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# $\mathcal{L}_1$ Adaptive Output Feedback Controller for Non Strictly Positive Real Multi-Input Multi-Output Systems in the Presence of Unknown Nonlinearities

Chengyu Cao and Naira Hovakimyan

***Abstract*— This paper presents an extension of the $\mathcal{L}_1$ adaptive output feedback controller to Multi-input Multi-output (MIMO) systems in the presence of nonlinear time-varying uncertainties without restricting the rate of their variation. As compared to earlier results in this direction, a new piece-wise continuous adaptive law is introduced along with a low-pass filtered control signal that allows for achieving arbitrarily close tracking of the input and the output signals of a reference system, the transfer function of which is not required to be strictly positive real (SPR). Stability of this reference system is proved using small-gain type argument. The performance bounds between the closed-loop reference system and the closed-loop $\mathcal{L}_1$ adaptive system can be rendered arbitrarily small by appropriate selection of the underlying filter and by reducing the time-step of integration. Simulations verify the theoretical findings.**

## I. INTRODUCTION

This paper extends the results of [1] to multi-input multi-output (MIMO) systems that do not verify the SPR condition for their input-output transfer function. Similar to [1], the $\mathcal{L}_\infty$-norms of both input/output error signals between the closed-loop adaptive system and the reference system can be rendered arbitrarily small by reducing the step-size of integration. The key difference from the earlier results in [2], [3] is the new piece-wise continuous adaptive law. The adaptive control is defined as output of a low-pass filter, resulting in a continuous signal despite the discontinuity of the adaptive law. For a brief literature review refer to [1]–[3].

The paper is organized as follows. Section II gives the problem formulation. In Section III, the closed-loop reference system is introduced. In Section IV, some preliminary results are developed towards the definition of the $\mathcal{L}_1$ adaptive controller. In Section V, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform performance bounds are presented in Section VI. In Section VII, simulation results are presented, while Section VIII concludes the paper. The small-gain theorem and some basic definitions from linear systems theory used throughout the paper are given in Appendix. Unless otherwise mentioned, $||\cdot||$ will be used for the 2-norm of the vector. Finally, for a given matrix $M \in \mathbb{R}^{m\times n}$, we let $\|M\|_a = \sqrt{\lambda_{\max}(M^\top M)}$ . We notice that $\|Mx\| \leq \|M\|_a \|x\|$ , for any $x \in \mathbb{R}^n$. Definitions of $\mathcal{L}_\infty$-norm of signals and $\mathcal{L}_1$-norm of systems can be found in [2], [3].

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Chengyu Cao is with University of Connecticut, Storrs, CT 06269; ccao@engr.uconn.edu

N. Hovakimyan are with University of Illinois at Urbana-Champaign, Urbana, IL 61801; nhovakim@illinois.edu

## II. PROBLEM FORMULATION

Consider the following MIMO system:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + u(t) + f(t, y(t)), \quad x(0) = x_0, \quad (1)\\ y(t) &= Cx(t), \quad y_0 = y(0) = Cx_0 , \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the system state (not measured), $u(t) \in \mathbb{R}^n$ is the input, $y(t) \in \mathbb{R}^m$ is regulated (measured) output, $A_m \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ are given matrices, with $A_m$ being Hurwitz and $C$ being full row rank, $f(t, y) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ is an unknown nonlinear map, subject to following assumptions.

*Assumption 1:* [Semiglobal Lipschitz condition] For any $\delta > 0$, there exist $L_\delta > 0$ and $B > 0$ such that $\|f(t, y) - f(t, \bar{y})\|_\infty \leq L_\delta \|y-\bar{y}\|_\infty$ , $\|f(t, 0)\|_\infty \leq B$ , for all $\|y\|_\infty \leq \delta$ and $\|\bar{y}\|_\infty \leq \delta$, uniformly in $t \geq 0$.

*Assumption 2:* [Semiglobal uniform boundedness of partial derivatives] For any $\delta > 0$, the partial derivatives of $f(t, y)$ w.r.t. $t$ and $y$ are piece-wise continuous and bounded for any $\|y\|_\infty \leq \delta$.

*Remark 1:* To streamline the subsequent derivations, in (1) the input matrix of the system has been set to identity. However, any full rank matrix $B$ can be straightforwardly accommodated in the design below.

The control objective is to design an adaptive controller to ensure that, for a given bounded piece-wise continuous reference input $r(t) \in \mathbb{R}^n$, $y(t)$ tracks the response $y_{des}(t) \in \mathbb{R}^n$ of the following desired system:

$$\begin{aligned} \dot{x}_{des}(t) &= A_m x_{des}(t) + r(t) , \\ y_{des}(t) &= Cx_{des}(t) , \quad x_{des}(0) = \hat{x}_0 , \end{aligned} \qquad (2)$$

where $x_{des}(t) \in \mathbb{R}^n$, and $\hat{x}_0$ is such that $C\hat{x}_0 = y_0$ . Obviously, $\hat{x}_0$ is not uniquely defined.

