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On Stability of Time Delay Hamiltonian Systems

Ramkrishna Pasumarthy and Chung-Yao Kao

***Abstract*— Stability of a class of nonlinear systems, called port-Hamiltonian systems, in the presence of time delay in the communication between the plant and controller is studied. The delay parameter is an unknown function which varies with time and for which the upper bounds on the magnitude and variation are known. The presence of delay may destroy the port-Hamiltonian structure of the system. Because of this, stability of the time delay systems is not obvious. We thus propose a theory to test the stability of port-Hamiltonian systems with time delay. The stability problem considered here, relies on the construction of a Lyapunov-Krasovskii (LK) functional based on the Hamiltonian of the port-Hamiltonian system. Based on the LK functional, we derive some sufficient conditions for the system to be asymptotically stable in presence of uncertain delays.**

I. INTRODUCTION

Port Hamiltonian models are natural candidates to describe many physical systems [12]. These classes of systems are basically defined with respect to a power conserving geometric structure capturing the basic interconnection laws and an Hamiltonian function given by the total stored energy of the system. A key feature of port-Hamiltonian systems is that a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system. This concept of interconnection is important from a control point of view, since implementing a control law or controlling a system is usually done with an external device via external port variables. An immediate example is the control by interconnection of port-Hamiltonian systems [2], [12], where the plant port-Hamiltonian system is connected to a controller port-Hamiltonian system via a feedback loop such that the closed-loop system has desired stability properties, by using various energy shaping techniques. Since the interconnection preserves the port-Hamiltonian structure, the control by interconnection method has some inherent robustness properties largely due to the structure of the port-Hamiltonian systems. Now assume that there is some time delay in the communication between the plant and the controller. The presence of time delays may often result in a closed-loop system which is not exactly in the port-Hamiltonian form. In other words the "$J - R$" structure is actually destroyed and hence does not reveal any information on the stability of the system.

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R. Pasumarthy is with the Department of Electrical Engineering, University of California at Los Angeles, Los Angeles, CA 90095-1594, USA. ramkrish@ee.ucla.edu.

C.-Y. Kao is with the Department of Electrical Engineering, National Sun Yat-Sen University, Kaohsiung, Taiwan cykao@mail.nsysu.edu.tw

Stability of time delay systems, on the other hand, has been an active area of research since decades. Various examples studied are those from economics, chemical processes, mechanics as well as in many Engineering domains. The stability analysis approaches for LTI systems are either time domain or frequency domain. Whereas the former is based on generating various Lyapunov-Krasovskii functionals [3], [5], [7], the Lyapunov-Razumikhin function approach, the input-output approach [1]; the later focuses on the small gain theorem based approach, or deriving stability in terms of transfer functions [4]. The results included deriving delay independent and delay dependent stability criteria both for systems with constant as well as time varying delays. For a comprehensive review of various stability criterion we refer to [11]. Significant advances have been made for the case of linear time-delay systems and according to the authors best knowledge, time delay nonlinear systems haven't been investigated as comprehensively as in the case for linear systems. Few delay dependent and delay independent criteria for general nonlinear systems have been reported in [8], [9], wherein the delay parameter is assumed to be constant.

In this paper we consider nonlinear systems with a specific structure, the class of which are called port-Hamiltonian systems. We model nonlinear systems with time varying delays, in the port-Hamiltonian framework, for which we know the upper bound on the magnitude of the delay and its variation. This paper is organized as follows: In Section II we present the concept of interconnection of two port-Hamiltonian systems and derive models for port-Hamiltonian systems in the presence of communication delays. Later in Sections III and IV we derive some sufficient conditions, in terms of matrix inequalities, for testing stability of port-Hamiltonian systems with communication delays. These conditions are derived based on the construction of a Lyapunov-Karasovskii functional by making use of the Hamiltonian of the time-delay port-Hamiltonian system.

Notation: We use symbols $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}^n$ to denote $n \times 1$ real vectors, and $\mathbb{R}^{n \times m}$ to denote $n \times m$ real matrices. For $b > a$, the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with topology of uniform convergence is denoted by $C([a, b], \mathbb{R}^n)$. For $\psi \in C([a, b], \mathbb{R}^n)$, the norm is defined as $\|\psi\| = \sup_{a \le \theta \le b} |\psi(\theta)|$, where $|\cdot|$ is the standard Euclidean norm of $\mathbb{R}^n$. Lastly, $C^r([a, b], \mathbb{R}^n) := \{\psi \in C([a, b], \mathbb{R}^n) : \|\psi\| < r\}$.

