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Direct Lyapunov Approach for Tracking Control of Underactuated Mechanical Systems

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***Abstract*—A direct Lyapunov method is applied to tracking problems for underactuated mechanical systems. The method involves reformulating the problem in terms of a sliding mode vector and then designing a control law that stabilizes the sliding mode vector to a lower bound. The design of the tracking control law utilizes the authors' previous work on stabilization of underactuated mechanical systems using a direct Lyapunov method. One of the attractive features of the approach presented is that it requires no inverse dynamics. The efficacy of the method is demonstrated with applications to the ball and beam, where the unactuated axis is made to track a specified path, and to the inverted pendulum cart, where the actuated axis is made to track a specified path while maintaining stability of the pendulum.**

I. Introduction

Underactuated mechanical systems have fewer actuators than degrees of freedom. Examples of underactuated systems include underwater vehicles, active control of fuel slosh in rockets, overhead cranes and crane loads, rockets, and satellites. Control applications of underactuated systems are divided into stabilization and tracking. Stabilization was once the primary focus of control researchers such as Bloch, Leonard, and Marsden (2000, 2001) with controlled Lagrangians; Olfati-Saber (1998, 2000, and 2001) with backstepping; Ortega, Spong, Gómez-Estern, Blankenstein (2002) in addition to Acosta, Ortega, Astolfi and Mahindrakar (2005) with interconnection damping assignment – passivity based control (IDA-PBC); Auckly, Kapistanki, and White (2000) with the λ method; and White, Foss, Patenaude, Xin, and García (2008) with the direct Lyapunov approach (DLA).

Tracking control of underactuated systems is quickly becoming an area of significant activity. Applications for tracking of underactuated systems include free flying robots, robots in remote or hazardous locations that suffer actuator failure on one or more joints, trajectory following by rockets, and avoiding obstacles while moving a crane load. All cited examples are holonomic dynamic systems. Applications involving wheeled vehicles such as mobile robots and unicycles are examples of non-holonomic, underactuated systems. The concentration here is holonomic systems.

Recent developments of tracking control design for holonomic, underactuated systems can be categorized into two main areas: matching and non-matching based. Other approaches to underactuated system tracking include the work of Driessen and Sadegh (2000) where optimal control techniques were used for minimum time path following of an underactuated manipulator. Their computations were made possible by linearization about the trajectory. Blajer and Kolodziejczyk (2007) developed a feed forward control scheme based on inverse dynamics for their gantry crane. Also, Jain and Rodríguez (1991) show how underactuated manipulators can be split into active and passive systems for the purposes of kinematics and dynamics.

Tracking control applications for fully actuated systems have influenced the approaches taken for underactuated systems. Several workers have considered inverse dynamics in developing a path for which the trajectory history of each axis is found in advance. The inverse dynamics for underactuated systems is complicated by the reduction in the possible paths specifically dictated by the lack of actuation.

Non-matching based techniques have proven popular in recent years. A notable contribution was made by Sandoz, Kokotović, and Hespanha (2008) with their trackablity filter scheme. This approach employs a filter which produces an augmented reference signal derived from a nominal input. This new signal is zero error trackable by the underactuated system provided that its zero dynamics are input to state stable (ISS). Alternatively, Ashrafiuon, and Erwin (2004) presented a sliding mode control approach which can drive an underactuated system onto a sliding surface. Lyapunov theory was used to develop the controller used to reach the sliding surface, however asymptotic stability of the sliding surface was not established for the general case. The absence of an effective method of determining the asymptotically stable surfaces could prove to be a limitation of this technique.

Research on extending matching equation based stabilization techniques to tracking has been increasing recently. An example of this is Singhal, Patayane, Banavar (2006) in which the authors derive and compare tracking controllers using the method of interconnection damping assignment-passivity based control (IDA-PBC) and a direct Lyapunov approach. These controllers were limited in

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application due to the zero acceleration assumption for the desired trajectory. In Wang and Goldsmith (2008), an alternate IDA –PBC formulation was presented that was applied to underactuated system regulation and the tracking control of some non-passive systems. In the tracking applications, all state variables had prescribed time histories and use was made of inverse dynamics.