## III. CLOSED-LOOP REFERENCE SYSTEM

Consider the following closed-loop reference system:

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + u_{ref}(t) + f(t, y_{ref}(t)) , \quad (3)\\ y_{ref}(t) &= Cx_{ref}(t) , \quad x_{ref}(0) = \hat{x}_0 , \\ u_{ref}(s) &= r(s) - F(s)\sigma_{ref}(s) , \qquad (4) \end{aligned}$$

where $\sigma_{ref}(s)$ is the Laplace transformation of $f(t, y_{ref}(t))$, and $F(s)$ is a low-pass filter with its DC gain $F(0) = 1$. Let $H(s) = C(s\mathbb{I} - A_m)^{-1}$ . For the proof of stability and uniform performance bounds the choice of $F(s)$ needs to ensure that there exists positive $\rho_r$ such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \|H(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + \|\hat{x}_0\|_\infty)}{\rho_r L_{\rho_r} + B}, \quad (5)$$

where $G(s) = H(s)(1 - F(s))$, and positive $\gamma_1$ such that

$$\gamma_1(1 - L_\rho \|G(s)\|_{\mathcal{L}_1}) > \|G(s)\|_{\mathcal{L}_1} \|\hat{x}_0 - x_0\|_\infty, \quad (6)$$

where

$$\rho = \rho_r + \gamma_1. \quad (7)$$

Since $H(s)$ is strictly proper and stable, $G(s) = H(s)(1 - F(s))$ is also strictly proper and stable.

*Remark 2:* The condition (5) is equivalent to the existence of $\rho_r$ such that $\rho_r(1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}) > \|H(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + \|\hat{x}_0\|_\infty) + B\|G(s)\|_{\mathcal{L}_1}$, which can always be satisfied if $\|G(s)\|_{\mathcal{L}_1}$ is small enough. Increasing the bandwidth of $F(s)$ will ensure that $\|G(s)\|_{\mathcal{L}_1}$ can be rendered arbitrarily small. We notice that for (5) to hold one needs to ensure that

$$L_{\rho_r} \|G(s)\|_{\mathcal{L}_1} < 1. \quad (8)$$

Since $L_\rho$ is continuous w.r.t. $\rho$, it follows from (8) that there always exists $\gamma_1$ such that $L_{\rho_r+\gamma_1}\|G(s)\|_{\mathcal{L}_1} < 1$. Thus, the condition in (6) can be verified by reducing $\|G(s)\|_{\mathcal{L}_1}$, which further implies that the constant $\gamma_1$ satisfying (6) can assume arbitrarily small values.

*Lemma 1:* If $F(s)$ verifies the condition in (5) and $\|y_0\|_{\mathcal{L}_\infty} < \rho_r$, then

$$\|y_{ref}\|_{\mathcal{L}_\infty} \quad < \quad \rho_r, \quad (9)$$

where $\rho_r$ is introduced in (5).

## IV. Preliminaries for the Main Result

Since $A_m$ is Hurwitz, there exists $P = P^\top > 0$ that satisfies the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q, \quad Q > 0$. From the properties of $P$ it follows that there exits non-singular $\sqrt{P}$ such that $P = (\sqrt{P})^\top \sqrt{P}$. Given the matrix $C(\sqrt{P})^{-1}$, let $D$ be a $(n-m)\times n$ matrix that contains the null space of $C(\sqrt{P})^{-1}$:

$$D(C(\sqrt{P})^{-1})^\top = 0, \quad (10)$$

and further let $\Lambda = \begin{bmatrix} C \\ D\sqrt{P} \end{bmatrix}$.

*Lemma 2:* For any $\xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$, where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{n-m}$, there exist positive definite $P_1 \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that $\xi^\top(\Lambda^{-1})^\top P \Lambda^{-1}\xi = y^\top P_1 y + z^\top P_2 z$.

Let $T$ be any positive constant, $\mathbf{I}_{m\times m} \in \mathbb{R}^{m\times m}$ be the identity matrix, and $\mathbf{0}_{m\times(n-m)} \in \mathbb{R}^{m\times(n-m)}$ be a zero matrix. Let $\phi(T) \in \mathbb{R}^{m\times(n-m)}$ be a matrix, which consists of $m+1$ to $n$ columns of $[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1} T}$ and let

$$\kappa(T) = \int_0^T \|[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1}(T-\tau)}\Lambda\|_a d\tau. \quad (11)$$

Further, define

$$\begin{aligned} \varsigma(T) &= \|\phi(T)\|_a \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T)\Delta, \\ \alpha &= \max\{\lambda_{\max}(\Lambda^{-\top} P \Lambda^{-1}) \left(\frac{2\Delta\|\Lambda^{-\top}P\|_a}{\lambda_{\min}(\Lambda^{-\top} Q \Lambda^{-1})}\right)^2, \\ &\quad \lambda_{\max}(P_2)\|\Lambda(\hat{x}_0 - x_0)\|^2\} \end{aligned} \quad (12)$$

where $\Delta = \rho L_\rho + B$. Letting $[\mathbf{I}_{m\times m}\ \ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1} t} = [\eta_1(t)\ \ \eta_2(t)]$, where $\eta_1(t) \in \mathbb{R}^{m\times m}$ is comprised of the first $m$ columns of $[\mathbf{I}_{m\times m}\ \ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1} t}$ and $\eta_2(t) \in \mathbb{R}^{m\times(n-m)}$ contains the remaining $(n-m)$ columns, we introduce the following functions

$$\beta_1(T) = \max_{t\in[0,\,T]} \|\eta_1(t)\|_a, \quad \beta_2(T) = \max_{t\in[0,\,T]} \|\eta_2(t)\|_a. \quad (13)$$