Given a function $H(x) : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, the notations $\nabla H(x)$ and $\nabla^2 H(x)$ are used to denote the gradient vector and the Hessian matrix of $H(x)$ at x, respectively. In this

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paper, it is often that the argument of a function is itself a function of time. In this case, the corresponding time-domain function is denoted by the same symbol enclosed by a pair of square brackets with a subindex t. For example, $[F]_t$ denotes the functions (of time) $F(x(t))$. Likewise, when F is evaluated at $x(t-\tau)$, the corresponding time-domain functions will be denoted as $[F]_{t-\tau}$.

Given a matrix M, the transposition of M is denoted by M'. When M is symmetric, the notation $M > 0$ is used to denote positive definiteness. The positive semi-definiteness, negative definiteness, and negative semi-definiteness are denoted using "$\geq$", "$<$", and "$\leq$", respectively.

Finally, for the sake of notational simplicity, the state and/or time dependency of a quantity will be dropped when it is clear from the context.

II. Motivation of the Problem and Problem formulation

We are interested in nonlinear systems with a specific structure, called port-Hamiltonian systems, which are of the form

$$\begin{aligned} \dot{x}(t) &= ([J]_t - [R]_t)\,[\nabla H]_t + g u(t) \\ y(t) &= g'[\nabla H]_t \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector at time t, $u(t) \in \mathbb{R}^m$, is the control action, and $H : \mathcal{X} \to \mathbb{R}$ is the total stored energy. $g \in \mathbb{R}^{n\times m}$ called the port matrix is assumed to be a constant matrix in the current analysis. $J : \mathcal{X} \to \mathbb{R}^{n\times n}$ and $R : \mathcal{X} \to \mathbb{R}^{n\times n}$ are matrix-valued functions, where J and R satisfy $J(x) = -J(x)'$ and $R(x) = R(x)' \geq 0$ for all $x \in \mathcal{X}$. $J(x)$ and $R(x)$ are often referred to as "the natural interconnection matrix" and " the damping matrix", respectively.

Because of the skew symmetry of $J(x)$ and semi-positive definiteness of $R(x)$, the time derivative of $H(x)$ along the solution of (1) satisfies

$$-[\nabla H]'_t[R]_t[\nabla H]_t + u(t)'y(t), \tag{2}$$

which shows that (1) is conservative (or "passive") if $H(x) \geq 0$ for all x. Physically, equation (2) corresponds to the fact that the internal interconnection structure of the system is power conserving, while u and y are the power-variables of the ports defined by g and thus $u'y$ is the externally supplied power and $-\nabla H(x)'R(x)\nabla H(x)$ the dissipated energy.

An important corollary of (2) is that, in the absence of input u, the energy of the autonomous system is non-increasing and will actually decrease in the presence of dissipation (the case where $R(x) \geq 0$ for all x). Since the energy function H is bounded from below, the system will eventually stop at a point of minimum energy.

Interconnection of port-Hamiltonian systems

Consider two port-Hamiltonian systems of the form (1)

$$\begin{aligned} \dot{x}_i(t) &= ([J_i]_t - [R_i]_t)\,[\nabla H_i]_t + g_i u_i(t) \\ y_i(t) &= g_i'[\nabla H_i]_t \end{aligned} \tag{3}$$

$i = 1, 2$, where J_1, R_1, H_1 are functions of x_1, and J_2, R_2, H_2 are functions of x_2. One of the two systems could be thought as a plant to be controlled and the other the controller. Interconnecting the two systems in (3) via the standard (power preserving) feedback interconnection

$$u_2 = y_1 + v_2, \quad u_1 = -y_2 + v_1, \tag{4}$$

where v_1 and v_2 are external signals injected at input ports of the first and the second Hamiltonian systems, respectively, we see that the composed system is still of the port-Hamiltonian form and and can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} [J_1 - R_1]_t & -g_1 g_2' \\ g_2 g_1' & [J_2 - R_2]_t \end{bmatrix} \begin{bmatrix} [\nabla H_1]_t \\ [\nabla H_2]_t \end{bmatrix} \\ &\quad + \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} g_1' & 0 \\ 0 & g_2' \end{bmatrix} \begin{bmatrix} [\nabla H_1]_t \\ [\nabla H_2]_t \end{bmatrix} \end{aligned}$$

with state space given by the product space $\mathcal{X}_1 \times \mathcal{X}_2$, and the total Hamiltonian $H(x_1, x_2) = H_1(x_1) + H_2(x_2)$. Assume that the total Hamiltonian $H(x_1, x_2) \geq 0$ for all x_1 and x_2. Since the port-Hamiltonian structure is preserved under the interconnection, the interconnected system remains passive and hence, at the absence of v_1 and v_2, stable in the sense of Lyapunov. The total Hamiltonian $H(x_1, x_2)$ is one Lyapunov function that proves stability of the interconnected system.

