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Consensus Problems with Directed Markovian Communication Patterns

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Abstract— **This paper is a continuation of our previous work and discusses the consensus problem for a network of dynamic agents with directed information flows and random switching topologies. The switching is determined by a Markov chain, each topology corresponding to a state of the Markov chain. We show that in order to achieve consensus almost surely and from any initial state, each union of graphs from the sets of graphs corresponding to the closed positive recurrent sets of states of the Markov chain must have a spanning tree. The analysis relies on tools from matrix theory, Markovian jump linear systems theory and random process theory. The distinctive feature of this work is addressing the consensus problem with "Markovian switching" topologies.**

I. INTRODUCTION

A consensus problem, which lies at the foundation of distributed computing, consists of a group of dynamic agents who seek to agree upon certain quantities of interest by exchanging information among them according to a set of rules. This problem can model many phenomena involving information exchange between agents such as cooperative control of vehicles, formation control, flocking, synchronization, parallel computing, etc. Thus the consensus problem has been widely studied in the literature. Distributed computation over networks has a long history in control theory starting with the work of Borkar and Varaiya [1], Tsitsikils, Bertsekas and Athans [17], [18] on asynchronous agreement problems and parallel computing. Olfati-Saber and Murray introduced in [12], [13] theoretical framework for solving consensus problems. Jadbabaie et al. studied in [6] alignment problems involving reaching an agreement. Relevant extensions of the consensus problem were done by Ren and Beard [11] and by Moreau in [9].

The communication networks between agents may change in time due to link failures, packet drops, node failure, etc. Many of the variations in topology may happen randomly which lead to considering consensus problems under a stochastic framework. Hatano and Mesbahi consider in [7] an agreement problem over random information networks where the existence of an information channel between a pair of elements at each time instance is probabilistic and independent of other channels. In [10], Porfiri and Stilwell provide sufficient conditions for reaching consensus almost surely in the case of a discrete linear system where the communication flow is given by a directed graph derived from a random graph process, independent of other time instances. Under a similar model of the communication topology, Salehi and Jadbabaie give necessary and sufficient conditions for almost sure convergence to consensus in [16].

In our previous work [8] we analyzed the consensus problem for a group of dynamic agents with undirected information flow and random switching topologies, where the switching process is governed by a Markov chain whose states correspond to possible communication topologies. In this paper we consider the same problem but for directed information flows which enhances the degree of generality and at the same it represents the more realistic situation. As pointed out in [8], the advantage of having Markovian switching topologies resides in their ability to model a topology change which depends on previous communication topologies. As motivation, in [14] it was showed that the distributed Kalman filtering problem decomposes in two dynamic consensus problems. If we make the assumption that the sensors are prone to failure (and assume exponential distribution for the life-time of the sensors), the communication topology becomes random with an underlying Markov process. Thus our proposed framework can be used as a starting point for treating distributed filtering for sensor networks with random information flow.

Notations: We will denote by $\mathbb{1}$ the n-dimensional vector of all ones. We will use the same symbol for a vector of all ones with n^2 entries. It will be clear from the context what dimension the vector $\mathbb{1}$ has. The symbol $\otimes$ denotes the Kronecker product.

The outline of the paper is as follows. In Section II we present the setup and formulation of the problem. In Section III we state our main result and give an intuitive explanation. In Section IV we provide first a set of theoretical tools used in proving the main result and then we proceed with the main proof.

II. PROBLEM FORMULATION

We consider a group of n agents with a discrete time dynamics (labeled as $\{1, 2, \ldots, n\}$) for which the information flow is modeled as a directed graph $G = (\mathcal{V}, \mathcal{E}, A)$ of order n. The set $\mathcal{V} = \{1, \ldots, n\}$ represents the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. We assume there are no self loops inside the graph. The adjacency matrix of the graph G is an $n \times n$ matrix with non-negative entries $A = (a_{ij})$, where a_{ij} is positive if an arc from i to j exists and zero otherwise. Also, all diagonal entries are assumed zero.