Further, let $\Phi(T)$ be the $n \times n$ matrix

$$\begin{aligned} \Phi(T) &= \int_0^T \mathrm{e}^{\Lambda A_m \Lambda^{-1}(T-\tau)}\Lambda d\tau \\ &= \Lambda A_m^{-1}\left(\mathrm{e}^{A_m T} - \mathbb{I}\right), \end{aligned} \quad (14)$$

$$\beta_3(T) = \max_{t\in[0,\,T]} \eta_3(t), \quad \beta_4(T) = \max_{t\in[0,\,T]} \eta_4(t), \quad (15)$$

where

$$\begin{aligned} \eta_3(t) &= \int_0^t \|[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda\Phi^{-1}(T) \\ &\qquad \mathrm{e}^{\Lambda A_m \Lambda^{-1} T}[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]^\top\|_a d\tau, \\ \eta_4(t) &= \int_0^t \|[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m \Lambda^{-1}(t-\tau)}\Lambda\|_a d\tau. \end{aligned}$$

Finally, let

$$\begin{aligned} \gamma_0(T) &= \beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \\ &\quad \beta_3(T)\varsigma(T) + \beta_4(T)\Delta. \end{aligned} \quad (16)$$

*Lemma 3:* The following limiting relationship is true: $\lim_{T\to 0} \gamma_0(T) = 0$.

## V. $\mathcal{L}_1$ Adaptive Output Feedback Controller

We consider the following state predictor (or passive identifier):

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + u(t) + \hat{\sigma}(t), \hat{y}(t) = C\hat{x}(t), \hat{x}(0) = \hat{x}_0, \quad (17)$$

where $\hat{\sigma}(t) \in \mathbb{R}^n$ is the vector of adaptive parameters. Letting $\tilde{y}(t) = \hat{y}(t) - y(t)$, the update law for $\hat{\sigma}(t)$ is given by

$$\begin{aligned} \hat{\sigma}(t) &= \hat{\sigma}(iT), \quad t \in [iT, (i+1)T) \\ \hat{\sigma}(iT) &= -\Phi^{-1}(T)\mu(iT), \quad i = 0, 1, 2, \cdots, \end{aligned} \quad (18)$$

where $\Phi(T)$ is defined in (14), and $\mu(iT) = \mathrm{e}^{\Lambda A_m \Lambda^{-1} T}\begin{bmatrix} \tilde{y}(iT) \\ \mathbf{0}_{(n-m)\times 1}\end{bmatrix}$, $i = 0, 1, 2, 3, \cdots$. The control signal is the output of the low-pass filter:

$$u(s) = r(s) - F(s)\hat{\sigma}(s). \quad (19)$$

The $\mathcal{L}_1$ adaptive controller consists of (17), (18) and (19), subject to the condition in (5)-(6).

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$. The error dynamics between (1) and (17) are

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + \hat{\sigma}(t) - f(t, y(t)), \quad (20)$$
$$\tilde{y}(t) = C\tilde{x}(t), \quad \tilde{x}(0) = \hat{x}_0 - x_0. \quad (21)$$

*Lemma 4:* Let $e(t) = y(t) - y_{ref}(t)$. If

$$\|y_t\|_{\mathcal{L}_\infty} \leq \rho, \quad (22)$$

where $\rho$ is defined in (6), then

$$\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|F(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|\hat{x}_0 - x_0\|_\infty}{1 - L_\rho\|G(s)\|_{\mathcal{L}_1}}. \quad (23)$$

## VI. Analysis of $\mathcal{L}_1$ Adaptive Controller

Consider the state transformation $\tilde{\xi} = \Lambda\tilde{x}$. It follows from (21) that

$$\dot{\tilde{\xi}}(t) = \Lambda A_m\Lambda^{-1}\tilde{\xi}(t) + \Lambda\hat{\sigma}(t) - \Lambda\sigma(t), \quad (24)$$
$$\tilde{y}(t) = [\tilde{\xi}_1(t)\cdots\tilde{\xi}_m(t)]^\top, \quad \tilde{\xi}(0) = \Lambda(\hat{x}_0 - x_0), \quad (25)$$

with $\tilde{y}(0) = 0$.

*Theorem 1:* Given the system in (1) and the $\mathcal{L}_1$ adaptive controller in (17), (18), (19) subject to (5), if we choose $T$ to ensure

$$\gamma_0(T) < \bar{\gamma}, \quad (26)$$

where

$$\bar{\gamma} = \frac{\gamma_1(1 - L_\rho\|G(s)\|_{\mathcal{L}_1}) - \|G(s)\|_{\mathcal{L}_1}\|\hat{x}_0 - x_0\|_\infty}{\|F(s)\|_{\mathcal{L}_1}}, \quad (27)$$

and $\gamma_1$ is an arbitrary positive constant introduced in (7), then

$$\|\tilde{y}\|_{\mathcal{L}_\infty} < \bar{\gamma} \quad (28)$$
$$\|y - y_{ref}\|_{\mathcal{L}_\infty} < \gamma_1, \quad (29)$$
$$\|u - u_{ref}\|_{\mathcal{L}_\infty} < \gamma_2. \quad (30)$$

with $\gamma_2 = L_\rho\|F(s)\|_{\mathcal{L}_1}\gamma_1 + \|F(s)\|_{\mathcal{L}_1}\bar{\gamma}$.

Thus, if one omits the initialization error of the state predictor, the tracking error between $y(t)$ and $y_{ref}(t)$, as well between $u(t)$ and $u_{ref}(t)$, is uniformly bounded by a constant proportional to $T$. The transient due to nonzero initialization error can be reduced by increasing the bandwidth of $F(s)$, and arbitrary improvement of the tracking performance can be further achieved by uniformly reducing $T$.

