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Probabilistic guarantees for rendezvous under noisy measurements

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Abstract— **This paper studies the performance of consensus-based rendezvous algorithms when the agent location measurements are subject to noise. In our previous work [1] we provided worst-case bounds on the convergence radius in the case of noisy location estimates. Even though worst-case results are tight, they are conservative. The aim of this paper is thus to investigate typical realizations of consensus-based rendezvous algorithms. We show that while the expected value of the convergence radius is finite, it is bounded by the noise covariance. We also show that there is a natural trade-off between the speed of convergence and the radius of convergence to rendezvous. The results are illustrated with simulations.**

I. Introduction

In robotic networks, *rendezvous* refers to the task of controlling agents in a formation towards a common meeting point, using only the observations of the neighboring agents. Several distributed algorithms for solving the rendezvous problem are currently available in the literature. For example, the approach originally presented in [2] has been extended to both synchronous [3] and asynchronous [4], [5] cases. The proposed algorithms are all distributed in the sense that each robot takes the decision based only on the observations of a certain subset of the agents in the formation.

Parallel to this work, there has been much research in the control community on consensus algorithms. Originally described in [6], and presented to the engineering community in [7] in the context of parallel computing [8], [9], consensus protocols has been introduced to the control community in [10]–[12] and have been extensively studied since. Variations of the consensus protocol have also been studied. For instance the so called gossip algorithms by the computer network community [13], also known as *aggregation protocols* [14], are examples of such alternative formulations.

Observe that if the agents move freely in $\mathbb{R}^n$, then achieving rendezvous for the network is equivalent for it to achieve consensus in $\mathbb{R}^n$ on their locations. This relationship between consensus and rendezvous has not been unnoticed to researchers [15]–[17]. In our paper [18] we investigated this relationship and proved that rendezvous algorithms that rely on the geometric properties of the convex hull are, in fact, a particular realizations of consensus protocols.

Most existing studies on the rendezvous assume that the evolution of the system is deterministic: there are no random influences on the measurements and the evolution of the state. This assumption is difficult to justify in real life applications, where both measurements and the evolution of the system have some degree of uncertainty. Some studies [19], [20] have analyzed the effect of noise for a particular version of the consensus protocol, and [21] considers the effect of uniformly distributed measurement noise for a particular class of rendezvous algorithms. In [1] we generalized these results and studied a general class of the consensus (and rendezvous) algorithms, where the only assumption on the noise is that it is zero-mean and bounded. We presented a deterministic performance guarantee that was tight for the worst case scenario, but for typical realizations it was quite conservative. Here we present a probabilistic analysis of the typical situation. Our analysis focuses on the study of the noisy consensus (for which the general solution still is, to the best of our knowledge, an open problem). We give a bound for the expected radius of the formation after each step, and offer some insights on the expected size of the convergence ball we obtained in [1]. We prove that if the noise covariance is uniformly bounded by σ^2, then the squared radius we get after each iteration differs by *at most* σ^2 with respect to the one derived in the deterministic case.

The paper is organized as follows. We first review consensus algorithms. Then, we characterize the contraction rate for consensus matrices, and review the results from [1] on worst-case convergence under noisy state estimation. We then study the convergence of the system in both the mean and mean-square senses. We demonstrate that although convergence in the mean is achieved as long as the estimation noise has zero mean, convergence in the mean-square sense is not possible for the general case. The paper concludes with simulations that verify the theoretical claims.

II. Preliminaries and notation

A. Consensus algorithms

Consensus protocols were introduced by DeGroot [6], and brought to the engineering community by Tsitsiklis in 1984 [7]. They were then re-discovered independently by the control community with the work of Jadbabaie *et al.* [10]. Subsequent research inspired by this work led to the continuous time version of the protocol [11], and was generalized in [12]. We refer the interested reader to the survey [22] and the references therein. In this paper, we focus on the discrete time consensus algorithm.

Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a vector, and let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix with the following properties:

1) $\mathbf{A}$ is primitive: there is a positive integer k such that $\mathbf{A}^k$ has all its entries positive.
2) $\mathbf{A}$ is stochastic: all its entries are non-negative, and the sum of the entries in each row is equal to 1.

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It follows from Geršgorin's circle theorem [23] and the Perron-Frobenius Theorem for primitive matrices [24] that $\mathbf{A}$ has all but one of its eigenvalues in the interior of the complex unit circle and that the remaining eigenvalue is equal to 1 and has $\mathbf{1}$, the vector in $\mathbb{R}^n$ which has all its entries equal to 1, as its associated eigenvector. From here it follows that the discrete time linear system given by $\mathbf{x}_m = \mathbf{A}^m \mathbf{x}_0$ is stable, and converges to an equilibrium point which is a scalar multiple of $\mathbf{1}$, the eigenvector associated with the eigenvalue 1 [25]. We call such a matrix $\mathbf{A}$ a *consensus matrix*.