Interconnected Hamiltonian systems with communication delays

The feedback interconnection (4) assumes ideal signal transmission between the two port-Hamiltonian systems, such that signals y_1 and y_2 arrive their respective destinations instantaneously, or with delays that are not significant enough to be taken into account. This assumption would not be realistic if the two systems are far apart from each other, or communicating with each other through channels that have busy traffic. In such cases, the signal transmission delays must be taken into account, and the feedback interconnection relationship shall be modelled as

$$\begin{aligned} u_2(t) &= y_1(t - \tau_1(t)) + v_2(t), \\ u_1(t) &= -y_2(t - \tau_2(t)) + v_1(t), \end{aligned} \tag{5}$$

where τ_1 and τ_2 model the forward and backward signal transmission delays, respectively. The closed-loop system now takes the following form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} [J_1 - R_1]_t & 0 \\ 0 & [J_2 - R_2]_t \end{bmatrix} \begin{bmatrix} [\nabla H_1]_t \\ [\nabla H_2]_t \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & g_1 g_2' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\nabla H_1]_{t-\tau_1} \\ [\nabla H_2]_{t-\tau_1} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ g_2 g_1' & 0 \end{bmatrix} \begin{bmatrix} [\nabla H_1]_{t-\tau_2} \\ [\nabla H_2]_{t-\tau_2} \end{bmatrix} + \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}. \end{aligned} \tag{6}$$

Let $x := \begin{bmatrix} x_1' & x_2' \end{bmatrix}'$, $v := \begin{bmatrix} v_1' & v_2' \end{bmatrix}'$, and $H(x) := H_1(x_1) + H_2(x_2)$. Then equation (6) can be expressed in the form

$$\begin{aligned} \dot{x}(t) = [J - R]_t[\nabla H]_t &+ T_1[\nabla H]_{t-\tau_1} \\ &+ T_2[\nabla H]_{t-\tau_2} + g v(t). \end{aligned} \tag{7}$$

The above equation no longer preserves the port-Hamiltonian structure. The energy balance equation now takes the form

$$\begin{aligned}[\dot{H}]_t = -[\nabla H]_t'[R]_t[\nabla H]_t + \sum_{i=1,2}[\nabla H]_t'T_i[\nabla H]_{t-\tau_i} \\ + y_1(t)'v_1(t) + y_2(t)'v_2(t)\end{aligned} \quad (8)$$

Since the second term in the right hand side of (8) may not be negative definite, the total Hamiltonian $H(x_1, x_2)$ does not help to reveal any information about the passivity/stability property of the interconnected system (6). To deduct these properties, one has to seek for other energy functions. Furthermore, in certain cases the presence of time delays may actually destroy these properties of the system – the feedback interconnection may no longer preserve passivity and stability.

Problem Formulation

The discussion in the previous session leads us to consider the following (autonomous) port-Hamiltonian system with time delays

$$\dot{x}(t) = ([J]_t - [R]_t)[\nabla H]_t + \sum_{i=1}^{m} T_i[\nabla H]_{t-\tau_i} \quad (9)$$

where $x \in \mathbb{R}^n$, $J : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $R : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy $J(x) = -J(x)'$ and $R(x) = R(x)' \geq 0$ for all x, $T_i \in \mathbb{R}^{n\times n}$ are constant matrices, and $H : \mathbb{R}^n \to \mathbb{R}^+$ satisfies $H(0) = 0$, $\nabla H(0) = 0$, and $H(x) > 0$ for all $x \neq 0$. Note that given the properties of H, $x(t) = 0$ is a solution of (9). We are interested in verifying stability property of this solution of (9). Equation (9) is a functional differential equation of retarded type. A corner stone for studying stability property of systems governed by such equations is the Lyapunov-Krasovskii theorem, by which the stability criteria presented in this paper will be derived. The theorem is briefly summarized below. The readers are referred to [3], [6] for details.

