2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Recovery of Linear Performance in Feedback Systems with Nonlinear Instrumentation

ShiNung Ching, Pierre T. Kabamba and Semyon M. Meerkov

Abstract— **The problem of controller design in linear systems is well-understood. Often, however, when linear controllers are implemented on a physical system, the anticipated performance is not met. In some cases, this can be attributed to nonlinearities in the instrumentation, i.e., sensors and actuators. Intuition suggests that to compensate for this instrumentation, one can boost, i.e., increase, the controller gain. This paper formally pursues this strategy, and develops the theory of *boosting*. It provides conditions under which the controller gain can be modified to offset the effects of instrumentation, thus recovering the performance of the intended linear design.**

I. Introduction

Consider the standard SISO linear feedback system shown in Figure 1, where $C(s)$ is the controller and $P(s)$ is the plant. The signals u, y and w denote the controller output, plant output, and standard white input disturbance, respectively. Assume that the controller is designed to achieve a certain level of disturbance rejection, specified by the output variance σ_y^2. In reality, however, the controller is implemented in the configuration shown in Figure 2, where $f(\cdot)$ and $g(\cdot)$ represent static nonlinearities in the actuator and sensor, respectively. This is referred to as a *Linear Plant/Nonlinear Instrumentation* (LPNI) system. The signals $\bar{u}$ and $\bar{y}$ indicate, respectively, the controller and plant output.

The performance of the LPNI system typically degrades in comparison with that of the original linear system in the sense that

$$\sigma_{\bar{y}} > \sigma_y. \tag{1}$$

The authors are with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109, USA kabamba@umich.edu

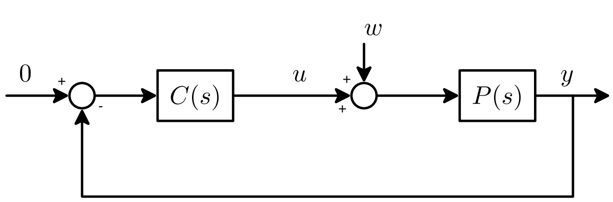

Fig. 1. Basic linear feedback system

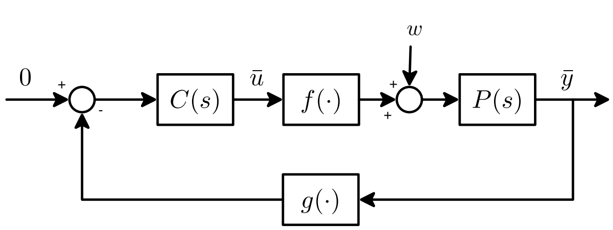

Fig. 2. Feedback system with nonlinear instrumentation

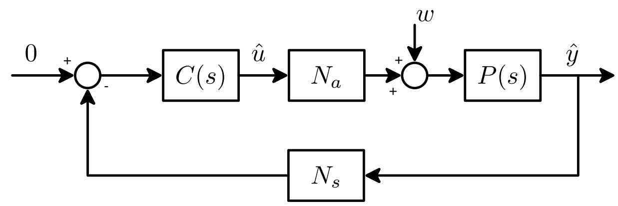

Fig. 3. Quasilinear feedback system

The main contribution of this paper is the introduction and development of the *boosting* method, that describes how, under certain conditions, the gain of $C(s)$ can be increased to eliminate this degradation. Thus, this paper shows how to recover the performance of a linear design in the presence of nonlinear instrumentation.

Quantifying (1) may seem daunting since analytical evaluation of $\sigma_{\bar{y}}$ requires solution of a Fokker-Planck equation, which is possible in only a few special cases [14]. Hence, a simplification is necessary. In this paper, the method of stochastic linearization (SL) is used for this purpose (see [2] and [12] for the original publications and [6] and [15] for monographs describing the method in detail).

According to SL, the LPNI system is replaced by the *quasilinear* system shown in Figure 3, where the gains N_a and N_s are quasilinearizations of $f(u)$ and $g(y)$, and the signals $\hat{u}$ and $\hat{y}$ are intended to approximate $\bar{u}$ and $\bar{y}$, respectively. The quasilinear gains are defined as

$$N_a := E\left[\frac{d}{d\hat{u}} f(\hat{u})\right], \tag{2}$$

and

$$N_s := E\left[\frac{d}{d\hat{y}} g(\hat{y})\right], \tag{3}$$

where the expectations are taken with respect to the probability density functions of $\hat{u}$ and $\hat{y}$, respectively. Henceforth, we assume that these quasilinear gains are nonzero.

