2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Reduced Order Observer Design for Nonlinear Systems with Control Applications

Zhengtao Ding

Abstract— **This paper presents a reduced-order observer design for nonlinear dynamic systems. The proposed reduced-order observer can be applied to a larger class of nonlinear dynamic systems than the systems with linear observer dynamics. The proposed observer design has also been used for control design of a class of nonlinear dynamic systems with the nonlinearity in the unmeasured system state variables. An example is included to demonstrate that the proposed observer and control design method can be used to stabilize a nonlinear dynamic system with unstable nonlinear zero dynamics.**

I. INTRODUCTION

Nonlinear observer design has attracted attention of control researchers from 1970s [1], [2]. One significant break through was the nonlinear observers with linear observer dynamics [3]. Many other results have been presented in literature (for example, see [4], [5], [6] etc). More recently, results have been reported on reduced-order design for nonlinear dynamic systems [7], [8]. In this paper, we present a basic design and analysis of reduced-order observers for general nonlinear dynamic systems, and then extend this basic design for some specific nonlinear systems, including nonlinear dynamic systems with the nonlinearity of unmeasured state variables.

One purpose of nonlinear observer design is to provide state estimate for control design using output feedback. Related results on nonlinear output feedback control can be found in a number of references [9], [10], [11], [12], [13], [14], [15], [16], [17]. However, there are very few results on output feedback control of nonlinear systems with the nonlinearities of the unmeasured system state variables or nonlinear unstable zero dynamics. In this paper, the proposed reduced order observer design will be applied to tackle some of these output feedback control problems. An example of output feedback control of a nonlinear systems with nonlinear unstable zero dynamics is included to demonstrate the proposed observer design for output feedback control.

II. A REDUCED-ORDER NONLINEAR OBSERVER DESIGN

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output, $f : \mathbb{R}^n \to \mathbb{R}^n$, is a nonlinear smooth vector field.

We have the following assumption of the system.

Zhengtao Ding is with Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, PO Box 88, Manchester M60 1QD, UK, Email:zhengtao.ding@manchester.ac.uk

Assumption 1. There exists a nonlinear map $\beta : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that

$$T = \begin{bmatrix} h(x) \\ \beta(x) \end{bmatrix}$$

is a diffeomorphism. Furthermore, with the state transformation defined by $y = h(x)$, $z = \beta(x)$, the dynamics of z can be written in the form

$$\dot{z} = p(z) + q(y). \tag{2}$$

with $p(0) = 0$. Furthermore, for the dynamics $\dot{z} = p(z)$, there exists a $V : \mathbb{R}^{n-m} \to \mathbb{R}$ such that for $z, \hat{z} \in \mathbb{R}^{n-m}$

$$\begin{gathered} c_1\|z\|^2 \le V(z) \le c_2\|z\|^2 \\ \frac{\partial V(z-\hat{z})}{\partial z}(p(z) - p(\hat{z})) \le -c_3\|z - \hat{z}\|^2 \\ \left\| \frac{\partial V(z)}{\partial z} \right\| \le c_4\|z\|^2 \end{gathered} \tag{3}$$

for some positive real constants c_i for $i = 1$ to 4.

With the conditions specified in Assumption 1 being satisfied, we can propose a reduced order observer design as

$$\dot{\hat{z}} = p(\hat{z}) + q(y) \tag{4}$$

$$\hat{x} = T^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix} \tag{5}$$

Theorem 1: The state of the original system z and of the observer $\hat{z}$ are bounded if the system output is bounded. The error of the reduced order observer (4) defined by $e = z - \hat{z}$ exponentially converges to zero and ehe estimate given in (4) asymptotically converges to of the state variable of (1).

Proof. First, let us establish the boundedness of z. We take $V(z)$ as a Lyapunov function candidate with $\hat{z} = 0$. From (2) and (3), we have

$$\begin{aligned} \dot{V} &\le -c_3\|z\|^2 + \frac{\partial V}{\partial z}q(y) \\ &\le -c_3\|z\|^2 + c_4\|z\|\|q(y)\| \\ &\le -c_3\|z\|^2 + c_4c_5\|z\|^2 \\ &\quad + \frac{4c_4}{c_5}\|q(y)\|^2 \end{aligned}$$

with c_5 being a positive real constant. Set $c_5 = \frac{c_3}{2c_4}$ and we have

$$\begin{aligned} \dot{V} &\le -\frac{c_3}{2}\|z\|^2 + \frac{8c_4^2}{c_3}\|q(y)\|^2 \qquad (6) \\ &\le -\frac{c_3}{2c_2}V + \frac{8c_4^2}{c_3}\|q(y)\|^2 \end{aligned}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

Therefore, from the comparison lemma [18], we can conclude that V is bounded if the output y of the original system is bounded, which implies the boundedness of z. The boundedness of $\hat{z}$ can be established in the same way.