Consider a functional differential equation of retarded type

$$\dot{x}(t) = f(x_t) \quad (10)$$

defined on the positive time interval $[0, \infty)$, where $f : \Omega \subseteq C([-\xi, 0], \mathbb{R}^n) \to \mathbb{R}^n$ and $x_t \in \Omega$ is defined as $x_t(\theta) := x(t + \theta)$ for $\theta \in [-\xi, 0]$. The initial condition of (10) is denoted by x_0, which is also a function in Ω. We assume that (10) has a unique solution for any initial condition x_0 and that $x(t) = 0$ is a solution of (10). The concept of stability of the solution $x(t) = 0$ is given below.

Definition 1: The solution $x(t) = 0$ of (10) is called stable if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for any initial condition x_0 which satisfies $\|x_0\| < \delta(\epsilon)$, the corresponding solution $x(t)$ satisfies $|x(t)| < \epsilon$ for all $t \geq 0$. It is called asymptotically stable if it is stable and $\delta(\epsilon)$ can be chosen such that $\|x_0\| < \delta(\epsilon)$ implies that $\lim_{t\to\infty} x(t) \to 0$. It is called globally asymptotically stable if it is asymptotically stable, and for any initial condition x_0 the corresponding solution $x(t)$ approaches to 0 as $t \to \infty$ no matter how large $\|x_0\|$ is.

Let $V : \Omega \to \mathbb{R}$. The derivative of V with respect to time along the solution of (10) initiated at initial condition ϕ is defined as

$$\dot{V}(\phi) := \frac{d}{dt}V(x_t) = \limsup_{\Delta t\to 0}\frac{1}{\Delta t}(V(x_{t+\Delta t}) - V(x_t))$$

where $x_t := x(t + \theta)$, $\theta \in [-\xi, 0]$, and $x(t)$ is the solution of (10) with the initial condition ϕ. We have the following theorem for stability of (10).

Theorem 2: (Lyapunov-Krasovskii): Suppose for any given r, f in (10) maps $C^r([-\xi, 0], \mathbb{R}^n)$ into a bounded set in $\mathbb{R}^n$. Let u, v, w be continuous nonnegative nondecreasing functions, where u and v satisfy $u(0) = v(0) = 0$, and $u(s) > 0$ $v(s) > 0$ for $s \neq 0$. If there exists $V : \Omega \to \mathbb{R}$ such that V and the derivative of V along the solution of (10) satisfy

$$u(|\phi(0)|) \leq V(\phi) \leq v(\|\phi\|), \text{ and } \dot{V}(\phi) \leq -w(|\phi(0)|)$$

for all $\phi \in \Omega$, then the solution $x(t) = 0$ of (10) is stable. If, in addition, $w(s) > 0$ for any $s \neq 0$, then the solution $x(t) = 0$ of (10) is asymptotically stable. Finally, if $\lim_{s\to\infty} u(s) = \infty$, then the stability is global.

As usual stability of any nonzero equilibrium $x(t) = x_*$, can be studied as the stability of $x(t) = 0$, by "shifting the equilibrium" to 0.

We are interested in verifying stability property of systems in the form of (9) under the following assumptions on the delay parameter τ_i:

A1 τ_i might be time-varying, and the rate of variation is upper bounded by one; i.e., $\dot{\tau}_i(t) \leq d < 1$ for all t and all i. In addition, all τ_i are nonnegative, bounded above, but the information on the upper bounds are not available.

A2 All τ_i satisfy $0 \leq \tau_i(t) \leq h$ and $\dot{\tau}_i(t) \leq d < 1$ for all t and for all i.

For the sake of clarifying the main idea and simplifying the presentation, we will consider a simplified version of (9) in the remaining sections,

$$\dot{x}(t) = ([J]_t - [R]_t)[\nabla H]_t + T[\nabla H]_{t-\tau}. \quad (11)$$

where only one delay term is present. Note, however, that the results we will present in the following sections can be trivially generalized to the case where there are any finite number of delay terms in equation (11).

III. STABILITY CRITERIA INDEPENDENT OF THE UPPER BOUND ON THE DELAY

In this section, we present a stability criterion for verifying stability property of (11) under the assumption A1.