The approach taken in this work is to develop a controller that does not require inverse dynamics. For a system having *n* degrees of freedom for which only *m* degrees are actuated, the controller design method assumes that *m* of the degrees of freedom have a specified smooth trajectory history. These histories might be determined by a rudimentary path planner given the initial and ending system configurations. The smoothness requirement stems from the necessity of determining the velocity and acceleration of each specified history. The attractiveness of the approach to be presented is that given the *m* specified histories, the control law will determine, at each point of time, suitable history values for the *n – m* degrees of freedom not having specified trajectory histories. This aspect of the control law removes the effort of having to generate these trajectories from inverse dynamics prior to the start of the motion.

Part of the control law presented by Slotine and Li (1988) is the starting point for the controller design. Given the form of the control law, the dynamics of the system are recast in terms of a sliding mode. The control law for the new dynamic equation is developed from a direct Lyapunov approach very similar to that presented in White et al. (2008). Because the tracking control law is to be applied to an underactuated system, *n – m* of the components of the control law vector must vanish. These *n – m* zero control law equations allow the determination of the desired accelerations associated with the degrees of freedom having unspecified histories. Integration of these accelerations determines the velocity and position of the unspecified axes. A lower bound on the sliding mode variables is established.

II. Analysis

A. Control Law

The mechanical system is described by the nonlinear matrix equation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{C}_D\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix}\boldsymbol{\tau}\\ 0\end{bmatrix} \tag{1}$$

where $\boldsymbol{q} \in \Re^n$ is a vector of generalized coordinates for the system's *n* degrees of freedom while its time derivative, denoted by $\dot{\boldsymbol{q}}$, contains the *n* generalized velocities. The right-hand side of (1) contains the vector $\boldsymbol{\tau} \in \Re^m$. It is assumed that the degrees of freedom are ordered so that the first *m* elements of the right side vector contain the nonzero inputs. Also in (1) $\boldsymbol{M}(\boldsymbol{q}) \in \Re^{n\times n}$ is the positive definite mass and/or inertia matrix, $C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \Re^n$ consists of centripetal and Coriolis forces and/or moments, $\boldsymbol{C_D} \in \Re^{n\times n}$ is the symmetric viscous damping matrix, and $\boldsymbol{G}(\boldsymbol{q}) \in \Re^n$ consists of forces and/or moments stemming from potential gradients.

The requirement of the control law is both to stabilize and to drive the system along the specified trajectory. The tracking controller presented by Slotine and Li (1988) was developed for fully actuated systems. In order to apply this sliding mode approach to underactuated systems, modifications of the original controller must be made. The control law for an underactuated system is

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\tau}\\ 0\end{bmatrix} &= \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{C}_D\dot{\boldsymbol{q}}_r \\ &- \overline{\boldsymbol{K}}_D\boldsymbol{s} + \begin{bmatrix}\boldsymbol{u}_1+\boldsymbol{F}\\ \boldsymbol{u}_2\end{bmatrix} + \boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})\end{aligned} \tag{2}$$

where $\dot{\boldsymbol{q}}_r$ and $\ddot{\boldsymbol{q}}_r$ are the reference velocity and acceleration vectors, respectively, $\overline{\boldsymbol{K}}_D \in \Re^{n\times n}$ is a positive definite Hermitian matrix, $\Phi(\boldsymbol{q})$ is a real scalar potential function of the generalized coordinates, $\boldsymbol{P}(\boldsymbol{q}) \in \Re^{n\times n}$ is a positive definite matrix defined below, and the gradient is computed with respect to the generalized coordinates. In comparing (2) to the control law presented by Slotine and Li (1988), it is seen that the gravitational term is not included and that there are additional terms which are necessitated by the underactuation. The input vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ together with $\boldsymbol{F}$, dealing with stabilization and tracking, will be defined later in the analysis. The gravitational vector $\boldsymbol{G}(\boldsymbol{q})$ of (1) will be seen at a later point to be related to part of the quantity $\boldsymbol{F}$ and the gradient of Φ. The reference velocity is defined as

$$\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d - \boldsymbol{\Lambda}\tilde{\boldsymbol{q}} = \dot{\boldsymbol{q}}_d - \boldsymbol{\Lambda}(\boldsymbol{q}-\boldsymbol{q}_d) \tag{3}$$

where $\boldsymbol{q}_d$ is the vector of desired coordinate positions and $\boldsymbol{\Lambda} \in \Re^{n\times n}$ is a constant, positive definite, symmetric matrix. The time derivative of (3) yields the reference acceleration vector. The quantity $\tilde{\boldsymbol{q}}$, consisting of the difference between the actual and desired coordinates, and its time derivative constitute the tracking errors. The sliding mode vector $\boldsymbol{s}$ is

$$\boldsymbol{s} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_r = \dot{\tilde{\boldsymbol{q}}} + \boldsymbol{\Lambda}\tilde{\boldsymbol{q}}\,. \tag{4}$$

If the control drives the sliding mode vector to the sliding surface where the vector $\boldsymbol{s}$ vanishes, we see that the tracking error then decays to zero.