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Definition 2.1: (Discrete-Time Markovian Random Graph) Let s be a given positive integer and let $\theta(k)$ be an homogeneous, finite-state, discrete-time Markov chain which takes values in the set $\mathcal{S} = \{1, \cdots, s\}$, with an $s \times s$ dimensional probability transition matrix $P = (p_{ij})$ (all rows sum up to one). Consider also a set of graphs of the same order $\mathcal{G} = \{G_i\}_{i=1}^s$. By a *discrete-time Markovian random graph* (DTMRG) we understand a map $\mathbf{G}$ from $\mathcal{S}$ to $\mathcal{G}$, such that

$$\mathbf{G}(\theta(k)) = G_{\theta(k)}$$

for all positive integer values of k.

In what follows this map will be denoted by $G_{\theta(k)}$. We note the $G_{\theta(k)}$ is a discrete finite-state Markovian process, whose probabilistic description is given by the probability transition matrix P.

We denote by $X(k)$ the n-dimensional vector representing the state of the agents. We assume that the information flow among agents is described by a DTMRG $G_{\theta(k)}$ and we consider a linear, discrete-time, stochastic dynamic system governing the evolution of the state vector:

$$X(k+1) = F_{\theta(k)} X(k), X(0) = X_0, \quad (1)$$

where the $n \times n$ random matrix $F_{\theta(k)}$ represents the state (local) updating rule corresponding to the communication graph given by $G_{\theta(k)}$ and it takes values in a finite set of matrices $\{F_i\}_{i=1}^s$. The initial condition X_0 is considered deterministic.

We define the *agreement space* as the subspace generated by the vector of all ones $\mathcal{A} = span(\mathbb{1})$.

Definition 2.2: We say that the vector $X(k)$ converges almost surely to consensus if it asymptotically reaches the agreement space in the almost sure sense

$$X(k) \xrightarrow{a.s} \mathcal{A}.$$

We say that the state vector $X(k)$ reaches *average consensus* almost surely if

$$X(k) \xrightarrow{a.s} av(X_0)\mathbb{1},$$

where $av(X_0) = \mathbb{1}^T X_0 / \mathbb{1}^T \mathbb{1}$.

Problem 2.1: Given a DTMRG $G_{\theta(k)}$ and the state updating rule $F_{\theta(k)}$, we derive necessary and sufficient conditions such that the state vector $X(k)$, evolving according to (1), converges almost surely to consensus for any initial state X_0.

Throughout this paper we will use the local updating rule (or protocol) constructed using Peron matrices:

$$F_i = I - \epsilon L_i, \ i \in \mathcal{S}, \quad (2)$$

where $L_i = D_i - A_i^T$ is the Laplacian of the graph G_i (with D_i and A_i being the inner degree matrix and the adjacency matrix respectively) and $\epsilon > \max_j\{d_{jj}\}$ (d_{jj} being the j^{th} entry of D_i diagonal). From now on we will refer to this updating rule as *protocol* A1. Note that protocol A1 is a *nearest neighbor updating rule* and it is appealing since it allows for a distributed implementation of the consensus problem.

Remark 2.1: In [11] an apparently more general local protocol is used:

$$F = (f_{ij}),$$

where $f_{ij} = \alpha_{ij} / \sum_{j=1}^n \alpha_{ij}$. In this protocol (referred from now on as protocol A2) the coefficients α_{ij} are positive if there exist an information flow from j to i or if $i = j$ and are zero otherwise. However we can obtain from this protocol a representation in terms of a Peron matrix if we chose $\epsilon = 1/\max_i\{\sum_j \alpha_{ij}\}$ and compute the adjacency matrix entries according to $a_{ji} = \frac{f_{ij}}{\epsilon}$. The inverse transition is also possible, in the sense that starting with protocol A1 we can obtain a representation as in protocol A2. The coefficients α_{ij} are obtained as a solution of a set of linear system of equations in terms of the adjacency matrix entries. These resulting coefficients need not be unique. Therefore we can claim that the two protocols are equivalent.