Proposition 3: Consider a port-Hamiltonian system with delay in the form of (11). Let

$$A(x) = \nabla^2 H(x)(J(x) - R(x)), \; A_1(x) = \nabla^2 H(x)T.$$

The system (11) is globally asymptotically stable if there exist symmetric real constant matrices $P > 0$ and $S > 0$ such that,

$$\begin{bmatrix} M_{11}(x) & M_{12}(x) \\ M_{12}(x)' & -(1-d)S \end{bmatrix} < 0, \quad \forall x \in \mathbb{R}^n, \quad (12)$$

where

$$M_{11}(x) = -R(x) + S + PA(x) + A(x)'P$$
$$M_{12}(x) = T/2 + PA_1(x)$$

If (12) holds for all x in an open neighborhood of the origin, say $\mathcal{X}$, then the stability is local.

Proof: To prove stability, consider the following Lyapunov-Krasovskii functional:

$$V(x_t) = [H]_t + [\nabla H]'_t P[\nabla H]_t + \int_{t-\tau(t)}^{t} [\nabla H]'_k S[\nabla H]_k dk, \tag{13}$$

where $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, and $x(t)$ is any solution of (11). Differentiating (13) along the solution of (11) and using the fact that $\frac{d}{dt}(\nabla H(x(t))) = \nabla^2 H(x(t))\dot{x}(t)$ we have

$$\dot{V} = \begin{bmatrix} [\nabla H]_t \\ [\nabla H]_{t-\tau(t)} \end{bmatrix}' \begin{bmatrix} M_{11}(x(t)) & M_{12}(x(t)) \\ M_{12}(x(t))' & -(1-\dot{\tau})S \end{bmatrix} \begin{bmatrix} [\nabla H]_t \\ [\nabla H]_{t-\tau(t)} \end{bmatrix}$$
$$\leq \begin{bmatrix} [\nabla H]_t \\ [\nabla H]_{t-\tau(t)} \end{bmatrix}' \begin{bmatrix} M_{11}(x(t)) & M_{12}(x(t)) \\ M_{12}(x(t))' & -(1-d)S \end{bmatrix} \begin{bmatrix} [\nabla H]_t \\ [\nabla H]_{t-\tau(t)} \end{bmatrix}$$

Hence, inequality (12) implies $\dot{V}(x_t) < 0$ for all non-zero x_t, and hence global asymptotic stability of (11). This concludes the proof. ■

IV. Delay dependant stability criteria

Next we investigate stability property of system (11) under assumption A2. To this end, note that one may express $[\nabla H]_{t-\tau(t)}$ as

$$[\nabla H]_{t-\tau(t)} = [\nabla H]_t - \int_{t-\tau(t)}^{t} \frac{d}{ds}[\nabla H]_s ds = [\nabla H]_t -$$
$$\int_{t-\tau(t)}^{t} [\nabla^2 H]_s \left((J-R)[\nabla H]_s + T[\nabla H]_{s-\tau(s)}\right) ds$$

Further assume that $J(x) - R(x) + T$, can be split into a skew symmetric matrix $\bar{J}(x)$ and a symmetric matrix $\bar{R}(x)$ which satisfies $\bar{R}(x) \geq 0$. We can then write (11) in a transformed form as

$$\begin{aligned}\dot{x}(t) =& (\bar{J} - \bar{R})[\nabla H]_t - T\int_{t-\tau(t)}^{t} [\nabla^2 H]_s \times \\ & \left((J-R)[\nabla H]_s + T[\nabla H]_{s-\tau(s)}\right) ds,\end{aligned} \tag{14}$$

where $J(x) - R(x) + T = \bar{J}(x) - \bar{R}(x)$. Since the system (11) is a special case of the transformed system (14), the stability of (14) guarantees the stability of (11), but the reverse as stated in [3] is not necessarily true. This is because the transformation (14), introduces additional dynamics which are not a part of the original system and which may become unstable even before the original system. Before we state our next result we state the following lemma, which will be useful in deriving stability condition for the system (14).

Lemma 4: Let $w_1(t)$ and $w_2(t)$ be defined as follows

$$w_1(t) = \int_{t-\tau(t)}^{t} \int_{k}^{t} f(s) ds dk,$$
$$w_2(t) = \int_{t-\tau(t)}^{t} \int_{k-\tau(k)}^{t} f(s) ds dk.$$

Then

$$\dot{w}_1(t) = \tau(t)f(t) - (1-\dot{\tau}(t))\int_{t-\tau(t)}^{t} f(s)ds, \tag{15}$$

$$\begin{aligned}\dot{w}_2(t) =& \tau(t)f(t) - (1-\dot{\tau}(t))\int_{t-\tau(t)}^{t} f(s-\tau(s))ds \\ & + \dot{\tau}(t)\int_{t-\tau(t)}^{t} f(s)ds\end{aligned} \tag{16}$$