It is shown, among others in [7], [12], that if $\{\mathbf{A}_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ are all consensus matrices, and their positive entries are uniformly bounded below by a positive real number $\alpha > 0$ (independent of i), then

$$\lim_{m\to\infty} \prod_{i=1}^{m} \mathbf{A}_i = \mathbf{v}^T \otimes \mathbf{1}. \tag{1}$$

where $\otimes$ denotes the Kronecker product and $\mathbf{v}$ is a vector such that it has non-negative entries that add up to one. The following lemma, taken from [24], can easily be seen to hold.

Lemma 1: Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be two non-negative matrices. If both $\mathbf{A}$ and $\mathbf{B}$ have its zero and positive entries in the same positions, then either both matrices are primitive, or none of them is. That is, for non-negative matrices the condition of being primitive depends only on the profile of the matrix.

B. Contraction rate for a consensus matrix

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ be a consensus matrix. If the matrix represents a connected graph, and its positive entries are bounded below by $\epsilon > 0$, we characterize the set $\mathcal{C}_\epsilon^n$ that describes the elements of such matrices as

$$\mathcal{C}_\epsilon^n = \mathcal{B}_\epsilon^n \cap \mathcal{S}_\epsilon^n \cap \mathcal{P}_\epsilon^n \cap \mathcal{D}_\epsilon^n, \tag{2}$$

where for $1 \le i, j \le n$

$$\mathcal{B}_\epsilon^n = \{a_{i,j} : a_{i,j} \in \{0\} \cup [\epsilon, 1]\},$$
$$\mathcal{S}_\epsilon^n = \{a_{i,j} : \sum_{k=1}^{n} a_{i,k} = 1\},$$
$$\mathcal{P}_\epsilon^n = \{a_{i,j} : \sum_{l_1,\ldots,l_{n-1}} a_{il_1} a_{l_1 l_2} \ldots a_{l_{n-1} j} \ge \epsilon^n\},$$
$$\mathcal{D}_\epsilon^n = \{a_{i,j} : a_{i,i} \in [\epsilon, 1]\},$$

The set $\mathcal{C}_\epsilon^n$ characterizes all the consensus matrices we are interested in. Observe that since $\mathcal{C}_\epsilon^n$ is the finite intersection of closed sets, it is closed. Since $\mathcal{B}_\epsilon^n$ is a bounded set, so is $\mathcal{C}_\epsilon^n$. Hence this set is compact.

For a particular $\mathbf{C} \in \mathcal{C}_\epsilon^n$, we denote its n eigenvalues (not necessarily distinct) by $\lambda_1, \ldots, \lambda_n$, where $|\lambda_n| \le \cdots \le |\lambda_2| < \lambda_1 = 1$. As a consequence of Rouché's theorem and the inverse mapping theorem for analytic functions [26], the roots of a polynomial are continuous functions of its coefficients. Since the coefficients of the characteristic polynomial of a matrix are a continuous function of its entries, the second largest eigenvalue λ_2 is a continuous function over the set $\mathcal{C}_\epsilon^n$, which is compact. Hence, there exists $\rho < 1$ such that $|\lambda_2| \le \rho$, for every $\mathbf{C} \in \mathcal{C}_\epsilon^n$. For technical reasons we will assume from now on that all the eigenvalues are distinct. Note that this is generically true since the set of matrices with repeated eigenvalues has measure zero.

Let $\mathbf{v}_i$ be the eigenvector associated to the eigenvalue λ_i. The set $\mathcal{V} = \{\mathbf{v}_i\}_{i=1}^n$ is linearly independent, hence $\operatorname{span}(V) = \mathbb{R}^n$. Let $\Delta = \operatorname{span}(\mathbf{1})$ be the diagonal on $\mathbb{R}^n$. Given $\mathbf{C} \in \mathcal{C}_\epsilon^n$, we denote by $\nabla_\mathbf{C}$ the complement of Δ which is invariant under the action of $\mathbf{C}$, $\nabla_\mathbf{C} = \operatorname{span}(\mathcal{V} \setminus \{\mathbf{1}\})$. Note that $\mathbf{C}$ acts as the identity on Δ.