The quasilinear system of Figure 3 provides an approximation of the original LPNI system of Figure 2. In contrast to the usual Jacobian linearization, this approximation does not require small signals. The price to pay is that the gains N_a and N_s depend not only on the nonlinearities, but on all elements of Figure 2, including $C(s)$, $P(s)$ and w. Recall that in Jacobian linearization, the linearized gains depend only on the derivatives of $f(u)$ and $g(y)$ at a local operating point, independent of all other elements in the system. Obviously, this is a simpler approximation, however, being local, it does not characterize the properties of LPNI systems subject to possibly large random disturbances.

978-1-4244-4524-0/09/$25.00 ©2009 AACC

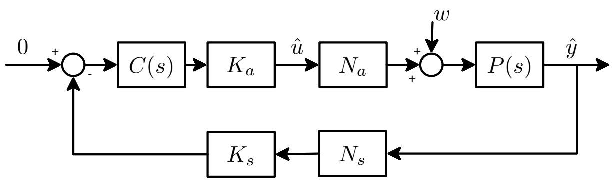

Fig. 4. Boosted quasilinear system

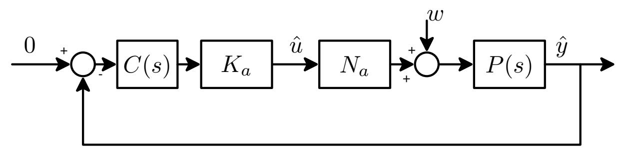

Fig. 6. a-Boosted quasilinear system

The issue of accuracy of SL is clearly of importance. Although no rigorous results are available in the literature, it has been shown in numerous examples that, under the assumption that $P(s)$ is low-pass filtering, SL approximates the behavior of the LPNI system with an accuracy well within 10% in terms of the standard deviations $\sigma_{\bar{y}}$, $\sigma_{\hat{y}}$ and $\sigma_{\bar{u}}$, $\sigma_{\hat{u}}$ [4], [7], [16], [17]. Further results on accuracy are presented in Section VI of this paper.

Note that SL has previously been used to study a variety of specific LPNI configurations. In particular, it has been used to analyze both tracking and disturbance rejection in the case of saturating actuators in [3], [8] and [17]. In [4], a method is presented to mitigate performance degradation by selecting appropriate parameters of a saturating actuator. None of these works, however, provides a method for controller design to *recover* the performance of an arbitrary linear design; this is carried out in this paper.

In general, analysis of LPNI systems has centered on the issue of stability (see, for instance, [10], [11], and the references therein), while issues of performance analysis are addressed to a lesser extent. One exception is the well-known method of anti-windup, used to mitigate the effects of actuator saturation on integral control schemes [1], [13]. These schemes are used in deterministic (i.e., step) reference tracking, and require additional elements in the feedback loop. In contrast, the boosting methodology treats disturbance rejection, and by using SL, enables performance recovery by simply increasing the controller gain.

The remainder of this paper is organized as follows: Section II provides the problem formulation. Sections III and IV present solutions of the boosting problems for actuators and sensors, respectively. These results are combined in Section V, which derives a separation principle and presents the case when $f(u)$ and $g(y)$ are simultaneously present. The accuracy of the results is validated statistically in Section VI. An illustrative example follows in Section VII. Conclusions are formulated in Section VIII.

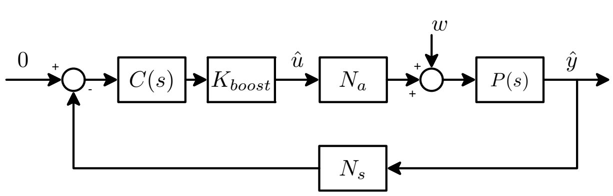

Fig. 5. Equivalent Boosted quasilinear system

II. Problem Formulation

The boosting methodology amounts to a modification of the controller $C(s)$ so that the quasilinear system completely recovers the performance of the original *linear* system, i.e.,

$$\sigma_{\hat{y}} = \sigma_y. \tag{4}$$

The technique achieves (4) by introducing scalar gains K_a and K_s, as shown in Figure 4. The idea is to compensate for the effects of $f(u)$ and $g(y)$ by selecting K_a and K_s to offset N_a and N_s respectively, so that