Combining equations (1) and (2) yields

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{C}_D\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r \\ &+ \boldsymbol{C}_D\dot{\boldsymbol{q}}_r - \overline{\boldsymbol{K}}_D\boldsymbol{s} + \begin{bmatrix}\boldsymbol{u}_1+\boldsymbol{F}\\ \boldsymbol{u}_2\end{bmatrix} + \boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})\,.\end{aligned} \tag{5}$$

Further manipulation and including (3) and (4) produces

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} + \boldsymbol{C}_D\boldsymbol{s} + \boldsymbol{G}(\boldsymbol{q}) &= -\overline{\boldsymbol{K}}_D\boldsymbol{s} \\ &+ \begin{bmatrix}\boldsymbol{u}_1+\boldsymbol{F}\\ \boldsymbol{u}_2\end{bmatrix} + \boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})\,.\end{aligned} \tag{6}$$

The goal of this effort is to use a direct Lyapunov method to complete the design of the control law for the system. The candidate Lyapunov function is

$$V=\frac{1}{2}\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{s} \quad (7)$$

where $\boldsymbol{K}_D \in \Re^{n\times n}$ is a symmetric, positive definite matrix defined as the product

$$\boldsymbol{K}_D=\boldsymbol{P}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q}), \quad (8)$$

where $\boldsymbol{P}(\boldsymbol{q})$ is a positive definite matrix defined in White et al. (2008) so that $\boldsymbol{K}_D$ has the previously specified properties. Computing the time derivative of (7) produces

$$\dot V=\boldsymbol{s}^T\boldsymbol{K}_D\dot{\boldsymbol{s}}+\frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{K}}_D\boldsymbol{s}=-\boldsymbol{s}^T\boldsymbol{K}_v\boldsymbol{s}+\Psi(\boldsymbol{s},\boldsymbol{u}) \quad (9)$$

where $\boldsymbol{u}$ refers to the vector $[\boldsymbol{u}_1\ \boldsymbol{u}_2]^T$ on the right side of (6), $\boldsymbol{K}_v \in \Re^{n\times n}$ is symmetric and, at least, positive semi-definite, and Ψ will be defined later in the analysis. Substituting the time derivative of $\boldsymbol{s}$ from (6) into (9), we obtain

$$\dot V=\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}^{-1}(\boldsymbol{q})\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s}-\boldsymbol{C}_D\boldsymbol{s}-\boldsymbol{G}(\boldsymbol{q})-\overline{\boldsymbol{K}}_D\boldsymbol{s}+\begin{bmatrix}\boldsymbol{u}_1+\boldsymbol{F}\\ \boldsymbol{u}_2\end{bmatrix}\right) \quad (10)$$
$$+\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})+\frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{K}}_D\boldsymbol{s}=-\boldsymbol{s}^T\boldsymbol{K}_v\boldsymbol{s}+\Psi(\boldsymbol{s},\boldsymbol{u}).$$

A matching equation method will be used to solve (10). Following a strategy similar to that of White, Foss, Patenaude, Guo, and García (2008), we decompose (10) into three matching equations. Before this is undertaken, we rewrite the quantity $\boldsymbol{F}$ as

$$\boldsymbol{F}=\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3 \quad (11)$$

where $\boldsymbol{F}_i$ will be used with the i[th] matching equation. The first matching equation is

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s}+\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}\right)+\frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{K}}_D\boldsymbol{s}=0 \quad (12)$$

while the second matching equation is

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D\right)\boldsymbol{s}+\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}\right)=-\boldsymbol{s}^T\boldsymbol{K}_v\boldsymbol{s} \quad (13)$$

and the third and final matching equation is

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}^{-1}(\boldsymbol{q})\left(-\boldsymbol{G}(\boldsymbol{q})+\begin{bmatrix}\boldsymbol{F}_3\\0\end{bmatrix}+\boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})\right)=0. \quad (14)$$

Following the procedure of White et al. (2008), the first two matching equations are rewritten as