*Remark 3:* Notice that the parameter $T$ is the fixed time-step in the definition of the adaptive law. The adaptive parameters in $\hat{\sigma}(t) \in \mathbb{R}^n$ take constant values during $[iT, (i+1)T)$ for every $i = 0, 1, \cdots$. Reducing $T$ imposes hardware (CPU) requirements, and Theorem 1 further implies that the performance limitations are consistent with the hardware limitations. This in turn is consistent with the earlier results in Refs. [1], [2], where improvement of the transient performance was achieved by increasing the adaptation rate in the continuous-time adaptive laws.

*Remark 4:* We notice that the following *ideal* control signal $u_{ideal}(t) = r(t) - \sigma_{ref}(t)$ is the one that leads to desired system response in (2) by canceling the uncertainties exactly. Thus, the reference system in (3)-(4) has a different response as compared to the *ideal* one. It only cancels the uncertainties within the bandwidth of $C(s)$, which can be selected compatible with the control channel specifications. This is exactly what one can hope to achieve with any feedback in the presence of uncertainties.

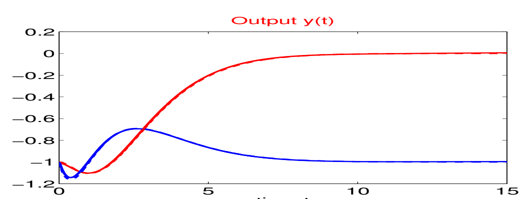
*Remark 5:* Consider the system

$$\dot{x}(t) = Ax(t) + u(t), \quad y(t) = Cx(t), \quad (31)$$

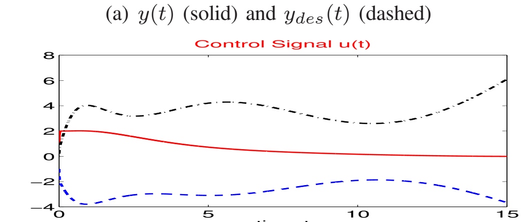
where $A$ is unknown. If the system is observable, then there exists $L$ and a Hurwitz $A_m$ such that $A_m = A - LC$. Hence, the system in (31) can be transformed into $\dot{x}(t) = A_m x(t) + u(t) + Ly(t)$, $y(t) = Cx(t)$, which is a particular case of the system in (1). Thus, if the system is known to be output feedback observable, the $\mathcal{L}_1$ adaptive controller can be applied to ensure guaranteed transient and steady-state performance.

## VII. Simulations

Consider the system $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} + u(t) + f(t, y(t))$, $y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)$, $y_0 = [-1 \quad -1]^\top$, where $f(t, y(t))$ is unknown nonlinear function. Let the desired system be: $\dot{x}_{des}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} + r(t)$, $y_{des}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_{des}(t)$, $x_{des}(0) = \hat{x}(0) = [-1 \quad -1 \quad 0]^\top$, where $r(t)$ is given reference input and $x_{des}(0)$ is chosen such that $Cx_{des}(0) = y_0 = [-1 \quad -1]^\top$. We consider the $\mathcal{L}_1$ adaptive output feedback controller defined via (17), (18) and (19), where $F(s) = \frac{100}{s+100}$, $T = 10^{-4}$.

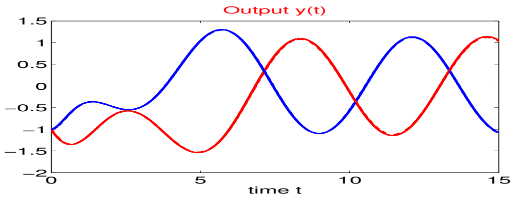



(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)

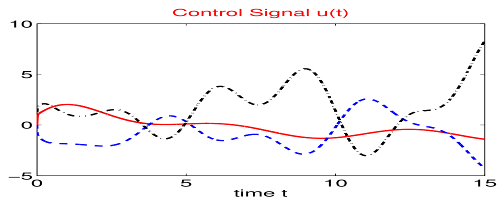



(b) Time-history of $u(t)$

Fig. 1. Performance for $r(t) = [1 \quad -1 \quad 0]$ and $f(t, y(t)) = [y_1(t) + \sin(0.1t) \quad y_2^2(t) + \cos(0.3t)]^\top \quad \sin(y_1(t)))$.

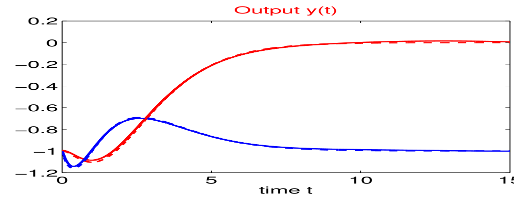



(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)

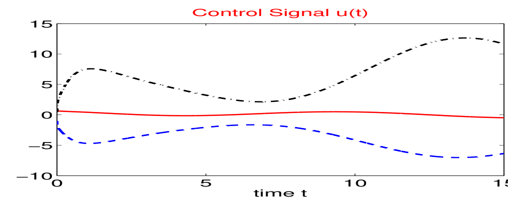



(b) Time-history of $u(t)$

Fig. 2. Performance for $r(t) = [\sin(t) \quad -\sin(t) \quad \cos(t)]$ and $f(t, y(t)) = [y_1(t) + \sin(0.1t) \quad y_2^2(t) + \cos(0.3t)]^\top \quad \sin(y_1(t))$.