Proof: To streamline the readability of the paper we place the proof in the Appendix. ■

The stability conditions for time delay system (14) can then be stated as follows:

Proposition 5: Denote $A_0(x) = T\nabla^2 H(x)(J(x) - R(x))$ and $A_1(x) = T\nabla^2 H(x)T$. The nonlinear system described by (14) is globally asymptotically stable if there exist real symmetric constant matrices $P > 0, S_0 > 0$ and $S_1 > 0$ such that

$$K = \begin{bmatrix} \frac{K_{11}}{h} + S_0 + S_1 & K_{12} & K_{13} \\ K'_{12} & -(1-d)S_0 + dS_1 & 0 \\ K'_{13} & 0 & -(1-d)S_1 \end{bmatrix} < 0 \tag{17}$$

for all $x \in \mathbb{R}^n$, where

$$\begin{aligned} K_{11} =& -\bar{R}(x) + P\nabla^2 H(x)(\bar{J}(x) - \bar{R}(x)) \\ & + (\bar{J}(x) - \bar{R}(x))'\nabla^2 H(x)P \\ K_{12} =& -\frac{1}{2}A_0(x) - P\nabla^2 H(x)A_0(x) \\ K_{13} =& -\frac{1}{2}A_1(x) - P\nabla^2 H(x)A_1(x) \end{aligned}$$

If (17) holds for all x in an open neighborhood of the origin, say $\mathcal{X}$, then the stability is local.

Proof: Let

$$\begin{aligned} V_1(x(t)) =& [H]_t + [\nabla H]'_t P[\nabla H]_t \\ V_2(x_t) =& \int_{t-\tau(t)}^{t} \int_{k}^{t} [\nabla H]'_s S_0 [\nabla H]_s ds dk \\ V_3(x_t) =& \int_{t-\tau(t)}^{t} \int_{k-\tau(k)}^{t} [\nabla H]'_s S_1 [\nabla H]_s ds dk \end{aligned}$$

We then have

$$\begin{aligned} & \frac{d}{dt}V_1 = \\ & \int_{t-\tau(s)}^{t} \{([\nabla H]'_t \frac{K_{11}}{\tau(t)} + [\nabla H]'_s K'_{12} + [\nabla H]'_{s-\tau(s)} K'_{13})[\nabla H]_t \\ & + [\nabla H]'_t K_{12}[\nabla H]_s + [\nabla H]'_t K_{13}[\nabla H]_{s-\tau(s)}\} ds \\ & \leq \\ & \int_{t-\tau(s)}^{t} \{([\nabla H]'_t \frac{K_{11}}{h} + [\nabla H]'_s K'_{12} + [\nabla H]'_{s-\tau(s)} K'_{13})[\nabla H]_t \\ & + [\nabla H]'_t K_{12}[\nabla H]_s + [\nabla H]'_t K_{13}[\nabla H]_{s-\tau(s)}\} ds. \end{aligned} \tag{18}$$

Similarly because of (15) we have,

$$\frac{d}{dt}V_2 = \int_{t-\tau(t)}^{t} \{[\nabla H]_t' S_0 [\nabla H]_t - (1-\dot{\tau})[\nabla H]_s' S_0 [\nabla H]_s\} ds$$
$$\leq \int_{t-\tau(t)}^{t} \{[\nabla H]_t' S_0 [\nabla H]_t - (1-d)[\nabla H]_s' S_0 [\nabla H]_s\} ds \quad (19)$$

and from (16) we have,

$$\begin{aligned}\frac{d}{dt}V_3 = &\int_{t-\tau(t)}^{t} \{[\nabla H]_t' S_1 [\nabla H]_t \\ &- (1-\dot{\tau})[\nabla H]_{s-\tau(s)}' S_1 [\nabla H]_{s-\tau(s)}\} ds \\ &+ \dot{\tau}\int_{t-\tau(t)}^{t} [\nabla H]_s S_1 [\nabla H]_s ds \\ \leq &\int_{t-\tau(t)}^{t} \{[\nabla H]_t' S_1 [\nabla H]_t \\ &- (1-d)[\nabla H]_{s-\tau(s)}' S_1 [\nabla H]_{s-\tau(s)}\} ds \\ &+ d\int_{t-\tau(t)}^{t} [\nabla H]_s S_1 [\nabla H]_s ds \end{aligned} \quad (20)$$