The dynamics of e is given by

$$\dot{e} = p(z(t)) - p(z(t) - e)$$

Taking $V(e)$ as the Lyapunov function candidate, we have

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial e}(p(z) - p(z-e)) \\ &\leq -c_3\|e\|^2 \leq -\frac{c_3}{c_2}V(e)\end{aligned}$$

Therefore we can conclude that the estimation error exponentially converge to zero. With $\hat{z}$ as an exponentially convergent estimate of z, we can further conclude that $\hat{x}$ is an asymptotic estimate of x. This concludes the proof of Theorem 1.

To demonstrate the proposed reduced order observer design, let us consider an example.

Example 1. Consider a second order system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - 3x_1^2x_2 - x_1^3 \\ \dot{x}_2 &= x_2 - 6x_2x_1^2 + 3x_2^2x_1 - x_2^3 \\ y &= x_1.\end{aligned}$$

Let us check if Assumption 1 is satisfied. For this, we need to find $\beta(x)$. Take

$$z = \beta(x) = x_2 - x_1.$$

We have

$$\begin{aligned}\dot{z} &= -x_2 - (x_2 - x_1)^3 \\ &= -(1 + z^2)z + y\end{aligned}$$

Let $V = \frac{1}{2}z^2$. It is easy to see the first and the third conditions in (3) are satisfied. For the second condition, we have

$$\begin{aligned}&\frac{\partial V(z-\hat{z})}{\partial z}(p(z) - p(\hat{z})) \\ =& -(z-\hat{z})(z - \hat{z} + z^3 - \hat{z}^3) \\ =& -(z-\hat{z})^2(1 + z^2 - z\hat{z} + \hat{z}^2) \\ =& -(z-\hat{z})^2(1 + \frac{1}{2}(z^2 + \hat{z}^2 + (z-\hat{z})^2)) \\ \leq& -(z-\hat{z})^2\end{aligned} \tag{7}$$

Therefore, the system satisfy the conditions specified (3). We design the reduced order observer as

$$\begin{aligned}\dot{\hat{z}} &= -(1 + \hat{z}^2)\hat{z} + y \\ \hat{x}_2 &= \hat{z} + y\end{aligned}$$

Simulation study has been carried out, and the simulation results are shown in Figures 1 and 2.

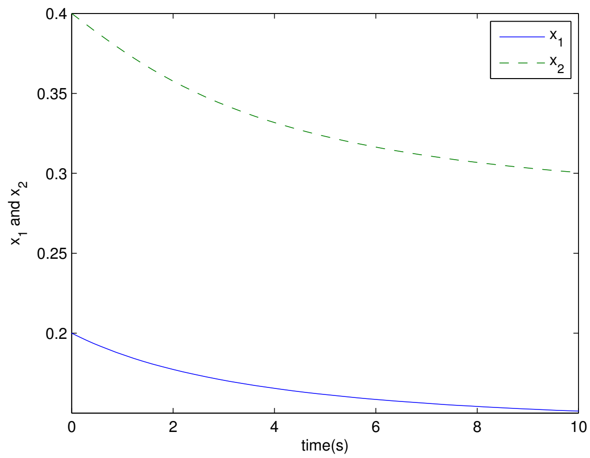

Fig. 1. The state variables

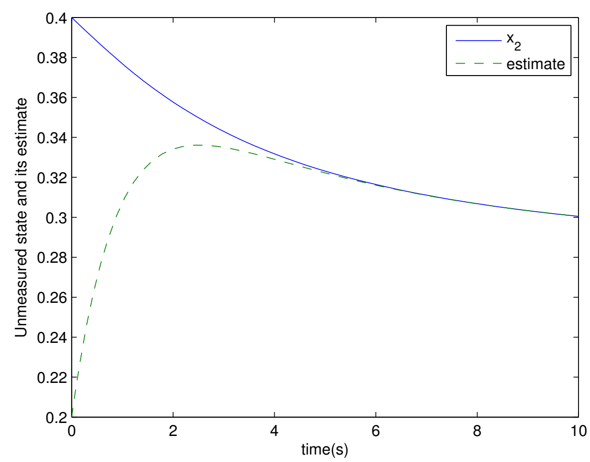

Fig. 2. Unmeasured state and its estimate

III. Reduced-Order Observer Design for Partially Linear Systems

Consider a single-input-single-output (SISO) nonlinear system

$$\begin{aligned}\dot{x} &= Ax + \phi(y,u) + f\varphi(d^Tx), \\ y &= c^Tx\end{aligned} \tag{8}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control, ϕ, is a known nonlinear smooth vector field, $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function, $c, d, f \in \mathbb{R}^n$ are constant vectors, and $A \in \mathbb{R}^{n\times n}$ is a constant matrix.