III. Main result

In this section we introduce the necessary and sufficient conditions for reaching consensus in the almost sure sense together with some intuitive explanations of these conditions. We defer the rigorous mathematical proof for Section IV.

Consider the problem setup presented in Section II. By the *Decomposition Theorem* of the states of a Markov chain (see [5]) the state space $\mathcal{S}$ can be partitioned uniquely as

$$\mathcal{S} = \{T \cup \ C_1 \cup \cdots \cup C_q\},$$

where T is the set of transient states and $C_1, \cdots, C_q$ are irreducible closed sets of (positive) recurrent states. Since $\theta(k)$ is a finite state Markov chain there exists at least one (positive) recurrent closed set. We make the assumption that the initial distribution is such that $\theta(0)$ can be in any state of the sets T or C_i. Let $\mathcal{G}_i = \{G_{j_1}, G_{j_2},, G_{j_{|C_i|}}\}$ be the sets of graphs corresponding to the states in the sets C_i with $i \in \{1, \ldots, q\}$ and where by $|C_i|$ we denote the cardinality of C_i.

Theorem 3.1: (*almost sure convergence to consensus*) Consider the stochastic system (1). Then, under protocol A1 (or A2), the state vector $X(k)$ converges almost surely to consensus for any initial state X_0 if and only if each union of graphs in the sets $\mathcal{G}_i$ corresponding to the closed sets C_i have a spanning tree.

We defer to the next section the proof of this theorem and rather provide here an intuitive explanation. Regardless of the initial state of $\theta(k)$, there exist a time instant after which $\theta(k)$ will be constrained to take values only in one of the closed sets C_i. Since C_i are irreducible and (positive) recurrent the probability of $\theta(k)$ to visit each of the states belonging to C_i will never converge to zero. Thus $\theta(k)$ will visit each of these states infinitely many times and consequently since the union of graphs corresponding to these states have a spanning tree, there will be a flow of information going from at least one agent to all others infinitely many times. Under protocol A1 (or A2) this is sufficient for the state vector $X(k)$ to converge to consensus [11]. On the other hand if we assume the existence of at least one set C_i such that the union of graphs in $\mathcal{G}_i$ does not have a spanning tree, then with non-zero probability $\theta(k)$ may be isolated in such a set after a while. Since the union of graphs in this set do not admit a spanning tree, then there will be at least one agent which will not exchange information with the others and hence impeding the convergence to consensus. Therefore we can find an initial state such that with non-zero probability, consensus is not reached.

Under certain assumptions almost sure convergence to *average consensus* can be achieved as well. The assumptions consist in having all state updating laws matrices F_i, be doubly stochastic. Within the protocol A1 this can be achieved if either the communication graphs are undirected (case in which matrices F_i are symmetric and therefore doubly stochastic) or directed, but *balanced* (for each node the inner degree is equal to the outer degree - in [13] it is shown that having balanced graphs is a necessary and sufficient condition for achieving average consensus for fixed topologies). Thus we can formulate the following corollary:

Corollary 3.1: (*almost sure convergence to average consensus*) Consider the stochastic system (1). Then, under protocol A1, if each union of graphs in the sets $\mathcal{G}_i$, corresponding to the closed sets C_i has a spanning tree and either all graphs are undirected or all are directed but balanced then the state vector $X(k)$ converges almost surely to average consensus for any initial states.

IV. Proof of the Main Result

In this section we detail the proof of *Theorem 3.1* and introduce a number of supporting results and their proofs. The proof of our theorem is based on the convergence properties of some matrices which arise from the analysis of the state vector's first and second moment. We start by stating a number of results from the literature which will be useful in our analysis and which deal with the properties of stochastic indecomposable, aperiodic matrices (SIA) (a stochastic matrix P is SIA if there exist a vector c such that $\lim_{k\to\infty} P^k = \mathbb{1}c^T$).