Recall that $\mathbb{R}^n = \Delta \oplus \nabla_\mathbf{C}$. Let $\mathbf{v} \in \mathbb{R}^n$. Write $\mathbf{v} = \mathbf{v}_\Delta + \mathbf{v}_{\nabla_\mathbf{C}}$. Observe that for the elements of $\mathbf{v}$ to be in consensus, we need $\mathbf{v}_{\nabla_\mathbf{C}} = 0$. If v_i, v_j are the components i and j of $\mathbf{v}$, then

$$\begin{aligned} |v_i - v_j| &= \left|\left((v_\Delta)_i + (v_{\nabla_\mathbf{C}})_i\right) - \left((v_\Delta)_j + (v_{\nabla_\mathbf{C}})_j\right)\right| \\ &= \left|(v_{\nabla_\mathbf{C}})_i - (v_{\nabla_\mathbf{C}})_j\right| \le \left|(v_{\nabla_\mathbf{C}})_i\right| + \left|(v_{\nabla_\mathbf{C}})_j\right| \\ &\le \sqrt{2}\, \|\mathbf{v}_{\nabla_\mathbf{C}}\|. \end{aligned} \tag{3}$$

Given $\mathbf{v}$, its norm in $\nabla_\mathbf{C}$ is a continuous function of $\mathbf{C} \in \mathcal{C}_\epsilon^n$. Since $\mathcal{C}_\epsilon^n$ is compact, there is a matrix $\mathbf{C}^\star \in \mathcal{C}_\epsilon^n$ for which the norm of $\mathbf{v}$ in $\nabla_{\mathbf{C}^\star}$ is maximum. Let $\mathbf{r} \in \nabla_\mathbf{C}$. Since $\nabla_\mathbf{C}$ is invariant under $\mathbf{C}$, then we can restrict the norm $\|\cdot\|$ in $\mathbb{R}^n$ to a norm in this invariant subspace. We denote such a norm as $\|\cdot\|_{\nabla_\mathbf{C}}$. In case $\nabla_\mathbf{C} = \Delta^\perp$, the orthogonal complement to the diagonal, we refer to this norm as the *residual error*. Observe that

$$\|\mathbf{C}\,\mathbf{r}\|_{\nabla_\mathbf{C}} \le \|\mathbf{C}\|_{\nabla_\mathbf{C}} \cdot \|\mathbf{r}\|_{\nabla_\mathbf{C}} \le \rho \|\mathbf{r}\|_{\nabla_\mathbf{C}}. \tag{4}$$

Therefore, if $\mathbf{v}' = \mathbf{C}\mathbf{v}$, from (3) and (4) we obtain

$$\left|(v')_i - (v')_j\right| \le \sqrt{2}\, \|(\mathbf{C}\mathbf{v})\|_{\nabla_\mathbf{C}} \le \sqrt{2}\rho\, \|\mathbf{v}_{\nabla_{\mathbf{C}^\star}}\|. \tag{5}$$

This represents a uniform bound on the decay rate between any two elements in $\mathbf{v}$. In particular, this bound holds for the elements that attain the *diameter* of $\mathbf{v}$ (the ones that maximize the left-hand side in (3)). This establishes the following result:

Theorem 2: The rate of convergence to consensus under matrices in $\mathcal{C}_\epsilon^n$ is at least exponential with the rate ρ.

This result (the exponential convergence rate) is well known in the literature on consensus algorithms, and for some particular matrices tighter convergence bounds have been obtained [27]–[29]. For the case when noise is present, in [20] some convergence rates are derived, but they rely on the particular structure of the consensus matrices they are considering.

III. Model

Let $\mathcal{R}$ be a robotic network with N robots as defined in [30], and let $\{q_i\}_{i=1}^N$ be the positions of the robots with respect to a fixed coordinate frame $\mathcal{Q}$. We assume that the robots have no knowledge about $\mathcal{Q}$.

We assume that each of the robots is capable of identifying robots that satisfy a certain criterion $\sim$, defining in this way the edges of the proximity graph corresponding to the formation. Observe that $\sim$ is not necessarily a symmetric relation. For each robot i, we let $\mathcal{N}_i = \{j \in \mathcal{R} | i \sim j\}$ be the set of *neighbors* of i. We assume that i is a neighbor of itself, $i \sim i$. Some examples of such relation $\sim$ are *being closer than certain distance* d or *being neighbors in the sense of Voronoi*.

In [18] we assumed that each robot i was able at any time to correctly estimate the positions of its neighbors with respect to its local coordinate frame $\mathcal{Q}_i$. We showed that convergence to rendezvous was independent of such a reference frame. We denote the position of robot j in the frame $\mathcal{Q}_i$ by $p_{i,j}$. In this paper, we focus on the case when if $i \sim j$, then the *estimated* position $\tilde{p}_{i,j}$ of robot j by robot i is $\tilde{p}_{i,j} = p_{i,j} + n_{i,j}$ where $n_{i,j}$ is some measurement noise.