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})-\overline{\boldsymbol{C}}_D-\boldsymbol{C}_D^{'}\right)\boldsymbol{s}+\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}\right)+\frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{K}}_D\boldsymbol{s}=0 \quad (15)$$

and

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D+\overline{\boldsymbol{C}}_D+\boldsymbol{C}_D^{'}\right)\boldsymbol{s}+\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}\right)=-\boldsymbol{s}^T\boldsymbol{K}_v\boldsymbol{s} \quad (16)$$

where the same matrices are subtracted from the first equation and added to the second and where $\boldsymbol{C}_D^{'}\in\Re^{n\times n}$ and $\overline{\boldsymbol{C}}_D\in\Re^{n\times n}$ are symmetric matrices defined in the following two sections. Note the sum of (15) and (16) is the same as the sum of (12) and (13).

a) The First Matching Equation

The vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are factored as

$$\boldsymbol{F}_i=\boldsymbol{F}_{im}\boldsymbol{s}. \quad (17)$$

Using this factorization, the vector $\boldsymbol{s}$ can be eliminated from either side of (15); however, in order for (15) to be true in the most general case, we must require the symmetric part of the resulting matrix equation to vanish. This leads to

$$\dot{\boldsymbol{K}}_D-\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)-\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D+$$
$$\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{1m}\\0\end{bmatrix}\right)+\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{1m}\\0\end{bmatrix}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D=0. \quad (18)$$

The elements of $\boldsymbol{F}_{1m}$ and $\overline{\boldsymbol{C}}_D$ are chosen so that the last two terms of (18) will equal

$$\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{1m}\\0\end{bmatrix}\right)+\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{1m}\\0\end{bmatrix}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D \quad (19)$$
$$=-\beta\left(\boldsymbol{K}_D-\boldsymbol{K}_{Df}\right)$$

where β is a negative constant and $\boldsymbol{K}_{Df}$ is the final form of the matrix $\boldsymbol{K}_D$, i.e. the form that $\boldsymbol{K}_D$ attains when equilibrium is reached. Using (19) the first matching equation becomes

$$\dot{\boldsymbol{K}}_D-\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)-$$
$$\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D-\beta\left(\boldsymbol{K}_D-\boldsymbol{K}_{Df}\right)=0 \quad (20)$$

which is evaluated numerically as part of the feedback process. The matrix $\boldsymbol{C}_D^{'}$ will be defined in the subsequent discussion of the second matching equation. A convenience of (20) is that choosing β large makes the matrix $\boldsymbol{K}_D$ essentially constant. Note that the $\boldsymbol{F}_{1m}$ from (19) times the vector $\boldsymbol{s}$ provides the control signal $\boldsymbol{F}_1$.

b) The Second Matching Equation

Again, using the factorization of (17) and "stripping off" the vector $\boldsymbol{s}$ from both sides of (16) produces

$$\boldsymbol{P}(\boldsymbol{q})\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D+\overline{\boldsymbol{C}}_D+\boldsymbol{C}_D^{'}\right)+\boldsymbol{P}(\boldsymbol{q})\begin{bmatrix}\boldsymbol{F}_{2m}\\0\end{bmatrix}=-\boldsymbol{K}_v. \quad (21)$$

The matrices $\overline{\boldsymbol{C}}_D$ and $\boldsymbol{C}_D^{'}$ present complications in the solution of (21). The matrix $\overline{\boldsymbol{C}}_D$ is already defined from the solution of the first matching equation, thus, $\boldsymbol{C}_D^{'}$ will be used to eliminate these two terms from the second matching equation, i.e. $\overline{\boldsymbol{C}}_D+\boldsymbol{C}_D^{'}=0$. Given these definitions, note that all of the matrices $\overline{\boldsymbol{C}}_D$ and $\boldsymbol{C}_D^{'}$ together with the matrix $\boldsymbol{F}_{1m}$ all vanish as equilibrium is approached and the first matching equation shows the time derivative of $\boldsymbol{K}_D$ vanishes.