Theorem 4.1: ([19]) Let $A_1, \ldots, A_s$ be a finite set of $n\times n$ SIA matrices with the property that for each sequence of matrices $A_{i_1}, \ldots, A_{i_j}$ of positive length the product matrix $A_{i_j}\cdots A_{i_1}$ is SIA. Then for each infinite sequence $A_{i_1}, A_{i_2}, \ldots$ there exist a vector c such that

$$\lim_{j\to\infty} A_{i_j}A_{i_{j-1}}\cdots A_{i_1} = \mathbb{1}c^T \tag{3}$$

Lemma 4.1: Given a positive integer s, consider a set of directed graphs $\{G_i\}_{i=1}^s$. If the union of graphs in the set have a spanning tree then the matrix product $F_{i_1}F_{i_2}\ldots F_{i_j}$ is SIA, where the finite set of indices $i_1, i_2, \ldots, i_j$ contains at least once each of the values $1, 2, \ldots, s$ and where each matrix F_i is the result of applying protocol A2 to graph G_i.

Remark 4.1: Lemma 4.1 is a slight modification of Lemma 3.9 in [11]. However the proof is identical and will be skipped (the proof is based on a result expressed in *Lemma 2*, [6]). Note that the matrix product $(F_{i_1}\otimes F_{i_1})\ldots(F_{i_j}\otimes F_{i_j})$ is also SIA since $(F_{i_1}\otimes F_{i_1})\ldots(F_{i_j}\otimes F_{i_j}) = (F_{i_1}\ldots F_{i_j})\otimes(F_{i_1}\ldots F_{i_j})$ and the Kronecker product of two SIA matrices is SIA as well.

Remark 4.2: The result of *Lemma 4.1* also holds if the matrices F_i were obtained by applying protocol A1 since the two protocols are equivalent (see *Remark 2.1*).

Lemma 4.2: Let s be a positive integer and let $\{A_{ij}\}_{i,j=1}^s$ be a set of $n\times n$ SIA matrices. Let $P = (p_{ij})$ be an $s\times s$ stochastic matrix corresponding to an homogeneous, irreducible, positive recurrent Markov chain and consider the $ns\times ns$ dimensional matrix Q whose $(i,j)^{th}$ block is defined as $Q_{ij} = p_{ji}A_{ij}$. Then

$$\lim_{k\to\infty}(Q^k)_{ij} = p_{ji}^{\infty}\mathbb{1}c_{ij}^T \tag{4}$$

where p_{ji}^{∞} is the $(j,i)^{th}$ entry of P^k for large values of k and c_{ij}^T is a vector with non-negative entries summing up to one.

Proof: The proof of this lemma is based on *Theorem 4.1*. We can express the $(i,j)^{th}$ block entry of matrix Q^k as follows:

$$\begin{aligned}(Q^k)_{ij} &= \sum_{1\leq i_1,\ldots i_{k-1}\leq n} p_{ji_1}A_{ii_1}p_{i_1i_2}A_{i_1i_2}\ldots p_{i_{k-1}i}A_{i_{k-1}j}\\ &= p_{ji}^{(k)}\sum_{1\leq i_1,\ldots i_{k-1}\leq n}\alpha_{i_1,\ldots i_{k-1}}A_{ii_1}A_{i_1i_2}\ldots A_{i_{k-1}j}\end{aligned} \tag{5}$$

where $p_{ji}^{(k)}$ is the $(j,i)^{th}$ entry of P^k and $\alpha_{i_1,\ldots i_{k-1}} = (p_{ji_1}p_{i_1i_2}\ldots p_{i_{k-1}i})/p_{ji}^{(k)}$ if $p_{ji}^{(k)} > 0$ and zero otherwise. Notice from (5) that each $(i,j)^{th}$ block of the matrix Q^k is a convex combination of products of SIA matrices multiplied by $p_{ji}^{(k)}$. As k goes to infinity, according to *Theorem 4.1*, each of the matrix product $A_{ii_1}A_{i_1i_2}\ldots A_{i_{k-1}j}$ will converge to a matrix of the form $\mathbb{1}c^T$, for some non-negative vector c. Hence we will have an infinite convex combination of matrices of the form $\mathbb{1}c^T$ which will result in a matrix of the same type. Note that the limit may converge to a set of limit points matrices rather then a single point, depending on the properties of the Markov chain. A set of limit points occurs if the Markov chain is periodic, which will force the terms p_{ji}^{∞} not to converge but rather to oscillate between some values. ∎