$$K_a N_a = K_s N_s = 1. \tag{5}$$

Note that in this case, N_a and N_s are functions of K_a and K_s, which makes the boosting problem nontrivial. Since K_s, K_a and $C(s)$ commute, boosting can be implemented by placing a single gain at the output of $C(s)$ as shown in Figure 5, where

$$K_{boost} := K_a K_s. \tag{6}$$

In addition, we establish a separation principle, which enables K_a and K_s to be evaluated from two simpler subproblems: (1) a-boosting, i.e., boosting to account for *only* a nonlinear actuator (assuming $g(y) = y$), and (2) s-boosting, i.e., boosting to account for *only* a nonlinear sensor (assuming $f(u) = u$).

A. *a-Boosting*

Consider the LPNI system with $g(y) = y$. Hence, the only nonlinearity is the actuator $f(u)$ and Figure 4 reduces to Figure 6. Here, since w is standard white noise, (2) becomes

$$N_a = \int_{-\infty}^{\infty} f'(x) \frac{1}{\sigma_{\hat{u}}\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma_{\hat{u}}^2}\right) dx. \tag{7}$$

Define the real analytic function

$$\mathcal{F}(\sigma) := \int_{-\infty}^{\infty} f'(x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma^2}\right) dx. \tag{8}$$

Since

$$\sigma_{\hat{u}} = \left\| \frac{P(s)\, C(s) K_a}{1 + P(s)\, N_a K_a C(s)} \right\|_2, \tag{9}$$

where $\|\cdot\|_2$ denotes the 2-norm of a transfer function, i.e.,

$$\|H\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega}, \tag{10}$$

(7) can be rewritten as

$$N_a = \mathcal{F}\left(\left\| \frac{P(s)\, C(s) K_a}{1 + P(s)\, N_a K_a C(s)} \right\|_2 \right). \tag{11}$$

The problem of a-boosting is to find K_a, if possible, such that

$$K_a N_a = 1, \tag{12}$$

where N_a itself depends on K_a through (11).

B. s-Boosting

Consider the LPNI system with $f(u) = u$. Hence, Figure 4 reduces to Figure 7, where

$$N_s = \int_{-\infty}^{\infty} g'(x) \frac{1}{\sigma_{\hat{y}}\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma_{\hat{y}}^2}\right) dx. \tag{13}$$

Define the function

$$\mathcal{G}(\sigma) := \int_{-\infty}^{\infty} g'(x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma^2}\right) dx. \tag{14}$$

Since

$$\sigma_{\hat{y}} = \left\| \frac{P(s)}{1 + P(s) N_s K_s C(s)} \right\|_2, \tag{15}$$

(13) can be rewritten as

$$N_s = \mathcal{G}\left(\left\| \frac{P(s)}{1 + P(s) N_s K_s C(s)} \right\|_2 \right). \tag{16}$$

The problem of s-boosting is to find K_s, if possible, such that

$$K_s N_s = 1, \tag{17}$$

where, again, N_s is a function of K_s through (16).

Remark 1: The structure of the LPNI system of Figure 2 implies that the problems of a- and s-boosting are not dual. Indeed, observe that for a-boosting, the gain K_a appears in the forward path between w and the input of the actuator nonlinearity, $\hat{u}$. For s-boosting, K_s does *not* appear in the path from w to the input of the sensor $\hat{y}$. Consequently, the numerator of the transfer function in (11) contains a boosting gain, whereas that in (16) does not. Thus, the two problems are different, and must be addressed separately.

III. a-Boosting

As implied by (11) and (12), the problem of a-boosting is equivalent to finding K_a that satisfies

$$K_a \mathcal{F}\left(\left\| \frac{P(s) C(s) K_a}{1 + P(s) N_a K_a C(s)} \right\|_2 \right) = 1. \tag{18}$$

Theorem 1: a-Boosting is possible if and only if

$$x \mathcal{F}\left(x \left\| \frac{P(s) C(s)}{1 + P(s) C(s)} \right\|_2 \right) = 1 \tag{19}$$

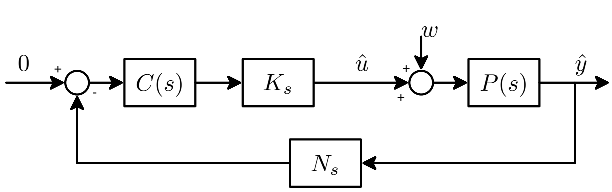

Fig. 7. s-Boosted quasilinear system

has a positive solution. Any positive solution of (19) yields a boosting gain

$$K_a = x. \tag{20}$$

Proof: *Sufficiency:* From (19) and (20),

$$K_a \mathcal{F}\left(K_a \left\| \frac{P(s) C(s)}{1 + P(s) C(s)} \right\|_2 \right) = 1. \tag{21}$$

It follows from (21) that (12) is a solution of (18). Thus, a-boosting is possible with the boosting gain K_a.