When $\varphi = 0$, the system (8) degenerates to the well known form of the nonlinear systems with the linear observer error dynamics, and nonlinear observer can be easily designed by using nonlinear output injection. With this additional term φ, the nonlinear output injection term can no longer be used to generate a linear observer error dynamics. In this section, we will convert the system to the form considered in the previous section, and they apply the reduced-order observer design for state observation. We specify certain conditions in the next assumption for the class of systems (1).

Assumption 2.

2.1 $\{c^T, A\}$ is observable.

2.2 A SISO linear system characterized by $\{A, f, c^T\}$ has relative degree 1, and is minimum phase.

We will show that with Assumption 2, we can find a $g(x)$ such that Assumption 1 is satisfied.

From Assumption 2.1, there exists a nonsingular state transformation

$$w = \mathcal{A}\mathcal{O}x := T_2 x$$

where

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & 1 \end{bmatrix}, \mathcal{O} = \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{n-1} \end{bmatrix},$$

with a_i, for $i = 1, \dots, n$, being the coefficients of the characteristic polynomial of A. It can be obtained that

$$\begin{aligned} \dot{w} &= A_o w + \bar{\phi}(w_1, u) + \bar{f}\varphi(d^T T_2^{-1} w) \\ y &= w_1 \end{aligned} \tag{9}$$

where A_o is the left companion matrix of the characteristic polynomial of A, and $\bar{\phi} = T_2\phi$, $\bar{f} = T_2 f$ and $\bar{b} = T_2 b$. Note that Assumption 2.2 ensures $\bar{f}_1 \neq 0$.

Let us introduce another state transformation $\eta = T_3 w$ where

$$T_3 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{\bar{f}_2}{\bar{f}_1} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\bar{f}_n}{\bar{f}_1} & 0 & \dots & 1 \end{bmatrix}.$$

Based on this state transformation, we let $z = \eta_{2:n}$, where the subscript $(2:n)$ denotes the vector form by the second to the nth elements of the original vector.

A direct evaluation gives

$$\dot{z} = Fz + \phi_w(y, u) \tag{10}$$

where $F \in \mathbb{R}^{(n-1)\times(n-1)}$ is the left companion matrix of the characteristic polynomial $s^{n-1} + \frac{\bar{f}_2}{\bar{f}_1}s^{n-2} + \dots + \frac{\bar{f}_n}{\bar{f}_1} = 0$,

$$f_w = \left[\frac{\bar{f}_2}{\bar{f}_1}, \dots, \frac{\bar{f}_n}{\bar{f}_1}\right]^T,$$

and

$$\phi_w(y, u) = \bar{\phi}_{2:n}(y, u) - \bar{\phi}_1(y, u) f_w - (a_{2:n} - a_1 f_w) y$$

with $a = [a_1, a_2, \dots, a_n]^T$. Comparing (10) with (2), we have $p(z) = Fz$ and $q(y) = \phi_w(y, u)$. Assumption 2.2 ensures that F is Hurwitz, and therefore there exist two positive definite matrices P and Q such that

$$F^T P + PF = -Q.$$

It can be easily shown that $V = z^T P z$ satisfy the conditions shown in (3). Hence, we design the reduced order observer as

$$\dot{\hat{z}} = F\hat{z} + \phi_w(y, u) \tag{11}$$

$$\hat{x} = T_2^{-1} T_3^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix}. \tag{12}$$

Since Assumption 1 is satisfied for (8) with $g(x) = \eta_{2:n}$, we can conclude from Theorem 1 that the observer error e converges to zero exponentially. In fact it is easy to see $\dot{e} = Fe$ for this case. Since $T = T_3 T_2$ is a linear transformation, the state estimate $\hat{x}$ converges to x exponentially.

The above result is summarized in the following lemma.

Lemma 2: For the system (8), the conditions specified in Assumption 2 ensure that Assumption 1 is satisfied. Furthermore, the reduced-order observer (11) ensures that the observer error exponentially converges to zero, and the estimate $\hat{x}$ in (12) exponentially converge to the state variable x.