Lemma 4.3: Let s be a positive integer and consider a set of directed graphs $\{G_i\}_{i=1}^s$ whose union admits a spanning tree. Let $\{F_i\}_{i=1}^s$ be a set of $n\times n$ matrices obtained by applying protocols A1 or A2 to graphs G_i. Consider also an homogeneous, irreducible (positive) recurrent finite-state Markov chain with an $s\times s$ transition probability matrix P and two matrices Q and $\tilde{Q}$ of dimensions $ns\times ns$ and $n^2s\times n^2s$ respectively whose blocks are constructed as: $Q_{ij} =$

$p_{ji}F_j$ and $\tilde{Q}_{ij} = p_{ji}F_j \otimes F_j$. Then

$$\lim_{k\to\infty} (Q^k)_{ij} = p_{ji}^{\infty} \mathbb{1} c_{ij}^T, \tag{6}$$

$$\lim_{k\to\infty} (\tilde{Q}^k)_{ij} = p_{ji}^{\infty} \mathbb{1} \tilde{c}_{ij}^T, \tag{7}$$

where p_{ji}^{∞} is the $(j,i)^{th}$ entry of P^k for large values of k and c_{ij} and $\tilde{c}_{ij}$ are two vectors of dimensions n and n^2 respectively with non-negative entries summing up to one.

Proof: We will first concentrate on the matrix Q. Our strategy consists in showing that there exist a k such that each (ij) block matrix of Q^k is given by a weighted SIA matrix: $(Q^k)_{ij} = p_{ji}^{(k)} A_{ij}^{(k)}$, where $A_{ij}^{(k)}$ is SIA. Then we can apply *Lemma 4.2* to obtain (6). The $(ij)^{th}$ block matrix of Q^k looks as in (5), with the main difference that in the current case $A_{ij} = F_j$:

$$\begin{aligned}(Q^k)_{ij} &= \sum_{1\leq i_1,\dots i_{k-1}\leq n} p_{ji_1}p_{i_1 i_2}\dots p_{i_{k-1}i} F_j F_{i_1}\dots F_{i_{k-1}} \\ &= p_{ji}^{(k)} A_{ij}^{(k)},\end{aligned} \tag{8}$$

where $A_{ij}^{(k)}$ is obtained as a convex combination of matrix products of the form $F_j F_{i_1}\dots F_{i_{k-1}}$. Then a sufficient condition for $A_{ij}^{(k)}$ to become SIA is that each matrix product $F_j F_{i_1}\dots F_{i_{k-1}}$ to be SIA. By *Lemma 4.1* and *Remark 4.2* this happens if $F_j F_{i_1}\dots F_{i_{k-1}}$ contains at least once each of the matrices F_j, $j \in \mathcal{S}$. Notice from (8) that to each possible path of length k from state j to state i (given by $p_{ji_1}p_{i_1 i_2}\dots p_{i_{k-1}i}$) we associate a matrix product $F_j F_{i_1}\dots F_{i_{k-1}}$. So basically we need to find a k such that for any two states j and i all possible paths (of length k) go through the states in the set $\mathcal{S} - \{j\}$ at least once before arriving to state i. The existence of such k is ensured by the irreducibility assumption. Indeed if such k does not exist then there would be a path between j and i arbitrary large that would never pass through all states in $\mathcal{S}$. If this is the case we would contradict the irreducibility assumption which corresponds to the connectedness property between any to states of the Markov chain. Therefore the irreducibility assumption guarantees the existence of a k such that each block matrix $(Q^k)_{ij}$ becomes a weighted SIA matrix. Hence by *Lemma 4.2* we obtain (6). Note that if the Markov chain is periodic, the probability p_{ji}^{∞} will not converge and hence in (6) we obtain a limiting set of points, whose cardinality is given by the value of the period. To show (7), we follow the same argument as before together with the part of Remark 4.1 concerning Kronecker products. ∎

At this point we are ready for the proof of *Theorem 4.1*.