Necessity: a-Boosting is possible with the boosting gain K_a. Thus, (12) and (18) hold. Clearly, substituting the former into the latter yields (19) and (20). ■

The existence and uniqueness of K_a depend on the specific form of $\mathcal{F}(\cdot)$. This is analyzed below for the saturation nonlinearity. Other nonlinearities can be treated analogously.

A. Actuator Saturation

Consider the a-boosted system of Figure 6 and let $f(\cdot)$ be a static saturation of authority α, i.e.,

$$f(u) = sat_\alpha(u) = \begin{cases} \alpha, & u > +\alpha \\ u, & -\alpha \le u \le \alpha \\ -\alpha, & u < -\alpha. \end{cases} \tag{22}$$

In this case, it is shown in [7] that (8) reduces to

$$\mathcal{F}(\sigma) = \text{erf}\left(\frac{\alpha}{\sqrt{2}\sigma} \right), \tag{23}$$

where erf denotes the standard *error function*,

$$\text{erf}(z) = \frac{2}{\pi} \int_0^z e^{-t^2} dt. \tag{24}$$

Hence,

$$N_a = \text{erf}\left(\frac{\alpha}{\sqrt{2} \left\| \frac{P(s)C(s)K_a}{1+P(s)N_a K_a C(s)} \right\|} \right). \tag{25}$$

It follows from Theorem 1 and (23) that a-boosting for the saturation nonlinearity (22) is possible if and only if the equation

$$x \text{erf}\left(\frac{c}{x} \right) = 1 \tag{26}$$

has a positive solution, where

$$c = \frac{\alpha}{\sqrt{2} \left\| \frac{P(s)C(s)}{1+P(s)C(s)} \right\|_2}. \tag{27}$$

Theorem 2: Equation (26) admits a unique positive solution if and only if

$$\alpha > \sqrt{\frac{\pi}{2}} \left\| \frac{P(s) C(s)}{1 + P(s) C(s)} \right\|_2. \tag{28}$$

Note that since

$$\sigma_u = \left\| \frac{P(s) C(s)}{1 + P(s) C(s)} \right\|_2 \tag{29}$$

and

$$\sqrt{\frac{\pi}{2}} \approx 1.25, \tag{30}$$

the following can be stated:

Rule-of-thumb 1: a-Boosting for a saturating actuator is possible if

$$\alpha > 1.25\sigma_u. \tag{31}$$

Remark 2: As it has been shown in [4],

$$\alpha > 2\sigma_u, \tag{32}$$

without boosting, leads to no more than 10% performance degradation of the linear design. In comparison, the above rule-of-thumb achieves complete performance recovery, with actuators that are less powerful than recommended in [4].

B. Performance recovery by redesigning $C(s)$

If (19) does not have a solution, i.e., a-boosting is impossible, a question arises: Can $C(s)$ be redesigned to achieve $\sigma_{\hat{y}} = \sigma_y$? The answer depends on the ability to find a controller that simultaneously achieves the performance specification *and* yields a solution to (19). Such a controller is said to be *boostable*.

In the case of actuator saturation, the boosting condition (28) implies that finding a boostable controller is a linear minimum-effort control problem, i.e., the problem of finding a controller that minimizes σ_u for a specified performance level σ_y. One method for accomplishing this is given in [9], where a controller $C_{opt}(s)$ is synthesized that yields the desired output σ_y with minimum control effort. If (28) is not satisfied by this $C_{opt}(s)$, then no linear boostable controller exists.

