

Simulations also show that even for initial conditions not satisfying the condition $|\theta_i(0) - \theta_j(0)| \leq \pi$, convergence is possible provided there is frequent multiagent connectivity. Notice that the states $\theta_1, \ldots, \theta_n$ such that $\theta_i = \theta_j + k_{ij}\pi$, with $k_{ij} \in \{0, 1\}$ constitute equilibrium points of the deterministic Kuramoto system. However, except for the case of $k_{ij} = 0$ for all i, j, all these states are unstable. Any perturbation will bring the system out of these bad equilibria. Therefore, it is very reasonable to expect that the current analysis for the modified Kuramoto system can be carried over to full sphere, $\mathbb{S}$.

VI. Conclusions

This paper presents some convergence results for multiagent systems subject to noise. The analysis makes use of a stochastic analogue of the LaSalle invariance principle for switching systems. Provided periodic connectivity of the multiagent system occurs, we can conclude that the expected value of the diameter of the multiagent system converges to zero with probability one. Future work will be devoted to study the effect of random graphs and multiagent connections in probability. We will also investigate the possible extension of the results to the sphere and its consequences for the modified Kuramoto system.

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