Based on the information that the robots gather from the observations of their neighbors they update their position, with respect to the coordinate frame $\mathcal{Q}$, as

$$q_i[m+1] = q_i[m] + u_i[m], \tag{6}$$

where the control law u_i for the motion of robot i is based on distributed consensus and is described next.

IV. Consensus-Based Rendezvous

Algorithm 1 Consensus-Based Rendezvous

Require: Agent i at time m

1: Identify the set of neighbors $\mathcal{N}_i = \{i_1, i_2, \ldots, i_{r_i(m)}\}$.
2: Evaluate the position p_{i,i_j}, $1 \le j \le r_i(m)$ of each neighbor, and its own position p_{i,i_0} with respect to a local coordinate frame $\mathcal{Q}_i$.
3: Compute $p_i = \sum_{j=0}^{r_i(m)} \lambda_{i,j} p_{i,i_j}$, where $\lambda_{i,j} > \epsilon > 0$ and $\sum_{j=0}^{r_i(m)} \lambda_{i,j} = 1$.
4: Set $u_i[m] = \varrho\,(p_i - p_{i,i_0})$, where $0 < \varrho < 1$.

Algorithm 1 is the *Consensus-Based Rendezvous* (CBR) as presented in [18] when no noise is present in the measurements. When estimation is subject to noise, the positions observed by the robot i are $\tilde{p}_{i,i_j}$ rather than p_{i,i_j}. Thus, Line 3 becomes

$$p_i = \sum_{j=0}^{r_i(m)} \lambda_{i,j} p_{i,i_j} + \sum_{j=0}^{r_{i(m)}} \lambda_{i,j} n_{i,j}, \tag{7}$$

where $n_{i,i} = 0$. The only assumption we make about the noise is that it has zero mean and bounded support.

The following discussion on the bound for the ball to which the robots converge under the CBR algorithm is taken from [1], and is included here for completeness.

As in [18], we can show that in the noisy scenario the updates can be made invariant with respect to the coordinate frame that each robot chooses. Furthermore, the orthonormality of the matrix describing the change of coordinates implies that the noise levels remain invariant.

Lemma 3 (Lemma 3 in [18]): The evolution of each robot is independent of the local frame $\mathcal{Q}_i$ it chooses to implement the CBR algorithm.

Due to this lemma we can thus assume that the location of the robots are described in the global frame $\mathcal{Q}$. We can thus stack together the equations for each robot, and write the evolution of the system in the matrix form as $\mathbf{q}[m+1] = \mathbf{I}\,\mathbf{q}[m] + \mathbf{U}[m] + \mathfrak{N}[m]$, where $\mathbf{I} \in \mathbb{R}^{N \times N}$ is the identity matrix, $\mathbf{q} \in \mathbb{R}^{N \times n}$, $\mathbf{U} \in \mathbb{R}^{N \times n}$ and the noise matrix $\mathfrak{N}[m] \in \mathbb{R}^{N \times n}$. Under the assumption that the proximity graph $\mathcal{G}(\mathcal{R})$ is connected, the induced matrix $\mathbf{A}_\mathcal{G}$, where the entry $a_{i,j} = \lambda_{i,j}$, is a consensus matrix. This makes $\mathbf{U}[m] = \varrho\,(\mathbf{A}_\mathcal{G} - \mathbf{I})\,\mathbf{q}[m]$, $\varrho \in (0, 1)$ and thus we can rewrite the discrete time system as

$$\mathbf{q}[m+1] = [(1-\varrho)\mathbf{I} + \varrho \mathbf{A}_\mathcal{G}]\,\mathbf{q}[m] + \mathfrak{N}[m]. \tag{8}$$

Since the matrix $[(1-\varrho)\mathbf{I} + \varrho \mathbf{A}_\mathcal{G}] = \mathbf{C}_\mathcal{G}[m]$ is also a consensus matrix, we rewrite (8) as

$$\mathbf{q}[m+1] = \mathbf{C}_\mathcal{G}[m]\mathbf{q}[m] + \mathfrak{N}[m]. \tag{9}$$

Remark 1: Although (9) has been derived under the assumption of a uniform ϱ for each robot, it is possible to derive an equivalent formulation if each robot has its own ϱ_i.

To simplify the notation, we will drop the dependency on the proximity graph $\mathcal{G}$. Under the action of $\mathbf{C}$, the evolution of the formation can be viewed as the joint evolution of n individual consensus systems in $\mathbb{R}^N$, all of them sharing the same consensus matrix. From now on, we thus consider a single vector in $\mathbb{R}^N$, knowing that the results extend to $\mathbb{R}^{N \times n}$.