(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)




(b) Time-history of $u(t)$

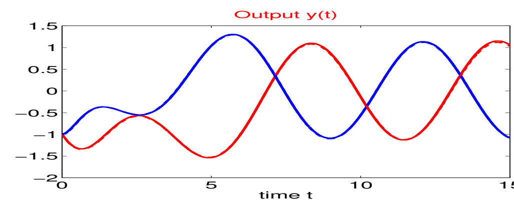
Fig. 3. Performance for $r(t) = [1 \quad -1 \quad 0]$ and $f(t, y(t)) = [\exp(y_1(t)) + 0.5\sin(0.5t) \quad y_2^2(t) + \cos(0.5t) \quad \sin(y_1(t))]^\top$.

First, let $f(t, y(t)) = \begin{bmatrix} y_1(t) + \sin(0.1t) \\ y_2^2(t) + \cos(0.3t) \\ \sin(y_1(t)) \end{bmatrix}$, and $x(0) = [-1 \quad -1 \quad -1]^\top$. The simulation results of $\mathcal{L}_1$ adaptive controller are shown in Figs 1(a)-1(b) for $r(t) = [1 \quad -1 \quad 0]^\top$. The simulation results in Figs 2(a)-2(b) correspond to $r(t) = [\sin(t) \quad -\sin(t) \quad \cos(t)]^\top$. We notice that $y(t)$ and $y_{des}(t)$ are almost the same for all $t \geq 0$, including the transient phase.

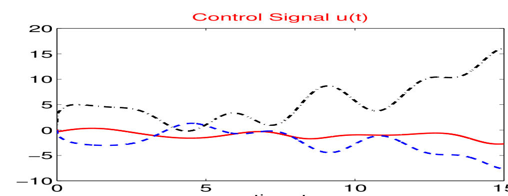
Next, we consider a different nonlinear uncertainty $f(t, y(t)) = \begin{bmatrix} e^{y_1(t)} + 0.5\sin(0.5t) \\ y_2^2(t) + \cos(0.5t) \\ \sin(y_1(t)) \end{bmatrix}$. The simulation results of $\mathcal{L}_1$ adaptive controller are shown in Figs 3(a)-3(b) for $r(t) = [1 \quad -1 \quad 0]^\top$. The simulation results in Figs 4(a)-4(b) correspond to $r(t) = [\sin(t) \quad -\sin(t) \quad \cos(t)]^\top$. We note that $y(t)$ and $y_{des}(t)$ are almost the same, independent of different nonlinearities. The $\mathcal{L}_1$ adaptive controller ensures desired tracking performance in the presence of unknown nonlinearities for different reference inputs without any re-tuning.

## VIII. Conclusions

We presented the $\mathcal{L}_1$ adaptive output feedback controller for MIMO reference systems that do not verify the SPR condition for their input-output transfer function. The new piece-wise constant adaptive law along with low-pass filtered control signal ensures uniform performance bounds for system's both input/output signals simultaneously. The performance bounds can be systematically improved by reducing the integration time-step.




(a) $y(t)$ (solid) and $y_{des}(t)$ (dashed)




(b) Time-history of $u(t)$

Fig. 4. Performance for $r(t) = [\sin(t) \quad -\sin(t) \quad \cos(t)]$ and $f(t, y(t)) = [\exp(y_1(t)) + 0.5\sin(0.5t) \quad y_2^2(t) + \cos(0.5t) \quad \sin(y_1(t))]^\top$.

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## IX. Appendix

**Proof of Lemma 1.** It follows from (3)-(4) that

$$y_{ref}(s) = H(s)(r(s) + \hat{x}_0 + (1 - F(s))\sigma_{ref}(s)). \tag{32}$$

If (9) is not true, since $\|y_{ref}(0)\|_\infty = \|Cx_{ref}(0)\|_\infty < \rho_r$ and $y_{ref}(t)$ is continuous, there exists $t$ such that

$$\|y_{ref_t}\|_{\mathcal{L}_\infty} \leq \rho_r, \tag{33}$$

$$y_{ref}(t) = \rho_r. \tag{34}$$

Using Assumption 1 and the upper bound in (33), we arrive at the following upper bound

$$\|\sigma_{ref_t}\|_{\mathcal{L}_\infty} \le L_{\rho_r}\|y_{ref_t}\|_{\mathcal{L}_\infty} + B\,. \tag{35}$$

Substituting (35) into (32), and noticing that $\|r_t\|_{\mathcal{L}_\infty} \le \|r\|_{\mathcal{L}_\infty}$, we obtain $\|y_{ref_t}\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}(L_{\rho_r}\rho_r + B) + \|H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H(s)\|_{\mathcal{L}_1}\|\hat{x}_0\|_\infty$. The condition in (5) can be solved for $\rho_r$ to obtain the following upper bound $\|G(s)\|_{\mathcal{L}_1}L_{\rho_r}\rho_r + \|H(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + \|\hat{x}_0\|_\infty) + \|G(s)\|_{\mathcal{L}_1}B < \rho_r$, which implies that $\|y_{ref_t}\|_{\mathcal{L}_\infty} < \rho_r$, and contradicts (34). This proves (9). □

**Proof of Lemma 2.** Using $P = (\sqrt{P})^\top\sqrt{P}$, one can write $\xi^\top(\Lambda^{-1})^\top P\Lambda^{-1}\xi = \xi^\top(\sqrt{P}\Lambda^{-1})^\top(\sqrt{P}\Lambda^{-1})\xi$. We notice that $\Lambda(\sqrt{P})^{-1} = \begin{bmatrix} C(\sqrt{P})^{-1} \\ D \end{bmatrix}$. Let $Q_1 = (C(\sqrt{P})^{-1})(C(\sqrt{P})^{-1})^\top, Q_2 = DD^\top$. From (10) we have $(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$. Non-singularity of $\Lambda$ and $\sqrt{P}$ implies that $(\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top$ is non-singular, and therefore $Q_1$ and $Q_2$ are also non-singular. Hence, $(\sqrt{P}\Lambda^{-1})^\top(\sqrt{P}\Lambda^{-1}) = (\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^\top)^{-1} = (\Lambda(\sqrt{P})^{-1})^{-\top}(\sqrt{P}\Lambda^{-1}) = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix}$. Denoting $P_1 = Q_1^{-1}$ and $P_2 = Q_2^{-1}$, completes the proof. □