IV. s-BOOSTING

As implied by (16) and (17), the problem of s-boosting is equivalent to finding K_s that satisfies

$$K_s \mathcal{G}\left(\left\| \frac{P(s)}{1 + P(s) N_s K_s C(s)} \right\|_2\right) = 1. \tag{33}$$

Since, unlike a-boosting, K_s enters the argument of $\mathcal{G}$ only as a factor of N_s, and for s-boosting $N_s K_s = 1$, the solution of (33) is always possible and is given by

$$K_s = \frac{1}{\mathcal{G}\left(\left\| \frac{P(s)}{1+P(s)C(s)} \right\|_2\right)}. \tag{34}$$

This result warrants further investigation since it suggests that linear performance may be recovered in the presence of *any* sensor nonlinearity. It turns out that, although an s-boosting gain can always be found, in some cases the accuracy of stochastic linearization may be poor. Thus, certain conditions should be satisfied before using s-boosting. These are developed in Section VI.

Below, we give explicit expressions for the function $\mathcal{G}$ in (34) for various types of sensor nonlinearities.

A. Sensor Saturation

In the case where $g(y)$ is a symmetric saturation of range α, the right hand side of (16) becomes

$$\mathcal{G}(\sigma) = erf\left(\frac{\alpha}{\sqrt{2}\sigma}\right). \tag{35}$$

B. Sensor Deadzone

Let $g(y)$ be a symmetric deadzone of the form

$$g(y) = \begin{cases} y - \frac{\Delta}{2}, & y > +\frac{\Delta}{2} \\ 0, & -\frac{\Delta}{2} \le y \le \frac{\Delta}{2} \\ y + \frac{\Delta}{2}, & u < -\frac{\Delta}{2}. \end{cases} \tag{36}$$

In this case, the right hand side of (16) becomes

$$\mathcal{G}(\sigma) = 1 - erf\left(\frac{\Delta/2}{\sqrt{2}\sigma}\right). \tag{37}$$

C. Sensor Quantization

Let $g(y)$ be a mid-tread quantizer of the form

$$g(y) = \frac{\Delta}{2} \sum_{k=1}^{m} \left[\operatorname{sgn}(2y + \Delta(2k-1)) \times \operatorname{sgn}(2y - \Delta(2k-1))\right]. \tag{38}$$

Then,

$$\mathcal{G}(\sigma) = Q_m\left(\frac{\Delta}{\sqrt{2}\sigma}\right), \tag{39}$$

where

$$Q_m(z) := \frac{2z}{\sqrt{\pi}}\left[\sum_{k=1}^{m} e^{-\frac{1}{4}(2k-1)^2(z)^2}\right]. \tag{40}$$

V. SIMULTANEOUS a- AND s-BOOSTING

The following separation principle ensures that the results of Sections III and IV remain applicable when actuator and sensor nonlinearities are simultaneously present.

Theorem 3: Simultaneous a- and s-boosting is possible if and only if each is possible independently. Moreover, the boosting gains K_a and K_s are the same as the individual a- and s-boosting gains, respectively.

Proof: Observe from Figure 3 that

$$K_a N_a = K_a \mathcal{F}\left(\left\| \frac{P(s) C(s) N_s K_s K_a}{1 + P(s) N_s K_s N_a K_a C(s)} \right\|_2\right) \tag{41}$$

and

$$K_s N_s = K_s \mathcal{G}\left(\left\| \frac{P(s)}{1 + P(s) N_s K_s N_a K_a C(s)} \right\|_2\right). \tag{42}$$

Subtituting

$$K_a N_a = K_s N_s = 1 \tag{43}$$

into (41) and (42) yields (18) and (33), which establishes the separation principle. ■

VI. ACCURACY OF STOCHASTIC LINEARIZATION IN SYSTEMS WITH BOOSTING

In this section, validation of the accuracy of stochastic linearization is performed in the context of boosting. Design guidelines are formulated to avoid cases where accuracy is poor.