Suppose that at time $m+1$, the system evolves according to the matrix $\mathbf{C} \in \mathcal{C}_\epsilon^N$. Let Δ be the diagonal in $\mathbb{R}^N$ and let $\nabla_\mathbf{C}$ be the complement of Δ invariant under the action of $\mathbf{C}$. Since $\mathbb{R}^N = \Delta \oplus \nabla_\mathbf{C}$ the evolution can be decomposed as

$$\begin{aligned} \mathbf{q}[m+1]_\Delta &= (\mathbf{C}[m]\mathbf{q}[m])_\Delta + \mathfrak{N}[m]_\Delta, \\ \mathbf{q}[m+1]_{\nabla_\mathbf{C}} &= (\mathbf{C}[m]\mathbf{q}[m])_{\nabla_\mathbf{C}} + \mathfrak{N}[m]_{\nabla_\mathbf{C}}. \end{aligned}$$

The norm $\|\cdot\|_{\nabla_\mathbf{C}}$ indicates how *far* the formation is from consensus, so it is enough to focus on the subspace $\nabla_\mathbf{C}$. We say that the formation reaches *ξ-consensus* if $\|\mathbf{q}\|_{\nabla_\mathbf{C}} < \xi$. With respect to reaching consensus, the effects of $\mathfrak{N}[m]_\Delta$ are negligible. Since $\nabla_\mathbf{C}$ is invariant under $\mathbf{C}$, it thus suffices to study the evolution only on this subspace:

$$\mathbf{q}[m+1]_{\nabla_\mathbf{C}} = (\mathbf{C}[m]\mathbf{q}[m])_{\nabla_\mathbf{C}} + \mathfrak{N}[m]_{\nabla_\mathbf{C}}. \tag{10}$$

For simplicity the index $\nabla_\mathbf{C}$ will be dropped from now on, but all the results apply only to $\nabla_\mathbf{C}$. Consider $\mathbf{C}\mathbf{q}[m] + \mathfrak{N}[m]$. From (4) we have that

$$\|\mathbf{q}[m+1]\| \le \rho \|\mathbf{q}[m]\| + \|\mathfrak{N}[m]\|. \tag{11}$$

Thus we have the following lemma:

Lemma 4: If at time m

$$\|\mathfrak{N}[m]\| < (1-\rho)\|\mathbf{q}[m]\|, \tag{12}$$

then $\|\mathbf{q}[m+1]\| < \|\mathbf{q}[m]\|$.

This result implies that as long as the noise is bounded (uniformly in time), the formation will converge to a finite ball. The bounded noise assumption is quite natural since the noise is the result of the measurement errors due to imperfect sensors, which have a finite range.

Theorem 5: Suppose that the noise is uniformly bounded by ς, this is $\|\mathfrak{N}[m]\| \leq \varsigma < \infty$ for every time m. Then, as long as $\|\mathbf{q}\| > \varsigma/(1-\rho)$, $\|\mathbf{q}\|$ will be decreasing, and the formation will converge to $\varsigma/(1-\rho)$-rendezvous.

Proof: This follows from LaSalle's principle and Lemma 4. ■

Note that Theorem 5 guarantees that the formation converges to a ball if it is not already inside it. Once inside, it can escape, but it will be then driven back into the ball.

As discussed in [1], for the bound given by Theorem 5 to be tight we require equality in the different steps, which leads us to the worst-case scenario. In a typical realization, the worst case scenario is far from typical, and the robots converge to a smaller ball. In the next section, we look at the typical realization, and offer some guarantees for its performance.

V. PROBABILISTIC GUARANTEES

The bound presented in Theorem (5) is tight, but conservative for a typical realization. We thus provide some probabilistic guarantees on this bound. In particular, we look at convergence in the mean and mean-square sense.

Although we are allowing each robot to choose the weights for the convex combinations (or the coefficients in the consensus matrices $\mathbf{C}$) at random, the stochastic properties we consider next are due to the noise in the measurements rather than to the matrices. Hence, for our noise analysis, we will assume that the matrices are known.

A. Convergence in the mean

We first show that, on average, the system will behave as its deterministic counterpart. This result holds as long as the noise has zero mean. We assume that the robots are points in $\mathbb{R}$, but the results extend directly to $\mathbb{R}^n$.

Claim 6: The noisy rendezvous system described by (9) converges in the mean. That is, there exists $q^\star$ such that $\lim_{k\to\infty} \mathbb{E}\left[|q_i[k] - q^\star|\right] = 0$ for each i.