Now, choose the following Lyapunov-Krasovskii functional

$$V(x_t) = V_1(x(t)) + V_2(x_t) + V_3(x_t).$$

Computing the time derivative, by using the relations (18), (19) and (20) we get

$$\frac{d}{dt}V = \int_{t-\tau(t)}^{t} \begin{bmatrix} [\nabla H]_t \\ [\nabla H]_s \\ [\nabla H]_{s-\tau(s)} \end{bmatrix}' K \begin{bmatrix} [\nabla H] \\ [\nabla H]_s \\ [\nabla H]_{s-\tau(s)} \end{bmatrix} ds$$

From (17) we have that $\dot{V}(x_t) < 0$ for all nonzero $x(t)$, which proves global asymptotic stability. ■

V. Examples

Example 6: Consider the following time delay port-Hamiltonian system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \nabla H_1 \\ \nabla H_2 \end{bmatrix} + \alpha \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} [\nabla H_1]_{t-\tau} \\ [\nabla H_2]_{t-\tau} \end{bmatrix}, \quad (21)$$

where $\alpha \geq 0$ and $\tau > 0$. Notice that the interconnection matrix J and the damping matrix R are constant in this case. Furthermore, the damping matrix R in this example satisfies $R > 0$. Under such cases, where the interconnection and damping matrices are constant and $R > 0$, we can simplify the stability condition (12) by choosing the following L-K functional (obtained by setting $P = 0$ in (13)).

$$V(x(t)) = [H]_t + \int_{t-\tau(t)}^{t} [\nabla H]_k' S [\nabla H]_k dk.$$

This results in the following simplified condition for stability, which is in the form of an LMI:

$$\begin{bmatrix} -R+S & T/2 \\ T/2 & -(1-d)S \end{bmatrix} < 0. \quad (22)$$

Using the simplified stability criteria (22), we see that the system is stable for any $\tau > 0$ as long as $0 < \alpha \leq 0.7$ as long as $d = 0$, i.e. for constant delays. Observe that the undelayed system is stable for any $\alpha \geq 0$ and hence we can say that the parameter α to some extent measures the size of the delayed term. The below table shows values of α for which the system is stable, for different values of d:

d	0.2	0.4	0.6	0.8	0.9	0.99
α	0.63	0.54	0.44	0.31	0.22	0.07

It is important to note that in this example we have not specified the structure of the Hamiltonian H which can either be quadratic in x (linear system) or non quadratic (nonlinear system). Hence we conclude that the stability criteria (22) also extends to a class of nonlinear systems, with constant interconnection and damping matrices which additionally satisfy $R > 0$.

Example 7: Consider the mathematical pendulum with Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + (1-\cos q),$$

actuated by a torque u, with output $y = p$, the angular velocity. With b a positive damping constant, the system can be written in the port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} \begin{bmatrix} \sin q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \;\; y = p.$$

The system is open-loop stable at the point $(0,0)$, which is usually not the point of interest. Lets consider the problem of stabilization of the pendulum, by interconnecting it with a controller, at the point $(q_*, 0, \xi_*)$, where ξ is the controller state. The port-Hamiltonian controller is of the form

$$\dot{\xi} = u_c; \;\; y_c = \nabla H_c$$

Interconnecting the plant (the pendulum) and the controller with the interconnection constraints $u = -y_c, \;\; u_c = y$, leads to the following closed-loop system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sin q \\ p \\ \nabla H_c \end{bmatrix}$$

with $H_c(\xi)$ free to be chosen. Usually the closed-loop Hamiltonian does not have a minimum at the desired equilibrium and hence does not qualify as a Lyapunov function. The idea is to generate dynamical invariants called Casimir functions and construct a Lyapunov function based on the Hamiltonian of the plant, the controller and the controller and the corresponding Lyapunov function, given as

$$H_d(q,p,\xi) = H_p(q,p) + H_c(\xi) + C(q,p,\xi)$$

This methodology and this example in particular has been extensively reported in [2], where it has been shown that the desired stability objectives are obtained by the following choices of controller Hamiltonian and the Casimir function,

$$H_c(\xi) = \frac{1}{2}\beta(\xi - \xi_* - \frac{1}{\beta}\sin q)^2$$
$$C(q,p,\xi) = \frac{1}{2}k(q - q_* - (\xi - \xi_*) - \frac{1}{k}\sin q_*)^2$$