A. Sufficiency

Proof: Note first that the stochastic system (1) represents a discrete-time Markovian jump linear system (the reader may consult for example [2] for a comprehensive introduction in the theory of Markovian jump linear systems). We define the error between the state vector $X(k)$ and the agreement space as:

$$e(k) = X(k) - \arg\min_{z\in\mathcal{A}} \|X(k) - z\|, \tag{9}$$

where we used Euclidean norm. Showing that the state vector converges almost surely to consensus is equivalent to showing that the error vector converges almost surely to zero or equivalently:

$$\|e(k)\|^2 \xrightarrow{a.s} 0.$$

Note that (9) can be also written as

$$e(k) = \left(I - \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}\right) X(k). \tag{10}$$

We will first address the case the Markov chain $\theta(k)$ is irreducible and positive recurrent. At the end of the proof, we will briefly address the case when $\theta(k)$ has transient states as well.

The proof has the following development. We analyze the convergence properties of the second moment of the state vector $X(k)$. From this analysis we will assert that the second moment of the error vector $e(k)$ converges exponentially to zero. Then by using the generalized Markov inequality together with the first Borel-Cantelli Lemma ([5]) we can conclude the almost sure convergence to zero of the error vector and implicitly the almost sure convergence of the state vector to the average consensus state.

Let the $n \times n$ symmetric matrix $V(k)$ denote the second moment of the state vector $X(k)$

$$V(k) = E[X(k)X(k)^T],$$

where we used E to denote the expectation operator. Using an approach similar to [2], consider the matrices $V_i(k)$

$$V_i(k) = E[X(k)X(k)^T \chi_{\{\theta(k)=i\}}],\ i \in \mathcal{S} \tag{11}$$

where $\chi_{\{\theta(k)=i\}}$ is the indicator function of the event $\{\theta(k) = i\}$. Then the second moment $V(k)$ can be expressed as the following sum:

$$V(k) = \sum_{i=1}^{s} V_i(k). \tag{12}$$

The set of discrete coupled Lyapunov equations governing the evolution of the matrices $V_i(k)$ is given by:

$$V_i(k+1) = \sum_{j=1}^{s} p_{ji} F_j V_j(k) F_j^T,\ i \in \mathcal{S}, \tag{13}$$

with initial conditions $V_i(0) = q_i X_0 X_0^T$ where $q = (q_i)$ is the initial distribution of the Markov chain, $q_i = Pr(\theta(0) = i)$.

We can further obtain a vectorized form of equations (13):

$$\eta(k+1) = \Lambda\eta(k), \tag{14}$$

where $\eta(k)$ is an n^2s dimensional vector formed by the columns of all matrices $V_i(k)$ and Λ is an $n^2s \times n^2s$ matrix given by

$$\Lambda = \begin{pmatrix} p_{11}F_1 \otimes F_1 & \dots & p_{s1}F_s \otimes F_s \\ \vdots & \ddots & \vdots \\ p_{1s}F_1 \otimes F_1 & \dots & p_{ss}F_s \otimes F_s \end{pmatrix}. \quad (15)$$

The initial vector $\eta(0)$ has the following structure

$$\eta(0)^T = [q_1 col_1(X_0X_0^T)^T, \dots, q_1 col_n(X_0X_0^T)^T, \dots$$
$$\dots, q_s col_1(X_0X_0^T)^T, \dots, q_s col_n(X_0X_0^T)^T],$$

where by col_i we understand the i^{th} column of the considered matrix. We notice that the current setup satisfies all the conditions of *Lemma 4.3* (the matrix Λ is the matrix $\tilde{Q}$ in the statement of the lemma) and hence we get

$$\lim_{k\to\infty}(\Lambda^k)_{ij} = p_{ji}^{\infty}\mathbb{1}c_{ij}^T, \quad (16)$$