A two step process is used to solve (21), the first being

$$\begin{bmatrix}\boldsymbol{F}_{2m}\\0\end{bmatrix}=-\boldsymbol{P}(\boldsymbol{q})^{-1}\boldsymbol{K}_{v1} \quad (22)$$

for which the solution is

$$\boldsymbol{K}_{v1}=\sum_{i=1}^{m}\alpha_i\boldsymbol{P}_i\boldsymbol{P}_i^T \quad (23)$$

where the α_i are constants chosen so that $\boldsymbol{K}_{v1}$ is positive semi-definite and $\boldsymbol{P}_i$ is the i[th] column of $\boldsymbol{P}(\boldsymbol{q})$. Applying (22) and (23) to (21) shows that

$$\boldsymbol{P}(\boldsymbol{q})\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D\right)=-\boldsymbol{K}_{v2} \quad (24)$$

where the sum of $\boldsymbol{K}_{v1}$ and $\boldsymbol{K}_{v2}$ is $\boldsymbol{K}_v$. The product of $\boldsymbol{P}(\boldsymbol{q})$ and the matrices in the parenthesis in (24) is not symmetric, however, the pre and post multiplication by $\boldsymbol{s}$ extracts the symmetric portion of the product matrices. Thus, we require

$$\frac{1}{2}\left[\boldsymbol{P}(\boldsymbol{q})\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D\right)+\left(-\boldsymbol{C}_D-\overline{\boldsymbol{K}}_D\right)\boldsymbol{P}(\boldsymbol{q})^T\right]=-\boldsymbol{K}_{v2} \quad (25)$$

Note that the $\boldsymbol{F}_{2m}$ from (22) times $\boldsymbol{s}$ provides the control signal $\boldsymbol{F}_2$. Because the matrices on the left of (25) are positive definite, the resulting matrix $\boldsymbol{K}_v$ is positive definite.

c) The Third Matching Equation

Stripping off the vector $\boldsymbol{s}$ from (14), we arrive at

$$\boldsymbol{K}_D\boldsymbol{M}^{-1}(\boldsymbol{q})\left(-\boldsymbol{G}(\boldsymbol{q})+\begin{bmatrix}\boldsymbol{F}_3\\0\end{bmatrix}+\boldsymbol{P}(\boldsymbol{q})^{-1}\nabla\Phi(\boldsymbol{q})\right)=0 \quad (26)$$

and the solution procedure was shown in White et al. (2008).

The remarkable result of this <u>tracking controller</u> development is that we have arrived at three matching equations that are (with the exception of $\overline{\boldsymbol{K}}_D$) identical to matching equations developed for <u>stabilization</u> as shown in the authors' previous work.

d) Remaining Terms

Removing the matching equations from (10), the remaining terms are

$$\boldsymbol{s}^T\boldsymbol{K}_D\boldsymbol{M}^{-1}(\boldsymbol{q})\begin{bmatrix}\boldsymbol{u}_1\\\boldsymbol{u}_2\end{bmatrix}=\Psi(\boldsymbol{s},\boldsymbol{u}). \quad (27)$$

In order to satisfy Lyapunov, we desire that the right side of (27) is non-positive. It will be seen that there are some circumstances that will render the right side of (27) positive, however, it will also be shown that bounds exist on the norm of the vector $\boldsymbol{s}$. More will be stated about the function Ψ in a later section of this paper.

B. Tracking Control

The quantities involved in the evaluation of (27) require further explanation. The control law is given by (2) and the constraint that the lower $n-m$ elements of the actuation vector vanish will be used to determine the vector $\boldsymbol{u}_2$. That the function $\Psi(\boldsymbol{s},\boldsymbol{u})$ is intended to be non-positive will be used to determine the vector $\boldsymbol{u}_1$.

The tracking discussion will pertain to the case where the motion constitutes a trajectory that is contained in the solution space of the system. In order to have the system track a prescribed trajectory, there are several possibilities. The first is to use the trajectory information to determine the time histories of the generalized coordinates. By knowing the time histories of the coordinates (assumed to be sufficiently smooth) the generalized velocities and accelerations are also known. There is a total of n degrees of freedom and the trajectory may specify either all or a subset of the generalized coordinates. If all coordinate histories are specified and if the desired motion is possible given the underactuation, this represents one extreme in the classes of possible tracking problems. At the other end, there is the situation where m coordinate histories are specified. The m history constraints provide conditions to determine the m actuations. Fewer than m constraints may lead to redundant solutions. If m coordinate histories are specified, then one or more of the other coordinate histories could be determined through inverse dynamics. In the general case, inverse dynamics is unattractive owing to the time and complexity involved in the solution process, thus, limiting the system's ability to respond rapidly to a given task. It should also be stated that in an underactuated system having m actuators, specifying m coordinate histories can in certain systems lead to redundant solutions for the other $n-m$ axes. This short discussion shows that there is a wealth of problem classes that can be considered.