Proof: Given $k \in \mathbb{N}$, it follows that

$$\begin{aligned} \mathbb{E}\left[\mathbf{q}[k+1]\right] &= \mathbb{E}\left[\mathbf{C}[k]\mathbf{q}[k] + \mathfrak{N}[k]\right] \\ &= \mathbb{E}\left[\mathbf{C}[k]\mathbf{q}[k]\right] + \mathbb{E}\left[\mathfrak{N}[k]\right] \\ &= \mathbf{C}[k]\mathbb{E}\left[\mathbf{q}[k]\right], \end{aligned} \quad (13)$$

and thus, by induction,

$$\mathbb{E}\left[\mathbf{q}[k+1]\right] = \left(\prod_{j=0}^{k} \mathbf{C}[k-j]\right) \mathbb{E}\left[\mathbf{q}[0]\right]. \quad (14)$$

But $\mathbb{E}\left[\mathbf{q}[0]\right] = \mathbf{q}[0]$, therefore

$$\lim_{k\to\infty} \mathbb{E}\left[\mathbf{q}[k+1]\right] = \mathbf{q}^\star, \quad (15)$$

where $\mathbf{q}^\star$ is the consensus value for the system if no noise is present. Since $\mathbf{q}^\star \in \Delta$, we can write $\mathbf{q}^\star = q^\star \mathbf{1}$. Choosing this $q^\star$ the result follows. ■

It is desirable to give some guarantees on the convergence of the system in the mean square sense, since this is related to the radius of the formation. Unfortunately, the convergence cannot be ensured for the general case[1]. In the next subsection it is shown that, although mean square convergence cannot be ensured, it is possible to impose a bound on the evolution of the residual error at each step, which can be related to quantifying how far the robots are from the rendezvous location.

B. Expected residual error

In Theorem 5 we derived a bound on the maximum diameter of the formation. As we discussed in [1] the bound is quite conservative for typical realizations. A much tighter bound can be derived by studying the mean-square properties of the formation. We will show that, at each step, the formation behaves *almost* as it would in the deterministic case, with an additional offset that is bounded by σ^2, the covariance for the measurement noises. From this derivation it will directly follow that the convergence of the system in the mean-square sense can not be guaranteed.

We first introduce some additional notation. If $\mathbf{q} \in \mathbb{R}^N$ we will denote by $\mu_{\mathbf{q}}$ the mean of the components of $\mathbf{x}$, $\mu_{\mathbf{q}} = \frac{1}{n}\mathbf{1}^T\mathbf{q}$.

Theorem 7: Given a sequence of $N \times N$ stochastic matrices $\{\mathbf{C}[k]\}_{k\in\mathbb{N}}$, let $\mathbf{q}[0] = \begin{bmatrix} q_1[0] & \cdots & q_N[0] \end{bmatrix}^T$ be a real vector in $\mathbb{R}^N$. If

$$\mathbf{q}[k+1] = \mathbf{C}[k]\mathbf{q}[k] + \mathfrak{N}[k] \quad (16)$$

where for $1 \leq i \leq N$ the noise $\mathfrak{N}[k] = \begin{bmatrix} \mathfrak{N}_1[k] & \cdots & \mathfrak{N}_N[k] \end{bmatrix} \in \mathbb{R}^N$ is a vector of independent random variables with $\mathbb{E}\left[\mathfrak{N}_i[k]\right] = 0$ and $\mathbb{E}\left[\mathfrak{N}_i^2[k]\right] = \sigma_i^2 < \sigma^2$, then

$$\begin{aligned} &\mathbb{E}\left[(q_i[k+1] - \mu_{\mathbf{q}}[k+1])^2 \middle| \mathbf{q}[k] \right] \\ &\qquad \leq (\hat{q}_i[k+1] - \mu_{\hat{\mathbf{q}}}[k+1])^2 + \frac{(N-1)\sigma^2}{N}, \end{aligned} \quad (17)$$

where $\hat{\mathbf{q}}[k+1] = \mathbf{C}[k]\mathbf{q}[k]$.

Proof: Recall that $\lambda_{i,j}[k]$ denotes the entry (i,j) of $\mathbf{C}[k]$. We thus rewrite $q_i[k+1]$ as

$$q_i[k+1] = \sum_{j\in\mathcal{N}_i} \lambda_{i,j}[k] q_j[k] + \mathfrak{N}_i[k]. \quad (18)$$

Since $\hat{q}_i[k+1] = \sum_{j\in\mathcal{N}_i} \lambda_{i,j} q_j[k]$, (17) can be rewritten as

$$\begin{aligned} &\mathbb{E}\left[(q_i[k+1] - \mu_{\mathbf{q}}[k+1])^2 \middle| \mathbf{q}[k] \right] \\ &= \mathbb{E}\left[\left((\hat{q}_i[k+1] - \mu_{\hat{q}}[k+1]) + (\mathfrak{N}_i[k] - \mu_{\mathfrak{N}}[k])\right)^2 \middle| \mathbf{q}[k] \right] \\ &= \mathbb{E}\left[\left((\hat{q}_i[k+1] - \mu_{\hat{q}}[k+1])^2 \middle| \mathbf{q}[k] \right] \\ &+ \mathbb{E}\left[2\left(\hat{q}[k+1] - \mu_{\hat{q}}[k+1]\right)\left(\mathfrak{N}_i[k] - \mu_{\mathfrak{N}}[k]\right) \middle| \mathbf{q}[k] \right] \\ &+ \mathbb{E}\left[(\mathfrak{N}_i[k] - \mu_{\mathfrak{N}}[k])^2 \middle| \mathbf{q}[k] \right]. \end{aligned} \quad (19)$$

[1]Although, for special realizations of the consensus matrices the result holds. For details, see [19].