where p_{ij}^{∞} is either the j^{th} component of the stationary distribution if the Markov chain is aperiodic or a set of recurring values if the chain is periodic; $\mathbb{1}$ is the n^2 dimensional vector of all ones and c_{ij} is an n^2 dimensional vector of non-negative entries summing up to one. As a consequence the vector $\eta(k)$ converges exponentially to a set of limiting points (whose cardinality is equal to the period of the chain):

$$\lim_{k\to\infty}\eta(k) = \{\eta_{\infty}^l\}_{l=1}^d \quad (17)$$

where d is the period of the chain and η_{∞}^l is a n^2s dimensional vector of the form ${\eta_{\infty}^l}^T = [\alpha_1^l\mathbb{1}^T \dots \alpha_s^l\mathbb{1}^T]$ with $\mathbb{1}$ an n^2 vector of all ones and α_i^l scalars which depend on the initial condition, on the probabilities p_{ij}^{∞} and on the limiting vectors c_{ij}.

By collecting the entries of $\lim_{k\to\infty}\eta(k)$ we obtain

$$\lim_{k\to\infty}V_i(k) = \{\alpha_i^l\mathbb{1}\mathbb{1}^T\}_{l=1}^d$$

and from (12) we finally obtain

$$\lim_{k\to\infty}V(k) = \{\beta^l\mathbb{1}\mathbb{1}^T\}_{l=1}^d \quad (18)$$

where $\beta^l = \sum_{i=1}^s \alpha_i^l$.

From (10) the second moment of the error vector can be expressed as

$$E[e(k)e(k)^T] = \left(I - \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}\right)E[X(k)X(k)^T]\left(I - \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}\right)$$

and from (18) we deduce that the second moment of the error vector converges asymptotically (and thus exponentially since linear systems are involved) to zero

$$\lim_{k\to\infty}E[e(k)e(k)^T] = 0,$$

Then by the generalized Markov inequality we can write

$$\sum_{k\geq 0}Pr(\|e(k)\|^2 > \epsilon) \leq \sum_{k\geq 0}\frac{E[\|e(k)\|^2]}{\epsilon} \quad (19)$$

for any positive ϵ. Since $E[\|e(k)\|^2] = trace(E[e(k)e(k)^T])$ converges exponentially to zero we can use the first Borel-Cantelli Lemma [5] to determine the almost sure convergence of the error vector to zero and implicitly the almost sure convergence of the state vector to consensus.

If the Markov chain has transient state as well, the result still holds. Due to space limitation, we will not address rigorously this case but rather present the idea behind it. Let us assume that there some transient states. Then, as time goes to infinity the probability to return to the transient states goes to zero. Hence the limiting values all blocks $(\Lambda^k)_{ij}$ where i is a transient state will be zero as time goes to infinity. For the (ij) blocks, where j is a transient state and i belongs to an irreducible, positive recurrent closed set, is not difficult to notice that, as time goes to infinity , $(\Lambda^k)_{ij}$ will take the same form as in (16). The convergence of the other blocks is covered by the case when the Markov chain is irreducible and positive recurrent, addressed in the beginning. Although not true in general, the intuition is the following. Let us assume that we start in a transient state. Then, since the state is transient then there exist a finite positive integer τ such that $M(\tau)$ belongs to a closed set C_i. Hence the second case can be regarded as having again an irreducible positive recurrent chain but with an initial condition given by:

$$X(\tau) = F_{i_1}F_{i_2}\dots F_{i_{\tau-1}}X_0$$

where $\{i_1, i_2, \dots, i_{\tau-1}\}$ is a set of indices representing states in the transient set.

Therefore, the burden of the proof lays on proving the first addressed case, which was rigorously treated.

■

B. Necessity

Proof: We show that if there exist at least one irreducible closed set corresponding to a set of graphs whose union does not have a spanning tree, then there exist some initial vectors X_0 such that the state vector does not converge in probability to consensus and hence does not converge in the almost sure sense either.

The complete probabilistic framework of the Markovian jump linear system (1) can be found for example in [2],pp.20. In our case the probabilistic description is rather simplified since X_0 was assumed deterministic.