In the current work, attention is directed to those systems where the number of specified coordinate histories equals the number of actuated axes. No inverse dynamics will be performed for those axes where the coordinate histories are not specified. Given this class of problems, there are three subclasses that need to be considered. The first subclass includes those problems where the coordinate histories are specified for the unactuated axes. The second subclass involves those problems where the specified coordinate histories describe the motion of actuated axes. The final subclass includes problems where some unactuated axes and some actuated axes have a total of m specified coordinate histories. In this paper, the first two mentioned subclasses will be treated as examples.

Attention will now return to (27) and its application. It is assumed that m degrees of freedom have been specified, leaving $n-m$ coordinates unspecified. In (2), the lower $n-m$ equations are solved for the reference accelerations of the unspecified coordinates. This step is always possible should the mass/inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ be full. However, because the unactuated axes have inertial coupling to the actuated axes (off diagonal terms of $\boldsymbol{M}(\boldsymbol{q})$), it would still be possible to perform this step. If this were not true, the system would be uncontrollable. That the lower $n-m$ rows of (2) are equal to zero allows for the reference accelerations to be found. In general, these $n-m$ equations are nonlinear and possibly unstable. The control $\boldsymbol{u}_2$ is used to stabilize these $n-m$ equations. In the examples to be presented, $\boldsymbol{u}_2$ is chosen to stabilize the equations through feedback linearization. Regardless of whether the solution of the lower $n-m$ equations is performed for actuated or non-actuated reference accelerations, the steps of the process are the same. Once the input vector $\boldsymbol{u}_2$ is determined, then $\boldsymbol{u}_1$ can be chosen to satisfy (27). Additional discussion on how to satisfy (27) is included in the forthcoming section.

C. Applying Equation (27)

It is desired that the vector $\boldsymbol{u}_1$ in (27) be chosen so that Ψ is less than zero or at least the right hand side of (9) is non-positive. If Ψ is to be other than positive, we must have

$$\boldsymbol{s}^T\boldsymbol{P}(\boldsymbol{q})\begin{bmatrix}\boldsymbol{u}_1\\\boldsymbol{u}_2\end{bmatrix}\le 0 \quad (28)$$

or if $\boldsymbol{P}(\boldsymbol{q})$ is partitioned as

$$\boldsymbol{P}(\boldsymbol{q})=\begin{bmatrix}\boldsymbol{P}_1 & \boldsymbol{P}_2\end{bmatrix} \quad (29)$$

we would then have

$$\boldsymbol{s}^T \boldsymbol{P}_1 \boldsymbol{u}_1 \le -\boldsymbol{s}^T \boldsymbol{P}_2 \boldsymbol{u}_2 . \tag{30}$$

Depending upon the dimension of $\boldsymbol{u}_1$, the ability to satisfy (27) might be limited. One possibility of satisfying (9) would be to determine $\boldsymbol{u}_1$ so that Ψ is zero. Experience shows that computing $\boldsymbol{u}_1$ this way makes $\boldsymbol{u}_1$ noisy and the noise is present in actuated states. It is proposed that a least squares approach be adopted in finding $\boldsymbol{u}_1$. We desire that

$$\begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} = -\boldsymbol{s} \tag{31}$$

which can be rewritten as

$$\boldsymbol{P}_1 \boldsymbol{u}_1 = -\boldsymbol{s} - \boldsymbol{P}_2 \boldsymbol{u}_2 \tag{32}$$

for which there are n equations and m unknowns. Solving (32) in the least squares sense yields

$$\boldsymbol{u}_1 = -\left(\boldsymbol{P}_1^T \boldsymbol{P}_1\right)^{-1} \boldsymbol{P}_1^T \left(\boldsymbol{s} + \boldsymbol{P}_2 \boldsymbol{u}_2\right). \tag{33}$$

This last relation provides a continuous dependence of $\boldsymbol{u}_1$ on $\boldsymbol{s}$ and $\boldsymbol{u}_2$. This last relation works well with the exception of when $\boldsymbol{u}_1$ is orthogonal to the columns of $\boldsymbol{P}_1$, in which case Ψ becomes $\boldsymbol{s}^T \boldsymbol{P}_2 \boldsymbol{u}_2$ and Ψ takes on the sign of this product.

The control vector $\boldsymbol{u}$ will become zero as the system comes to rest. As time increases, the tracking error will tend toward zero. In (9), we see that the first term on the right hand side is quadratic in $\boldsymbol{s}$ and the second term is linear in $\boldsymbol{s}$. As $\boldsymbol{s}$ becomes small, it becomes increasingly difficult to ensure that the sum of the two terms is non-positive. From this we see that there is a lower bound to in how small $\boldsymbol{s}$ can become.