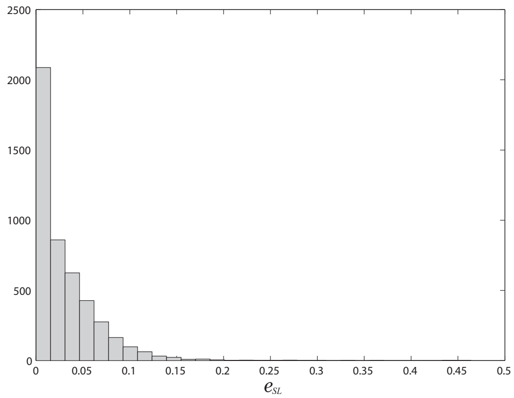

Fig. 8. Histogram of e_{SL} for a-boosting

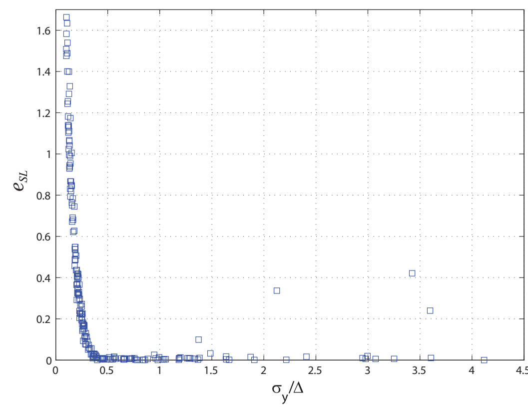

Fig. 9. e_{SL} as function of σ_y/Δ

A. Accuracy of a-Boosting

The accuracy of stochastic linearization for saturating actuators has been studied in detail in [5], [7], [8], and has been shown to be good for a wide range of systems. To validate accuracy in the context of boosting, the following statistical study is performed: We consider 2500 first-order and 2500 second-order plants of the form:

$$P_1(s) = \frac{1}{Ts+1}, \tag{44}$$

$$P_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}. \tag{45}$$

The controller is $C(s) = K$ and the actuator is a saturation of the form (22). The system parameters are randomly and equiprobably selected from the following sets:

$T \in [0.01, 10]$,
$\omega_n \in [0.01, 10], \zeta \in [0.05, 1]$,
$K \in [1, 20]$,
$\alpha \in (\alpha_{min}, 2\alpha_{min}]$,

where α_{min} is the right hand side of (28). Boosting is performed for each system, and the LPNI system is simulated to identify the error of stochastic linearization, defined as

$$e_{SL} = \frac{|\sigma_{\bar{y}} - \sigma_{\hat{y}}|}{\sigma_{\hat{y}}}. \tag{46}$$

The histogram of e_{SL} is shown in Figure 8. Clearly, accuracy is very good: 71.4% of the systems yield $e_{SL} < 0.05$ and only 9.2% of systems yield $e_{SL} > 0.1$. Further analysis reveals that these latter cases occur when the signals $\bar{u}$ and $\bar{y}$ are highly non-Gaussian. This is consistent with the assumption of stochastic linearization, namely that those signals should be approximately Gaussian.

Remark 3: In general, stochastic linearization is accurate when the closed loop linear system provides a sufficient amount of low-pass filtering [15]. A similar situation holds for the method of describing functions.

B. Accuracy of s-Boosting

A similar statistical study is performed to validate the accuracy of stochastic linearization in the context of s-boosting. Here, the sensor is assumed to be a mid-tread quantizer of the form (38), and $C(s) = K$. We consider 1000 first- and 1000 second-order plants of the form (44) and (45), with system parameters chosen equiprobably from the sets:

$T \in [0.01, 10]$,
$\omega_n \in [0.01, 10], \zeta \in [0.05, 1]$,
$K \in [1, 20]$,
$m \in [1, 10]$,
$\Delta \in (0, 4\sigma_y]$,

where σ_y is the nominal linear performance to be recovered. As illustrated in Figure 9, simulation reveals that accuracy degrades significantly as the ratio σ_y/Δ decreases. This is expected, since when σ_y/Δ is small, most of the output signal lies in the quantizer deadzone. Hence, the nonlinear system operates in an effectively open loop regime. Our experience indicates that to avoid this situation, the following should be observed:

$$\frac{\sigma_y}{\Delta} > 0.33. \tag{47}$$

This leads to:

Rule-of-thumb 2: s-Boosting for a quantized sensor is possible if

$$\Delta < 3\sigma_y. \tag{48}$$

Remark 4: Recall that Δ is the *total* deadzone width, and hence (48) stipulates that the deadzone 'amplitude', i.e., $\Delta/2$, should be no greater than 1.5 standard deviations. This rule-of-thumb may seem generous, since intuition would suggest that $\Delta/2$ should be, at most, one standard deviation. The extra deadzone width allowance comes from boosting, which increases the loop gain.

When (47) is satisfied, the accuracy of s-boosting is similar to that of a-boosting. Again, accuracy is generally very good, and fails in those scenarios where the plant has insufficient filtering characteristics.

Remark 5: Similar results hold when $g(y)$ is the symmetric deadzone (36). In general, Rule-of-thumb 2 should be observed for any sensor nonlinearities that exhibit small gain near the origin.