IV. Application to Control Design Output Feedback Systems

For the control design, we need to further assume the structure information of the system. In particular, we replace $\phi(y, u)$ in (8) by $\phi(y) + b\sigma(y)u$, i.e., the system is now described by

$$\begin{aligned} \dot{x} &= Ax + \phi(y) + f\varphi(d^T x) + b\sigma(y)u, \\ y &= c^T x \end{aligned} \tag{13}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinear function and $\sigma(y) \neq 0, \forall y \in \mathbb{R}$, $b \in \mathbb{R}^n$ is constant vector. For the control design, we introduce two more conditions in the following assumptions.

Assumption 3.

3.1 $\{d^T, A\}$ is observable.

3.2 A SISO linear system characterized by $\{A, b, d^T\}$ has relative degree n.

Since $\{d^T, A\}$ is observable, we define a state transformation

$$\xi = T_1 x$$

where $T_1 = [d, A^T d, \dots, (A^{n-1})^T d]^T$, and it is nonsingular. Under the coordinate ξ, we have $y = c^T T_1^{-1}\xi := l^T \xi$, and we define $\psi(y) := T_1 \phi(y)$.

Assumption 4. The nonlinear function ψ satisfies the following condition

$$\frac{\partial \psi_i(l^T \xi)}{\partial \xi_j} = 0$$

for $i = 1, \dots, n-1$ and $j > i$.

Remark 1: The condition specified in Assumption 4 is similar to the triangular condition imposed on the strict feedback form [12], and it is required for control design.

For the convenience of the control design, we transform the system (13) to the state variable ξ. It can be obtained as

$$\begin{aligned} \dot{\xi} &= A_c \xi + \psi(y) + g\varphi(\xi_1) + h\sigma(y)u, \\ y &= l^T \xi \end{aligned} \tag{14}$$

where A_c is the lower companion matrix of the characteristic polynomial of A, $h = T_1 b$ and in this section, we denote $g = T_1 f$. Based on Assumption 3.2, we have $h_i = 0$ for $i = 1, \ldots, n-1$ and $h_n \neq 0$. Hence from the structure of A_c and h, we can write the dynamics for the individual states as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \psi_1(y) + g_1\varphi(\xi_1) \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + \psi_{n-1}(y) + g_{n-1}\varphi(\xi_1) \\ \dot{\xi}_n &= h_n\sigma(y)u + \psi_n(y) + g_n\varphi(\xi_1) \\ &\quad -\sum_{i=1}^{n} a_i \xi_{n-i+1} \end{aligned} \tag{15}$$

Based on the state estimate for z, we have the estimate for ξ given by

$$\hat{\xi} = T_1 T_2^{-1} T_3^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix}$$

Let $\tilde{\xi} = \xi - \hat{\xi}$ and it can be obtained that

$$\tilde{\xi} = T_1 T_2^{-1} T_3^{-1} \begin{bmatrix} 0 \\ e \end{bmatrix} = T_1 T_2^{-1} \begin{bmatrix} 0 \\ e \end{bmatrix}$$

and

$$\dot{\tilde{\xi}} = T_1 T_2^{-1} \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} T_2 T_1^{-1} \tilde{\xi} := G\tilde{\xi}$$

Since ξ is not available, control design will be carried out with $\hat{\xi}$. For the control design, we introduce the following notations:

$$\begin{aligned} \zeta_1 &:= \hat{\xi}_1 \\ \zeta_i &:= \hat{\xi}_i - \alpha_{i-1}(\hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, y) \text{ for } i = 2, \ldots, n \end{aligned} \tag{16}$$

where α_i are the stabilizing functions to be designed. In order to express the control designed for Step n in the same form as the other steps, we define $\xi_{n+1} := h_n\sigma(y)u - \sum_{i=1}^{n} a_i \xi_{n-i+1}$ and $\hat{\xi}_{n+1} := h_n\sigma(y)u - \sum_{i=1}^{n} a_i \hat{\xi}_{n-i+1}$. Thus, we can denote $\zeta_{n+1} = \hat{\xi}_{n+1} - \alpha_n$ and $\tilde{\xi}_{n+1} = -\sum_{i=1}^{n} a_i \tilde{\xi}_{n-i+1}$. For the convenience of notations, we introduce

$$\begin{aligned} \bar{g}_1 &:= g_1 \\ \bar{g}_i &:= g_i - \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + l_j \frac{\partial \alpha_{i-1}}{\partial y}\right) g_j \end{aligned} \tag{17}$$

for $i = 2, \ldots, n$.

The follow property of smooth functions is useful for the control design and the stability analysis.