D. Bounds on $\|\boldsymbol{s}\|$

If the desired generalized coordinates are set to zero, the control problem becomes one of stabilization. It has been shown in Patenaude (2008) that the resulting system in (6) is asymptotically stable. The previous discussion demonstrated that the tracking errors and the control signals do not decay to zero. Treating the controls $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ as non-vanishing perturbations and invoking Lemma 9.3 of Khalil (2002), an ultimate bound exists on the magnitude of the sliding mode $\boldsymbol{s}$.

III. Examples

A. The Ball and Beam

The presented control law was applied to a ball and beam system. The system geometry and dynamic equations are shown in Fig. 1 with definitions of the physical parameters.

In this example, the radial position of the ball was chosen as $r_d(t) \equiv a_1\left(1-\cos(\omega t)\right) + a_1$. The initial angular velocity, angular position, and ball velocity were set to zero. The desired beam angle was found from the lower n-m rows of (2). For this example $\overline{\boldsymbol{K}}_D$ was set to zero. Figure 2 shows the desired and actual ball position as a function of time.

The values for the physical parameters and the chosen constants are $\bar{I}$ *=0.4 Kg m*^{2}*,* m *= 1.5 Kg,* R_o *= 0.02 m,* $\boldsymbol{C}_D$*= 0.16 N sec./m, g = 9.81 m/sec*^{2}*,* β *= - 1000,* a_1 *= .1 m,* ω*=.3 rad/sec,* α_1 = 1.0, $\boldsymbol{K}_{Df11}$ = 5, $\boldsymbol{K}_{Df12}$ = -25, $\boldsymbol{K}_{Df22}$ = 606, $\boldsymbol{\Lambda}_{11}$= 0.005, $\boldsymbol{\Lambda}_{12}$=0, $\boldsymbol{\Lambda}_{22}$= 2, and $\overline{\boldsymbol{K}}_D$ = 0.

The control $\boldsymbol{u}_2$ is chosen as

$$\begin{aligned} \boldsymbol{u}_2 \equiv & -mg\sin(\theta) + mr\dot{\theta}\dot{\theta}_d + c_d\left(-\dot{r}_d + \boldsymbol{\Lambda}_{22}(r - r_d)\right) \\ & -\frac{7}{5}m\left(\ddot{r} - \boldsymbol{\Lambda}_{22}(\dot{r} - \dot{r}_d)\right). \end{aligned} \tag{34}$$