We now analyze each of the three right-hand side terms of (19). From the definition of $\hat{\mathbf{q}}[k+1]$, the first term is simply

$$\mathbb{E}\left[\left(\hat{q}_i[k+1]-\mu_{\hat{\mathbf{q}}}[k+1]\right)^2\Big|\,\mathbf{q}[k]\right] = \left(\hat{q}_i[k+1]-\mu_{\hat{q}}[k+1]\right)^2. \quad (20)$$

Since the $\mathfrak{N}_i$ are independent from $\mathbf{q}[k]$, we obtain for the second term

$$\mathbb{E}\left[2\left(\hat{\mathbf{q}}[k+1]-\mu_{\hat{\mathbf{q}}}[k+1]\right)\left(\mathfrak{N}_i[k]-\mu_{\mathfrak{N}}[k]\right)|\,\mathbf{q}[k]\right]=0. \quad (21)$$

Finally, for the third term, since the $\mathfrak{N}_i$ are independent from $\mathbf{q}[k]$, we can ignore the conditional part of the expectation, and thus obtain

$$\mathbb{E}\left[\left(\mathfrak{N}_i[k]-\mu_{\mathfrak{N}}[k]\right)^2\Big|\,\mathbf{q}[k]\right] = \mathbb{E}\left[\mathfrak{N}_i[k]^2\right]-2\mathbb{E}\left[\mathfrak{N}_i[k]\mu_{\mathfrak{N}}[k]\right]+\mathbb{E}\left[\mu_{\mathfrak{N}}^2[k]\right]. \quad (22)$$

The first term on the right hand side of (22) is simply the variance of the noise, which is bounded by σ^2. For the second and third terms we will use that the $\mathfrak{N}_i$ are uncorrelated. Therefore,

$$\mathbb{E}\left[\mathfrak{N}_i[k]\mu_{\mathfrak{N}}[k]\right]\le\frac{\sigma^2}{N}, \quad (23)$$

and

$$\mathbb{E}\left[\mu_{\mathfrak{N}}^2[k]\right]=\frac{1}{N^2}\mathbb{E}\left[\sum_{r=1}^{n}\mathfrak{N}_r^2+\sum_{r\neq j}\mathfrak{N}_r\mathfrak{N}_j\right]\le\frac{\sigma^2}{N}. \quad (24)$$

Putting these results together, we can express (17) as

$$\begin{aligned}\mathbb{E}&\left[\left(q_i[k+1]-\mu_{\mathbf{q}}[k+1]\right)^2\Big|\,\mathbf{q}[k]\right]\\ &\le\left(\hat{q}_i[k+1]-\mu_{\hat{\mathbf{q}}}[k+1]\right)^2+\left(\sigma^2-\frac{\sigma^2}{N}\right)\\ &=\left(\hat{q}_i[k+1]-\mu_{\hat{\mathbf{q}}}[k+1]\right)^2+\frac{(N-1)\sigma^2}{N}, \quad (25)\end{aligned}$$

which we wanted to prove. ■

Remark 2: It is easy to see (for instance, following [20]) that $\mathbf{q}-\mu_{\mathbf{q}}\mathbf{1}\in\Delta^{\perp}$. Therefore, (17) quantifies the mean-square value for the residual error after each iteration.

Remark 3: The bound in (25) is tight for the case when the $\mathfrak{N}_i$ are independent identically distributed random variables with $\mathbb{E}\left[\mathfrak{N}_i^2\right]=\sigma^2$. This implies that after *any* step, the residual error is expected to be, in the mean square sense, about σ^2 away from the residual error in the deterministic case. This also implies that convergence in the mean square sense is not possible in general.

VI. Physical constraints on σ

As described in [1], the effective value of σ will depend on the maximum velocity of the robots.