Lemma 3: For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, the following expression holds for any $\rho_1, \rho_2 \in \mathbb{R}$,

$$|\varphi(\rho_1 + \rho_2) - \varphi(\rho_1)| \leq \gamma_0(\rho_2) + |\rho_1|\gamma_1(\rho_1)\gamma_2(\rho_2) \tag{18}$$

where $\gamma_i : \mathbb{R} \to \mathbb{R}$ for $i = 0, 1, 2$ are functions with $\gamma_i(\rho) \geq 0$ for any $\rho \in \mathbb{R}$, and $\gamma_1(\cdot)$ is smooth. Furthermore, if for a function $\rho(t) : \mathbb{R}^+ \to \mathbb{R}$ with $|\rho(t)| < \rho_3 e^{-\rho_4 t}$ for positive reals constants ρ_3 and ρ_4, then there exist some positive real constants ρ_5 and ρ_6 such that $\gamma_0(\rho(t)) \leq \rho_5 e^{-\rho_6 t}$ and $\gamma_2(\rho(t)) \leq \rho_5 e^{-\rho_6 t}$.

Proof. Proof is omitted due to the page limit.

The control design will be carried out in n steps.

Step 1. We start the control design from ζ_1. Its dynamics are given by

$$\begin{aligned} \dot{\zeta}_1 &= \dot{\xi}_1 - \dot{\tilde{\xi}}_1 \\ &= \xi_2 + \psi_1(y) + g_1\varphi(\xi_1) - G_1\tilde{\xi} \\ &= \zeta_2 + \alpha_1 + \psi_1(y) + g_1\varphi(\xi_1) - G_1\tilde{\xi} + \tilde{\xi}_2 \end{aligned}$$

where G_1 denotes the first row of matrix G. From Assumption 3, we have

$$\begin{aligned} & \zeta_1 g_1(\varphi(\xi_1) - \varphi(\hat{\xi}_1)) \\ = \; & \zeta_1 g_1(\varphi(\hat{\xi}_1 + \tilde{\xi}_1) - \varphi(\hat{\xi}_1)) \\ \leq \; & |\zeta_1 g_1|(\gamma_0(\tilde{\xi}_1) + |\hat{\xi}_1|\gamma_1(\hat{\xi}_1)\gamma_2(\tilde{\xi}_1)) \\ \leq \; & k\zeta_1^2 g_1^2 + \frac{1}{4k}\gamma_0^2(\tilde{\xi}_1) + k\zeta_1^2 g_1^2 \hat{\xi}_1^2 \gamma_1^2(\hat{\xi}_1) + \frac{1}{4k}\gamma_2^2(\tilde{\xi}_1) \\ = \; & k\zeta_1^2 g_1^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) + \frac{1}{4k}\gamma_0^2(\tilde{\xi}_1) + \frac{1}{4k}\gamma_2^2(\tilde{\xi}_1) \end{aligned}$$

where we have used the fact that $|ab| \leq ka^2 + \frac{1}{4k}b^2$ for any positive real a, b, k. Based on the above, we design the first stabilizing function as

$$\begin{aligned} \alpha_1 &= -(r_1 + k)\zeta_1 - \psi_1(y) - g_1\varphi(\hat{\xi}_1) \\ &\quad -k\zeta_1 \bar{g}_1^2(1 + \gamma_1^2(\hat{\xi}_1)) \end{aligned} \tag{19}$$

where r_1 is among the set of positive real design parameters $\{r_i\}$ for $i = 1, \ldots, n$.

Step 2. The dynamics of ζ_2 is described by

$$\begin{aligned} \dot{\zeta}_2 &= \dot{\hat{\xi}}_2 - \dot{\alpha}_1(\hat{\xi}_1, y) \\ &= \dot{\xi}_2 - G_2\tilde{\xi} - \frac{\partial \alpha_1}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 - \frac{\partial \alpha_1}{\partial y} l_1 \dot{\xi}_1 \\ &= \dot{\xi}_2 - [\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1]\dot{\xi}_1 - G_2\tilde{\xi} + \frac{\partial \alpha_1}{\partial \hat{\xi}_1} G_2\tilde{\xi} \end{aligned}$$

Remark 2: To obtain the above expression, we have used $\dot{y} = \frac{\partial \alpha_1}{\partial y} l_1 \dot{\xi}_1$ not $\dot{y} = \sum_{j=1}^{n} \frac{\partial \alpha_1}{\partial y} l_j \dot{\xi}_j$. This is due to the triangular condition specified in Assumption 2. From Assumption 2, we have $\frac{\partial \psi_1(y)}{\partial y} = 0$ if $l_2 \neq 0$. Similarly we will use $\dot{y} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y} l_j \dot{\xi}_j$ at Step i, because if $l_i \neq 0$, then from Assumption 2 we must have $\frac{\partial \psi_j(y)}{\partial y} = 0$ for $j = 1, \ldots, i-1$. The differences of $\dot{y}$ shown in the expressions do not mean that $\dot{y}$ is different at different steps. In fact, for a given y, the expression is unique, and many terms are actually zero. We use different expressions of $\dot{y}$ in different step for the convenience of notations for a generic expression of y.