Let $A_k(\epsilon)$ define the following event $A_k(\epsilon) = \{\omega_k : \|e(k)\|^2 > \epsilon\}$ for some positive ϵ, where $e(k)$ is the error vector defined in (9).

Suppose that there exist an irreducible and positive recurrent set C_{i^*} such that the union of the graphs from the corresponding set $\mathcal{G}_{i^*}$ does not have a spanning tree. Conditioning on the initial state of the Markov chain, the probability of the event $A_k(\epsilon)$ can be expressed as:

$$Pr(A_k(\epsilon)) = \sum_{j=1}^{s}Pr(A_k(\epsilon)|\theta(0) = j)Pr(\theta(0) = j).$$

By the assumption on the initial distribution of $\theta(k)$ we have that $Pr(\theta(0) \in C_{i^*}) > 0$. Since the union of graphs corresponding to the set C_{i^*} does not have a spanning tree, then there exist at least two agents such that there is no paths containing these two agents. Therefore we can find an initial vector such that consensus is not reached implying that we can find an ϵ such that $Pr(A_k(\epsilon)|\theta(0) \in C_{i^*})$ does not converge to zero. As a consequence the state vector does not converge to consensus almost surely since it does not converge in probability to consensus. ∎

Notice that the assumptions stated in *Corollary 3.1* imply that the matrices F_i are doubly stochastic matrices. Then using results from [8] (i.e. versions of *Lemmas 4.2* and *4.3* particularised for doubly stochastic matrices), the proof of *Corollary 3.1* follows.

Remark 4.3: The implications of modeling the communication flows between agents as Markovian random graphs are more subtle then they may appear at first glance. In [16], Salehi and Jadbabaie pointed out that in the case of independent and identically distributed random graphs a necessary and sufficient condition for reaching the agreement space almost surely is that $|\lambda_2(E[F(k)])| < 1$, where λ_2 is the second largest eigenvalue of $E[F(k)]$ and $F(k)$ is the updating rule at time k. In the case of Markovian random graphs a similar condition such as $|\lambda_2(E[F_{\theta(k)}])| < 1$, $k \geq 0$ or $|\lambda_2(\lim_{k\to\infty} E[F_{\theta(k)}])| < 1$ is not necessary. For example consider a Markovian random graph $G_{\theta(k)}$ which can take only two values $\{G_1, G_2\}$. Assume that the graphs G_1 and G_2 have both three vertices. In the first graph node 1 and 2 are connected and in the second one nodes 2 and 3 are connected. Clearly the union of G_1 and G_2 has a spanning tree. Assume that the underlying Markov chain $\theta(k)$ of $G_{\theta(k)}$ has the probability transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus the process $G_{\theta(k)}$ oscillates between the two graphs at each time instant. Choosing for instance to start with graph G_1 we get that

$$E[F_{\theta(k)}]) = \begin{cases} F_1 & if\ k\ is\ odd \\ F_2 & otherwise \end{cases}$$

Clearly since G_1 or G_2 are not connected the multiplicity of eigenvalue 1 of both F_1 and F_2 will not be one. Therefore $\lambda_2(E[F_{\theta(k)}]) < 1$ will not hold for any k. Moreover, since $E[F_{\theta(k)}]$ does not actually converge the previous condition will not hold in the limit either.

V. Conclusion

In this paper we extended our previous work to the analysis of a stochastic consensus problem for a group of agents with directed information flow, using a Markovian random graph as model for the stochastic communication topology. This model has a higher degree of generality (e.g. it includes the i.i.d. random graphs models consider until know in the literature) and also represents the more realistic situation. We showed that a necessary and sufficient condition for the state vector to converge to consensus almost surely consists in having a spanning tree for each union of graphs in the sets corresponding to the positive recurrent closed set of states of the Markov chain. We also showed that average consensus can be reached as well provided the graphs are undirected or directed but balanced. Under the Markovian random graph modeling, the dynamic stochastic equation determining the evolution of the agents became a Markovian jump linear system, which proved to be instrumental in showing the almost sure convergence. Our analysis relied on several tools from algebraic matrix theory, matrix theory and Markovian jump linear systems theory.

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