**Proof of Lemma 3.** Notice that since $\beta_1(T)$, $\beta_3(T)$, $\Delta$ and $\alpha$ are bounded, it is sufficient to prove that

$$\lim_{T\to 0}\varsigma(T) = 0\,, \tag{36}$$

$$\lim_{T\to 0}\beta_2(T) = 0\,, \tag{37}$$

$$\lim_{T\to 0}\beta_4(T) = 0\,. \tag{38}$$

Since $\lim_{T\to 0}[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]e^{\Lambda A_m\Lambda^{-1}T} = [\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]$, then $\lim_{T\to 0}\phi(T) = \mathbf{0}_{m\times(n-m)}$, which implies $\lim_{T\to 0}\|\phi(T)\|_a = 0$. Further, it follows from the definition of $\kappa(T)$ in (11) that $\lim_{T\to 0}\kappa(T) = 0$. Since $\Delta$ and $\alpha$ are bounded, $\lim_{T\to 0}\varsigma(T) = 0$, which proves (36). Since $\eta_2(t)$ is continuous, it follows from (13) that $\lim_{T\to 0}\beta_2(T) = \lim_{t\to 0}\|\eta_2(t)\|_a$. Since $\lim_{t\to 0}[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]e^{\Lambda A_m\Lambda^{-1}t} = [\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]$, we have $\lim_{t\to 0}\|\eta_2(t)\| = 0$, which proves (37). Similarly $\lim_{T\to 0}\beta_4(T) = \lim_{t\to 0}\|\eta_4(t)\| = 0$, which proves (38). Boundedness of $\Delta$, $\alpha$ and $\beta_3(T)$ implies $\lim_{T\to 0}\Big(\beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T)\varsigma(T) + \beta_4(T)\Delta\Big) = 0$, which completes the proof. □

**Proof of Lemma 4.** Let $\tilde{\sigma}(s) = \hat{\sigma}(s) - \sigma(s)$, where $\sigma(t) = f(t, y(t))$. It follows from (21) that

$$\tilde{y}(s) = H(s)\tilde{\sigma}(s) + H(s)(\hat{x}_0 - x_0)\,. \tag{39}$$

It follows from (19) that

$$u(s) = r(s) - F(s)\sigma(s) - F(s)\tilde{\sigma}(s)\,, \tag{40}$$

and the system in (1) consequently takes the form: $y(s) = H(s)\Big(r(s) + x_0 + (1 - F(s))\sigma(s) - F(s)\tilde{\sigma}(s)\Big)$. Using the expression for $y_{ref}(s)$ from (32), and letting $d_e(s)$ be the Laplace transform of $d_e(t) = f(t, y(t)) - f(t, y_{ref}(t))$, one can derive

$$e(s) = H(s)\left((1 - F(s))d_e(s) + x_0 - \hat{x}_0 - F(s)\tilde{\sigma}(s)\right). \tag{41}$$

It follows from Assumption 1 and (22) that $\|d_{e_t}\|_{\mathcal{L}_\infty} \le L_\rho\|e\|_\infty$. Hence, it follows from (39) and (41) that $\|e_t\|_{\mathcal{L}_\infty} \le L_\rho\|G(s)\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty} + \|F(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|\hat{x}_0 - x_0\|_\infty$, which proves (23). □

**Proof of Theorem 1.** The proof will be done by contradiction. Assume that (29) is not true. Then, since $\|y(0) - y_{ref}(0)\|_\infty = 0 \le \gamma_1$, $y(t)$, $y_{ref}(t)$, are continuous, there exists $\tau \ge 0$ such that

$$\|y(\tau) - y_{ref}(\tau)\|_\infty = \gamma_1\,, \tag{42}$$

while

$$\|(y - y_{ref})_\tau\|_{\mathcal{L}_\infty} \le \gamma_1\,. \tag{43}$$

At first, we will prove that if (43) holds, then

$$\|\tilde{y}_\tau\|_{\mathcal{L}_\infty} \le \bar{\gamma}\,. \tag{44}$$

We prove the bound in (44) by a contradiction argument. Since $\tilde{y}(0) = 0$ and $\tilde{y}(t)$ is continuous, then assuming the opposite implies that there exists $t' \le \tau$ such that $\|\tilde{y}(t)\| < \bar{\gamma}, \quad \forall\, 0 \le t < t'$, $\|\tilde{y}(t')\| = \bar{\gamma}$, which leads to

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} = \bar{\gamma}\,. \tag{45}$$

It follows from (43) that $\|y_{t'}\|_{\mathcal{L}_\infty} \le \rho$, and hence Assumption 1 implies that

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \le \Delta\,. \tag{46}$$

It follows from (24) that

$$\begin{aligned}\tilde{\xi}(iT + t) &= e^{\Lambda A_m\Lambda^{-1}t}\tilde{\xi}(iT) + \int_{iT}^{iT+t} e^{\Lambda A_m\Lambda^{-1}(iT+t-\tau)} \\ &\quad \Lambda\hat{\sigma}(iT)d\tau - \int_{iT}^{iT+t} e^{\Lambda A_m\Lambda^{-1}(iT+t-\tau)}\Lambda\sigma(\tau)d\tau \\ &= e^{\Lambda A_m\Lambda^{-1}t}\tilde{\xi}(iT) + \int_0^t e^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\hat{\sigma}(iT)d\tau \\ &\quad - \int_0^t e^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\sigma(iT + \tau)d\tau\,.\end{aligned} \tag{47}$$