Using the dynamics of ξ_1 and ξ_2, and $\xi = \hat{\xi} + \tilde{\xi}$, we have

$$\begin{aligned}\dot{\zeta}_2 &= \zeta_3 + \alpha_2 + \psi_2(y) + \bar{g}_2\varphi(\xi_1) \\ &\quad -[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1](\hat{\xi}_2 + \psi_1(y)) \\ &\quad +\tilde{\xi}_3 - [\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1]\tilde{\xi}_2 \\ &\quad -G_2\tilde{\xi} + \frac{\partial \alpha_1}{\partial \hat{\xi}_j} G_1 \tilde{\xi} \qquad (20)\end{aligned}$$

The stabilizing function α_2 is designed as

$$\begin{aligned}\alpha_2 &= -\zeta_1 - (r_2 + k)\zeta_2 - \psi_2(y) - \bar{g}_2\varphi(\hat{\xi}_1) \\ &\quad +[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1](\hat{\xi}_2 + \psi_1(y)) \\ &\quad -k\zeta_2\bar{g}_2^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) \\ &\quad -k\zeta_2[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1]^2 - k\zeta_2[\frac{\partial \alpha_1}{\partial \hat{\xi}_1}]^2 \qquad (21)\end{aligned}$$

Step i. Similar to the procedures shown in Step 2, in the subsequent steps, for $i = 3, \dots, n$, we have

$$\begin{aligned}\dot{\zeta}_i &= \zeta_{i+1} + \alpha_i + \psi_i(y) + \bar{g}_i\varphi(\xi_1) \qquad (22) \\ &\quad -\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j](\hat{\xi}_{j+1} + \psi_j(y)) \\ &\quad +\tilde{\xi}_{i+1} - \sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]\tilde{\xi}_{j+1} \\ &\quad -G_i\tilde{\xi} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} G_j \tilde{\xi} \qquad (23)\end{aligned}$$

The stabilizing function α_i is obtained as

$$\begin{aligned}\alpha_i &= -\zeta_{i-1} - (r_i + k)\zeta_i - \psi_i(y) - \bar{g}_i\varphi(\hat{\xi}_1) \\ &\quad -\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j](\hat{\xi}_{j+1} + \psi_j(y)) \\ &\quad -k\zeta_i\bar{g}_i^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]^2 \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j}]^2 \qquad (24)\end{aligned}$$

When $i = n$, we have α_n as defined in (24). The control input is then designed by setting $\zeta_{n+1} = 0$, which results in

$$u = \frac{\alpha_n + \sum_{i=1}^{n} a_i\hat{\xi}_{n-i+1}}{h_n\sigma(y)} \qquad (25)$$

Remark 3: In the control design, we use k to be denote a generic positive real design constant, instead of using $k_{i,j}$ for different i and j in α_i, for $i = 1, \dots, n$.

We have the stability result for the proposed control design.

Theorem 4: Under Assumptions 2 to 4, the output feedback control (25) based on the output y and observer state $\hat{z}$ obtained in (11) globally and asymptotically stabilize the dynamic system (13).

Proof. Proof is omitted due to the page limit.

We introduce the control design from an estimated state variable to demonstrate the application of the reduced-order observer design proposed in this paper. One of the advantages of backstepping design from an estimated state variable is its capability to deal with nonlinear systems with unstable zero dynamics. In the remaining part of this section, we include an example to illustrate the proposed control design for the stabilization of such a system.