For a robot i, consider its next location p_i as defined in (7). Since we are dealing with physical robots with an upper bound on how fast they can move, there is a d so that the distance between successive points in time is bounded by d, $\|q_i[m+1]-q_i[m]\|\le d$ for every m. This means that although the point p_i obtained in (7) might satisfy $|p_i-q_i[m]|>d$, the maximum velocity constraint of the robot will limit it to be at most d units away from where it started during each time interval. Let $|p_i-q_i[m]|=D>d$. The point $q_i[m+1]$ that the robot reaches is then at most

$$\begin{aligned}q_i[m+1] &= q_i[m]+d\left(p_i-q_i[m]\right)/D\\ &= (1-d/D)q_i[m]+(d/D)p_i. \quad (26)\end{aligned}$$

Since p_i is as presented in (7), we can rewrite (26) as

$$\begin{aligned}q_i[m+1]&=(1-d/D)q_i[m]\\ &+(d/D)\left(\sum_{j=0}^{r_i(m)}\lambda_{i,\,j}q_{i,\,i_j}[m]+\sum_{j=0}^{r_{i(m)}}\lambda_{i,j}n_{i,\,j}\right) \quad (27)\end{aligned}$$

This implies that the noise $\mathfrak{N}[m]_i=\sum_{j=0}^{r_{i(m)}}\lambda_i n_{i,\,j}$ will be reduced by a factor of $d/D<1$.

In other words, the smaller the time interval and the slower the robots are, the smaller the d and the more robust the system will be with respect to noise. On the other hand, the slower the robots are, the slower the convergence will be.

VII. Simulations

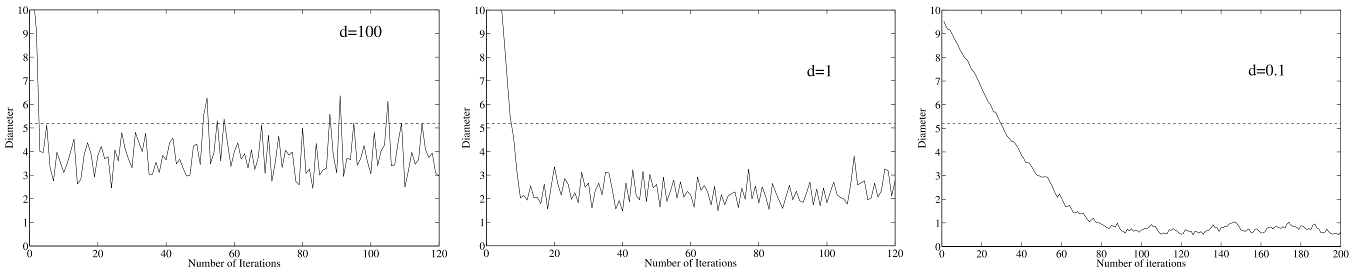
Note that the theoretical results are invariant to change of scale. So, for the simulations, we will omit any explicit reference to units.

For the simulations we present here, we implemented our algorithm by uniformly deploying 30 robots in a square of side 10. We assumed a uniform noise distribution between $[-9,\,9]$ (giving thus a variance of $\tilde{\sigma}^2=324/12=27$) for the relative measurements between the robots (the $n_{i,\,j}$ in (7)), which is quite large compared to the size of the region. We chose the uniform distribution for the noise because among all the distributions with a given bounded support, this is the one that provides the least information about the process. We ran the system by setting d in (26) to .1, 1 and 100 respectively, and assuming a proximity graph induced by an r-disk graph with $r=6$. Observe that because we were considering the $n_{i,\,j}$ rather than the individuals $\mathfrak{N}_i[k]$, the value of σ^2 in our bound satisfies $\sigma^2\le\tilde{\sigma}^2=27$.

Figure 1 shows the evolution of the diameter for the same noise level and different values of d. As the figure shows, for typical realizations our bound reflects the behavior of the residual error. As we discussed, the smaller the d the more robust the system is to noise (hence, the smaller the σ that affects the formation). As expected, the ξ-rendezvous level depends on d but, because of the dependency of ρ on d, the final effect is quite complex.

VIII. Conclusion

We derived probabilistic bounds on the performance of consensus-based rendezvous algorithms when the agent location measurements are subject to noise. We showed that the bound for the residual error of convergence is bounded by σ at each iteration with respect to the behavior in the deterministic case. Furthermore, under the assumptions that the noise has bounded support and zero mean, the system converges in the mean to the deterministic value for the rendezvous. We also showed that in general it is not possible



(a) Diameter evolution when $d = 100$. (b) Diameter evolution when $d = 1$. (c) Diameter evolution when $d = 0.1$.

Fig. 1. Evolution of the formation diameter when the noise is uniformly distributed in $[-9, 9]$, and the robots have different maximum speeds. Dashed line marks the theoretical bound. Observe that the larger the maximum traveled distance d, the less robust the system is to noise, but the faster it converges.

to guarantee the convergence in the mean square sense for consensus protocols. Further work will focus on deriving better bounds by studying typical realizations of consensus matrices.

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