The constants β and k are chosen to satisfy

$$\cos q_* + k > 0, \quad \beta \cos q_* + k \cos q_* + k\beta > 0.$$

We further add damping in the system of the form $\dot{\xi} = -z\nabla H_c, z > 0$. Now assume that there is some delay in communication between the plant and the controller, which means that the plant and the controller are interconnected by the interconnection constraints (5). Further we assume in this example that $\tau_1(t) = \tau_2(t) = \tau(t)$. This would result in the following closed-loop time delay system:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -b & 0 \\ 0 & 0 & -z \end{bmatrix} \begin{bmatrix} \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \\ \frac{\partial H_d}{\partial \xi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} [\frac{\partial H_d}{\partial q}]_{t-\tau(t)} \\ [\frac{\partial H_d}{\partial p}]_{t-\tau(t)} \\ [\frac{\partial H_d}{\partial \xi}]_{t-\tau(t)} \end{bmatrix}. \tag{23}$$

We then apply Proposition 5 to study the stability of the delayed system, having known that the system without delay has desired stability properties. The stability condition also requires computing the Hessian of the Lyapunov function of the system without delay, and is given by

$$\nabla^2 H_d(q,p,\xi) = \begin{bmatrix} \cos q + K & 0 & -K \\ 0 & 1 & 0 \\ -K & 0 & \beta + K \end{bmatrix}. \tag{24}$$

Suppose we want to study the stability of the pendulum at the upright position $q = \pi$, then to satisfy the conditions for stability of the system without delay we choose, $K = 2, \beta = 3$. We further choose $z = k = 1$. To verify stability in the presence of delay, we use the stability condition (17). The below table shows allowable time delays h for various values of $d \in [0, 1)$, for which the system (23) is globally asymptotically stable.

d	0	0.2	0.4	0.6	0.8	0.9
h	0.132	0.112	0.089	0.064	0.036	0.02

Remark 8: Contrast to stability criterion for linear time delay systems which can be formulated as LMIs, the conditions (12,17) for nonlinear systems involve matrix-valued functions. In general to compute the feasibility of these matrix inequalities is nontrivial, especially when the bounds on the state dependent terms are unknown or if they are unbounded. In the case of the nonlinear pendulum we observe that the only state dependant term is $\cos q$ (see (24)), which is always bounded and takes values between $[-1, 1]$. Since we know the bounds on the state dependent term we solve the matrix inequality (17) iteratively for values of $\cos q$ ranging between $[-1, 1]$ and look for values of h which are valid $\forall q$.

VI. CONCLUSIONS

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for nonlinear time delay systems, in the port-Hamiltonian framework. Sufficient conditions for stability, based on matrix inequalities, are derived. The advantage of this approach is the construction of the Lyapunov-Krasovskii functional by making use of the Hamiltonian (or the total energy) of the given system. A few issues however remain open, namely the solvability of matrix inequalities when the state variables are unbounded, or do not lie within a specified range. An answer to this problem could be found by using SOS tools [10], whose applicability in the present context, remains to be investigated.

APPENDIX: PROOF OF LEMMA 4

Let

$$F(t) = \int_{-\infty}^{t} f(s)ds$$

Then,

$$\begin{aligned}
\frac{d}{dt}w_1(t) &= \frac{d}{dt}\int_{t-\tau(k)}^{t}\int_{k}^{t} f(s)dsdk \\
&= \frac{d}{dt}\int_{t-\tau(t)}^{t}\left(\int_{-\infty}^{t} f(s)ds - \int_{-\infty}^{k} f(s)ds\right)dk \\
&= \frac{d}{dt}\int_{t-\tau(t)}^{t}\left(F(t) - F(k)\right)dk \\
&= \frac{d}{dt}\int_{t-\tau(t)}^{t} F(t)dk - \frac{d}{dt}\int_{t-\tau(t)}^{t} F(k)dk \\
&= \frac{d}{dt}\tau(t)F(t) - \frac{d}{dt}\left(\int_{-\infty}^{t} F(k)dk - \int_{-\infty}^{t-\tau(t)} F(k)dk\right) \\
&= \tau(t)f(t) + \dot{\tau}F(t) - F(t) + (1-\dot{\tau})F(t-\tau(t))) \\
&= \tau(t)f(t) - (1-\dot{\tau})(F(t) - F(t-\tau)) \\
&= \tau(t)f(t) - (1-\dot{\tau})\int_{t-\tau(t)}^{t} f(s)ds
\end{aligned}$$

This proves (15). Similarly we can prove (16).

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