Since $\tilde{\xi}(iT) = \begin{bmatrix} \tilde{y}(iT) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{z}(iT) \end{bmatrix}$, it follows from (47) that

$$\tilde{\xi}(iT + t) = \chi(iT + t) + \zeta(iT + t)\,, \tag{48}$$

where $\chi(iT + t) = e^{\Lambda A_m\Lambda^{-1}t}\begin{bmatrix} \tilde{y}(iT) \\ 0 \end{bmatrix} + \int_0^t e^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\hat{\sigma}(iT)d\tau$, $\zeta(iT+t) = e^{\Lambda A_m\Lambda^{-1}t}\begin{bmatrix} 0 \\ \tilde{z}(iT) \end{bmatrix} - \int_0^t e^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\sigma(iT + \tau)d\tau$. In what follows, we prove that for all $iT \le t'$ one has

$$\|\tilde{y}(iT)\| \le \varsigma(T)\,, \tag{49}$$

$$\tilde{z}^\top(iT)P_2\tilde{z}(iT) \le \alpha\,, \tag{50}$$

where $\varsigma(T)$ and $\alpha$ are defined in (12). Since $\tilde{\xi}_1(0) = 0$, it is straightforward that $\|\tilde{y}(0)\| \le \varsigma(T)$. We further note that $\tilde{z}^\top(0)P_2\tilde{z}(0) \le \lambda_{\max}(P_2)\|\tilde{z}(0)\|^2 \le \lambda_{\max}(P_2)\|\tilde{\xi}(0)\|^2 \le \lambda_{\max}(P_2)\|\Lambda(\hat{x}_0 - x_0)\|^2 \le \alpha$. For any $(j + 1)T \le t'$, we will prove that if

$$\|\tilde{y}(jT)\| \le \varsigma(T)\,, \tag{51}$$

$$\tilde{z}^\top(jT)P_2\tilde{z}(jT) \le \alpha\,, \tag{52}$$

then (51)-(52) hold for $j + 1$ too. Hence, (49)-(50) hold for all $iT \le t'$.

Assume (51)-(52) hold for $j$, and in addition, $(j + 1)T \le t'$. It follows from (48) that $\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T)$, where

$$\chi((j + 1)T) = e^{\Lambda A_m\Lambda^{-1}T}\begin{bmatrix} \tilde{y}(jT) \\ 0 \end{bmatrix} + \int_0^T e^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda\hat{\sigma}(jT)d\tau\,, \tag{53}$$

$$\zeta((j + 1)T) = e^{\Lambda A_m\Lambda^{-1}T}\begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} - \int_0^T e^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda\sigma(jT + \tau)d\tau\,. \tag{54}$$

Substituting the adaptive law from (18) in (53), we have

$$\chi((j+1)T) = 0\,. \tag{55}$$

It follows from the definition of $\zeta(iT+t)$ in (48) that $\zeta(t)$ is the solution of the following dynamics:

$$\dot{\zeta}(t) = \Lambda A_m \Lambda^{-1}\zeta(t) - \Lambda\sigma(t)\,, \tag{56}$$

$$\zeta(jT) = \begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix}, \quad t \in [jT,\ (j+1)T]. \tag{57}$$

Consider the following function $V(\zeta(t)) = \zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\zeta(t)$ over $t \in [iT,\ (i+1)T]$. Since $\Lambda$ is non-singular and $P$ is positive definite, $\Lambda^{-\top}P\Lambda^{-1}$ is positive definite and, hence, $V(\zeta)$ is a positive definite function. It follows from (56) that over $t \in [jT,\ (j+1)T]$, $\dot{V}(t) = \zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\Lambda A_m\Lambda^{-1}\zeta(t) + \zeta^\top(t)\Lambda^{-\top}A_m^\top\Lambda^\top\Lambda^{-\top}P^\top\Lambda^{-1}\zeta(t) - 2\zeta^\top(t)\Lambda^{-\top}P\Lambda^{-1}\Lambda\sigma(t) = -\zeta^\top(t)\Lambda^{-\top}Q\Lambda^{-1}\zeta(t) - 2\zeta^\top(t)\Lambda^{-\top}P\sigma(t)$. Using the upper bound from (46), over $t \in [iT,\ (i+1)T]$ one can derive

$$\dot{V}(t) \le -\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})\|\zeta(t)\|^2 + 2\|\zeta(t)\|\|\Lambda^{-\top}P\|_a\Delta\,. \tag{58}$$

Notice that for all $t \in [jT,\ (j+1)T]$, if

$$V(t) \ge \alpha\,, \tag{59}$$

we have $\|\zeta(t)\| \ge \sqrt{\dfrac{\alpha}{\lambda_{\max}(\Lambda^{-\top}P\Lambda^{-1})}} \ge \dfrac{2\Delta\|\Lambda^{-\top}P\|_a}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})}$, and the upper bound in (58) yields

$$\dot{V}(t) \le 0\,. \tag{60}$$

It follows from Lemma 2 and the relationship in (57) that $V(\zeta(jT)) = \tilde{z}^\top(jT)P_2\tilde{z}(jT)$, which further along with the upper bound in (52) leads to the following

$$V(\zeta(jT)) \le \alpha\,. \tag{61}$$

It follows from (59)-(60) and (61) that $V(t) \le \alpha\,,\ \forall\, t \in [jT, (j+1)T]$, and therefore

$$V((j+1)T) = \zeta^\top((j+1)T)(\Lambda^{-\top}P\Lambda^{-1})\zeta((j+1)T) \le \alpha\,. \tag{62}$$