VII. Illustrative Example

To illustrate the efficacy of boosting, consider the system of Figure 1 with the plant

$$P(s) = \frac{1}{s} \tag{49}$$

and the controller

$$C(s) = 4\frac{s+0.5}{s+2}. \tag{50}$$

The resulting output standard deviation is

$$\sigma_y = 0.5. \tag{51}$$

Assume that the control input is constrained by a saturation with authority $\alpha = 2$ and the sensor exhibits a deadzone of width $\Delta = 0.1$. Simulation of the system with this nonlinear instrumentation results in

$$\sigma_{\bar{y}} = 0.581, \tag{52}$$

a degradation of 16%.

It is easily verified that $\sigma_u = 1.2247$, and hence (28) is satisfied. Thus, the conditions of Theorem 3 are met and boosting can be used to recover the original linear performance. Solving (26) and (34) results in

$$K_{boost} = 1.325. \tag{53}$$

Using this boosting gain in a MATLAB simulation yields the desired result:

$$\sigma_{\bar{y}} = 0.503. \tag{54}$$

Thus, a successful recovery of the designed performance is demonstrated.

VIII. Conclusions

This paper develops the method of boosting, whereby a scalar gain is applied to an existing controller in order to recover linear performance in the presence of nonlinear instrumentation. Necessary and sufficient conditions under which boosting is possible are provided. Boosting enables controller design using well-known linear techniques, such as LQR/LQG, while at the same time accounting for the effects of sensors and actuators.

References

[1] K. J. Astrom and L. Rundqwist. Integrator windup and how to avoid it. *Control Conference, American*, 26, 1982.

[2] R. C. Boonton. Nonlinear control systems with random inputs. *IRE Trans. on Circuit Theory*, CT-1:9–18, 1954.

[3] S. Ching, S. M. Meerkov, and P. T. Kabamba. Root locus for random reference tracking in systems with saturating actuators. *IEEE Transactions on Automatic Control*, 54:79–91, 2009.

[4] Y. Eun, C. Gökçek, P. T. Kabamba, and S. M. Meerkov. Selecting the level of actuator saturation for small performance degradation of linear designs. *Proceedings of the 40th IEEE Conference on Decision and Control*, pages 1769–1774, 2001.

[5] Y. Eun, P. T. Kabamba, and S. M. Meerkov. Analysis of random reference tracking in systems with saturating actuators. *IEEE Transactions on Automatic Control*, 50(11):1861–1866, 2005.

[6] A. Gelb and W. E. V. Velde. *Multiple Input Describing Function and Nonlinear Design*. New York: McGraw-Hill, 1968.

[7] C. Gökçek, P. T. Kabamba, and S. M. Meerkov. Disturbance rejection in control systems with saturating actuators. *Nonlinear Analysis*, 40:213–226, 2000.

[8] C. Gökçek, P. T. Kabamba, and S. M. Meerkov. An LQR/LQG theory for systems with saturating actuators. *IEEE Transactions on Automatic Control*, 46(10):1529–1542, 2001.

[9] K. M. Grigoriadis and R. E. Skelton. Minimum-energy covariance controllers. *Automatica*, 33-4:569–578, 1997.

[10] T. Hu and Z. Lin. *Control Systems with Actuator Saturation: Analysis and Design*. Birkhauser, Boston, 2001.

[11] V. Kapila. *Actuator Saturation Control*. Marcel Dekker, 2001.

[12] I. E. Kazakov. Approximate method for the statistical analysis of nonlinear systems. Technical Report VVIA 394, Trudy, 1954.

[13] M. V. Kothare, P. J. Campo, M. Morari, and C. N. Nett. A unified framework for the study of anti-windup designs. *Automatica (Journal of IFAC)*, 30:1869–1883, 1994.

[14] D. Liberzon and R. W. Brockett. Nonlinear feedback systems perturbed by noise: Steady-state probability distributions and optimal control. *IEEE Transactions on Automatic Control*, 45:1116–1130, 2000.

[15] J. Roberts and P. Spanos. *Random Vibration and Statistical Linearization*. Wiley, 1990.

[16] J. Skrzypczyk. Accuracy analysis of statistical linearization methods applied to nonlinear dynamical systems. *Reports on Mathematical Physics*, 36:1–20, 1995.

[17] W. M. Wonham and W. F. Cashman. A computational approach to optimal control of stochastic saturating systems. *Int. Journal of Control*, 10:77–98, 1969.