Example 2. Consider a nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + (x_1 + x_2 + x_3)^2 \\ \dot{x}_2 &= x_3 + 2(x_1 + x_2 + x_3)^2 - y^3 + u \\ \dot{x}_3 &= 3(x_1 + x_2 + x_3)^2 + y^3 - u \\ y &= x_1\end{aligned}$$

Comparing with the structure of (13), we have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \phi(y) = \begin{bmatrix} 0 \\ -y^3 \\ y^3 \end{bmatrix},$$

$$\varphi(d^T x) = (x_1 + x_2 + x_3)^2, \sigma = 1.$$

It can be verified that Assumptions 2, 3 and 4 are satisfied. Furthermore, the zero dynamics is obtained as

$$\dot{x}_0 = x_0 + 6x_0^2$$

where $x_0 = x_2 + x_3$. This zero dynamics is nonlinear and unstable. For the state transformation, we have $T_2 = I$ and

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

We have

$$\begin{aligned}z_1 &= x_2 - 2y \\ z_2 &= x_3 - 3y\end{aligned}$$

and it is easy to obtain that

$$\dot{z} = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} z + \begin{bmatrix} -y - y^3 + u \\ -6y + y^3 - u \end{bmatrix}.$$

We therefore design the reduced order observer as

$$\dot{\hat{z}} = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} -y - y^3 + u \\ -6y + y^3 - u \end{bmatrix}$$

and the estimates of the unmeasured states are given by

$$\begin{aligned}\hat{x}_2 &= \hat{z}_1 + 2y \\ \hat{x}_3 &= \hat{z}_2 + 3y\end{aligned}$$

For the state transformation T_1, we have

$$\xi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x$$

and the system is transformed to

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + 6\xi_1^2 \\ \dot{\xi}_2 &= \xi_3 + 5\xi_1^2 \\ \dot{\xi}_2 &= 3\xi_1^2 + y^3 - u \\ y &= \xi_1 - \xi_2\end{aligned}$$

For the nonlinear function $\varphi(\xi_1) = \xi_1^2$, we take $\gamma_1 = 2$. The control design starts from $\hat{\xi}_1$, for which we have $\hat{\xi}_1 = y + \hat{x}_2 + \hat{x}_3$. From the control design proposed earlier, we have

$$\begin{aligned}\alpha_1 &= -(r_1+k)\zeta_1 - g_1\hat{\xi}_1^2 - kg_1^2\zeta_1(1+4\hat{\xi}_1^2) \\ \alpha_2 &= -\zeta_1 - (r_2+k)\zeta_2 - \bar{g}_2\hat{\xi}_1^2 + \tfrac{\partial\alpha_1}{\partial\hat{\xi}_1}\hat{\xi}_2 \\ &\quad -k\zeta_2\bar{g}_2^2(1+4\hat{\xi}_1^2) - k\zeta_2[\tfrac{\partial\alpha_1}{\partial\hat{\xi}_1}]^2 \\ \alpha_3 &= -\zeta_2 - (r_3+k)\zeta_3 - y^3 - \bar{g}_3\hat{\xi}_1^2 \\ &\quad +\tfrac{\partial\alpha_2}{\partial\hat{\xi}_1}\hat{\xi}_2 + \tfrac{\partial\alpha_2}{\partial\hat{\xi}_2}\hat{\xi}_3 - k\zeta_3\bar{g}_3^2(1+4\hat{\xi}_1^2) \\ &\quad -k\zeta_3([\tfrac{\partial\alpha_2}{\partial\hat{\xi}_1}]^2 + [\tfrac{\partial\alpha_2}{\partial\hat{\xi}_2}]^2) \\ u &= -\alpha_3\end{aligned}$$

where $g_1 = 6$, $\bar{g}_2 = 5 - 6\frac{\partial\alpha_1}{\partial\hat{\xi}_1}$ and $\bar{g}_3 = 3 - 6\frac{\partial\alpha_2}{\partial\hat{\xi}_1} - 5\frac{\partial\alpha_2}{\partial\hat{\xi}_2}$. In the simulation study, we set $r_1 = r_2 = r_3 = 1$ and $k = 0.01$. The simulation results are shown in Figures 3 and 4.

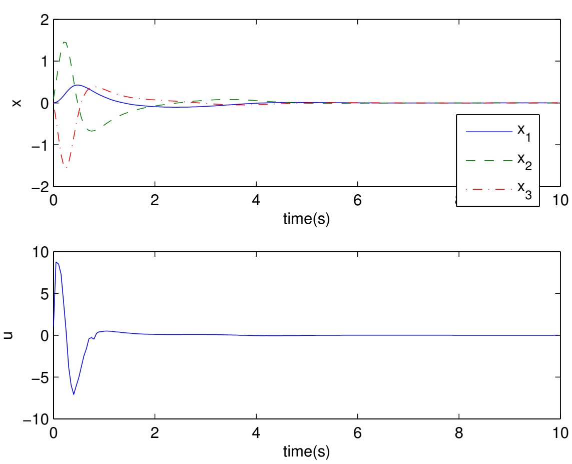

Fig. 3. The state variables and the input

V. Conclusions

In this paper, we have presented a basic design of reduced-order observers for nonlinear dynamic systems. The conditions have been identified for the convergence of the observer errors. More detailed design procedures have been proposed for a specific class of nonlinear systems, which is larger than the nonlinear systems with linear observer errors for the full order observers. The proposed reduced order observer design has been applied to output feedback stabilization of nonlinear systems with nonlinearity of the unmeasured state and nonlinear unstable zero dynamics.