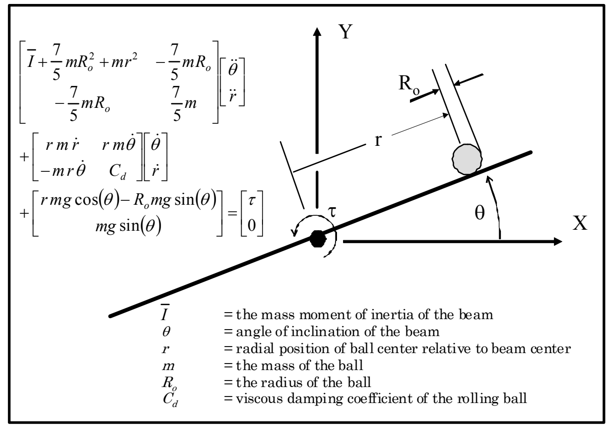

Figure 1: Ball and Beam System

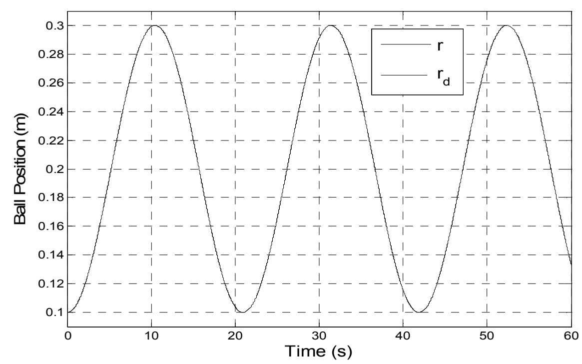

Figure 2: Desired and Actual Ball Position

B. The Inverted Pendulum Cart

The inverted pendulum cart geometry and the dynamic equations of motion are shown in Figure 3 with definitions of the physical parameters. The values for the physical parameters and the chosen constants are J = 0.4 *Kg m*^{2}*,* m = 1.5 *Kg,* $\overline{m}$ = 5.0 *Kg,* l= 0.7 *m,* g = 9.81 *m/sec*^{2}*,* β *= -1000,* a_1 = .2 *m,* ω=.35 *rad/sec,* $\boldsymbol{K}_{Df11}$ = 200, $\boldsymbol{K}_{Df12}$ = - 300, $\boldsymbol{K}_{Df22}$ = 550, α_1 = 1, $\boldsymbol{\Lambda}_{11}$ = 0.05, $\boldsymbol{\Lambda}_{12}$ = 0, $\boldsymbol{\Lambda}_{22}$= 1, and $\overline{\boldsymbol{K}}_D$ = $\boldsymbol{M}(\boldsymbol{q})$.

In this example, the initial conditions of the system are zero. The x trajectory was chosen as $x_d \equiv a_1\left(1-\cos(\omega t)\right)$. The desired pendulum angle was determined from the lower $n - m$ rows of (2). Figure 4 shows the desired and actual cart position as a function of time.

The control $\boldsymbol{u}_2$ for the inverted pendulum cart is

$$\begin{aligned} \boldsymbol{u}_2 \equiv & \frac{1}{2}ml\cos(\theta)\left(\ddot{x}_d - \boldsymbol{\Lambda}_{11}(\dot{x} - \dot{x}_d)\right) + \frac{1}{2}ml\sin(\theta) \\ & \frac{1}{2}ml\cos(\theta)\left(\dot{x} - \dot{x}_d + \boldsymbol{\Lambda}_{11}(x - x_d)\right) + J\left(\dot{\theta} - \dot{\theta}_d + \boldsymbol{\Lambda}_{22}\dot{\theta}\right). \end{aligned} \tag{35}$$

In both examples, it was observed that by adjusting the elements of $\boldsymbol{\Lambda}$, it was possible to emphasize the tracking performance of one variable over the other. In both cases, the tracking of the desired axis received the higher priority.

IV. Conclusion

This work has presented a means of designing tracking controllers for underactuated systems. The approach consists of a variation of a well-accepted tracking control law for fully actuated systems coupled with a stabilizing control law design technique presented by the authors in a previous investigation. It was shown that the tracking error would continue to decay until Lyapunov was no longer satisfied. Even though asymptotic tracking cannot be proved, the examples presented show that the performance is excellent. Another benefit of the approach is that inverse dynamics was not necessary at any point of the controller development or implementation.

$$\begin{bmatrix} \overline{m} & -\frac{ml}{2}\cos(\theta) \\ -\frac{ml}{2}\cos(\theta) & J \end{bmatrix}\begin{bmatrix}\ddot{X}\\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix}0 & \frac{ml}{2}\sin(\theta)\dot{\theta} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{X}\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ -\frac{mgl}{2}\sin(\theta)\end{bmatrix} = \begin{bmatrix}F\\ 0\end{bmatrix}$$

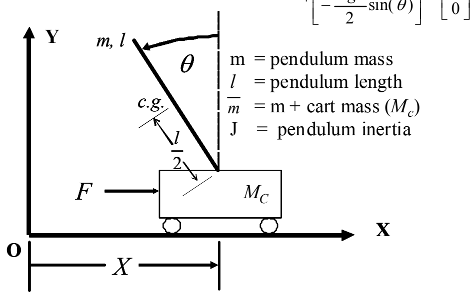

Figure 3: Inverted Pendulum Cart System

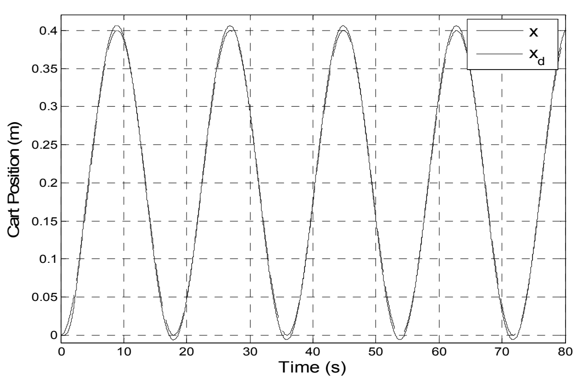

Figure 4: Desired and Actual Cart Position

The examples consisted of one system where the actuated axis was made to closely track a specified trajectory and another system where the unactuated axis was made to closely track a specified trajectory. Both examples showed the same level of fine performance.

Future goals in this investigation include quantification of the tracking error limits and developing asymptotic tracking.

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