Since

$$\tilde{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T)\,, \tag{63}$$

the equality in (55) and the upper bound in (62) lead to the following inequality $\tilde{\xi}^\top((j+1)T)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((j+1)T) \le \alpha$. Using the result of Lemma 2 one can derive that $\tilde{z}^\top((i+1)T)P_2\tilde{z}((i+1)T) \le \tilde{\xi}^\top((i+1)T)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((i+1)T) \le \alpha$, which implies that the upper bound in (52) holds for $j+1$.

It follows from (25), (55) and (63) that $\tilde{y}((j+1)T) = [\mathbf{I}_{m\times m}\mathbf{0}_{m\times(n-m)}]\zeta((j+1)T)$, and the definition of $\zeta((j+1)T)$ in (54) leads to the following expression: $\tilde{y}((j+1)T) = [\mathbf{I}_{m\times m}\quad \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m\Lambda^{-1}T}\begin{bmatrix} 0 \\ \tilde{z}(jT) \end{bmatrix} - [\mathbf{I}_{m\times m}\quad \mathbf{0}_{m\times(n-m)}]\int_0^T \mathrm{e}^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda\sigma(jT+\tau)d\tau$. The upper bounds in (46) and (52) allow for the following upper bound: $\|\tilde{y}((j+1)T)\| \le \|\phi(T)\|\ \|\tilde{z}(iT)\| + \int_0^T \|[\mathbf{I}_{m\times m}\ \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m\Lambda^{-1}(T-\tau)}\Lambda\|_a\ \|\sigma(jT+\tau)\|d\tau \le \|\phi(T)\|\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T)\Delta = \varsigma(T)$, where $\phi(T)$ and $\kappa(T)$ are defined in (11), and $\varsigma(T)$ is defined in (12). This confirms the upper bound in (51) for $j+1$. Hence, (49)-(50) hold for all $iT \le t'$.

For all $iT+t \le t'$, where $0 \le t \le T$, using the expression from (47) we can write that $\tilde{y}(iT+t) = [\mathbf{I}_{m\times m}\quad \mathbf{0}_{m\times(n-m)}]\mathrm{e}^{\Lambda A_m\Lambda^{-1}t}\tilde{\xi}(iT) + [\mathbf{I}_{m\times m}\quad \mathbf{0}_{m\times(n-m)}]\int_0^t \mathrm{e}^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\hat{\sigma}(iT)d\tau - [\mathbf{I}_{m\times m}\quad \mathbf{0}_{m\times(n-m)}]\int_0^t \mathrm{e}^{\Lambda A_m\Lambda^{-1}(t-\tau)}\Lambda\sigma(iT+\tau)d\tau$. The upper bound in (46) and definitions of $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ and $\eta_4(t)$ allow for the following upper bound $\|\tilde{y}(iT+t)\| \le \|\eta_1(t)\|_a\ \|\tilde{y}(iT)\| + \|\eta_2(t)\|_a\ \|\tilde{z}(iT)\| + \eta_3(t)\|\tilde{y}(iT)\| + \eta_4(t)\Delta$. Taking into consideration (51)-(52) and recalling the definitions of $\beta_1(T)$, $\beta_2(T)$, $\beta_3(T)$, $\beta_4(T)$ in (13)-(15), for all $0 \le t \le T$ and for any non-negative integer $i$ subject to $iT+t \le t'$, we have $\|\tilde{y}(iT+t)\| \le \beta_1(T)\varsigma(T) + \beta_2(T)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T)\varsigma(T) + \beta_4(T)\Delta$. Since the right hand side coincides with the definition of $\gamma_0(T)$ in (16), then for all $t \in [0, t']$ we have $\|\tilde{y}(t)\| \le \gamma_0(T)$, which along with the assumption on $T$ introduced in (26) yields $\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} < \bar{\gamma}$. This clearly contradicts the statement in (45). Therefore, $\|\tilde{y}_\tau\|_{\mathcal{L}_\infty} < \bar{\gamma}$, which proves (44).

It follows from (43) that $\|y_\tau\|_{\mathcal{L}_\infty} < \rho$, and hence Lemma 4 implies that $\|e_\tau\|_{\mathcal{L}_\infty} \le \frac{\|F(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|\hat{x}_0 - x_0\|_\infty}{1 - L_\rho\|H(s)(1-F(s))\|_{\mathcal{L}_1}}$. Further, it follows from (27) and (44) that $\|y_\tau\|_{\mathcal{L}_\infty} < \gamma_1$, which contradicts (42). Hence, (29) has to be true. Further, since (44) holds for any $\tau$, (28) is proved.

It follows from (40) and (4) that $u(s) - u_{ref}(s) = -F(s)d_e(s) - F(s)\tilde{\sigma}(s)$. It follows from Assumption 1 and the upper bound in (29) that $\|u - u_{ref}\|_{\mathcal{L}_\infty} \le L_\rho\|F(s)\|_{\mathcal{L}_1}\|y - y_{ref}\|_{\mathcal{L}_\infty} + \|F(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty}$, which along with (28)-(29) leads to the second bound in (30). The proof is complete. □