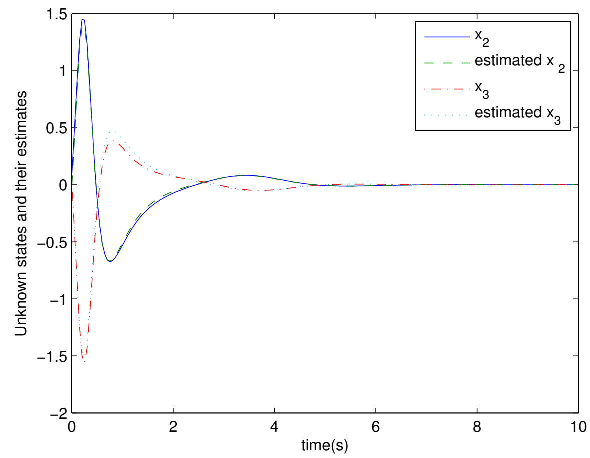

Fig. 4. Unmeasured state and its estimate

References

[1] F. Thau, "Observing the states of nonlinear dynamical systems," *International Journal of Control*, vol. 18, pp. 471–479, 1973.

[2] S. Kou, D. Ellitt, and T. Tarn, "Exponential observers for nonlinear dynamical systems," *Information and Control*, vol. 29, pp. 204–216, 1975.

[3] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Sys. Contr. Lett.*, vol. 3, pp. 47–52, 1983.

[4] D. Bestie and M. Zeitz, "Canonical form observer design fr non-linear time-variable systems," *International Journal of Control*, vol. 38, pp. 47–52, 1983.

[5] X. Xia and W. Gao, "Nonlinear observer design by canonical form," *International Journal of Control*, vol. 47, pp. 1081–1100, 1988.

[6] A. Krener and W. Kang, "Locally convergent nonlinear observers," *SIAM Journal on Control and Optimization*, vol. 42, no. 1, pp. 155–177, 2003.

[7] V. Sundarapandian, "Reduced order observer design for nonlinear systems," *Applied Mathematics Letters*, vol. 19, pp. 936–941, 2006.

[8] C. Kravaris, G. Savvoglidis, M. Kornaros, and N. Kazantzis, "Nonlinear reduced-order observer design for state and disturbance estimation," in *Proc. 13th Mediterranean Conference on Control and Automation*, Limassol, Cyprus, 2005, pp. 1321–1326.

[9] R. Marino and P. Tomei, "Dynamic output feedback linearization and global stabilization," *Systems and Control Letters*, vol. 17, no. 2, pp. 115–121, 1991.

[10] L. Praly and Z. Jiang, "Stabilization by output feedback for systems with iss inverse dynamics and uncertainties," *Systems and Control Letters*, vol. 21, no. 1, pp. 19–33, 1993.

[11] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive, and Robust*. London: Prentice-Hall, 1995.

[12] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: John Wiley & Sons, 1995.

[13] Z. Ding, "Output regulation of uncertain nonlinear systems with nonlinear exosystems," *IEEE Trans. Automatic Control*, vol. 51, no. 3, pp. 498–503, 2006.

[14] L. Liu and J. Huang, "Global robust output regulation of output feedback systems with unknown high-frequency gain sign," *IEEE Trans. Automatic Control*, vol. 51, no. 4, pp. 625–631, 2006.

[15] F. Mazenc, L. Praly, and W. Dayawansa, "Global stabilization by output feedback: examples and counterexamples," *Systems and Control Letters*, vol. 23, no. 2, pp. 119–125, 1994.

[16] C. Qian and W. Lin, "Output feedback control of a class of nonlinear systems: A nonseparation principle paradigm," *IEEE Trans. Automatic Control*, vol. 47, no. 10, pp. 1710–1715, 2002.

[17] H. Choi and J. Lim, "Global exponential stabilization of a class of nonlinear systems by output feedback," *IEEE Trans. Automatic Control*, vol. 50, no. 2, pp. 255–257, 2005.

[18] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey: Prentice Hall, 2002.