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The PI-Controller for infinite dimensional linear systems in Banach state spaces

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***Abstract*— The PI controller for plants with unbounded control and observation operators is discussed. This is a generalization of pervious work considering bounded control operators. Our approach is mainly based on regular linear systems in the Salamon–Weiss sense.**

***Index Terms*— PI-Controller, unbounded control operators, semigroup, infinite–dimension.**

I. INTRODUCTION

The PI-controllers attracted the attention of many researchers in systems theory and engineers for many years. This is a natural way to stabilize and regulate models in engineering. The theory has been started for finite dimensional systems, *e.g.* [3]. Pohjolainen [12] extended the finite dimensional theory of PI-controllers to the infinite dimensional linear systems. He considered a linear system governed by an analytic semigroup on Banach space, a bounded control operator and an admissible unbounded observation operator; see also [10]. The techniques used in the aforementioned works are mainly based on the state-space approach. By using a frequency domain approach, the authors of [11] have solved the PI-controller problem for an infinite dimensional linear system with a general semigroup and bounded control and observation operators. In the paper [24] the authors generalize the results of [12] by considering general semigroups instead of analytic semigroups, but the state space is Hilbert. Recently a generalization of the paper [24] to the Banach state spaces is established in [2]. There, the authors have used the restriction that the control operator is bounded. The case of unbounded control operator with bounded observation operator was considered by Pohjolainen [13]. The recent paper [14], the authors investigated the PI-control problem for well-posed linear systems in Hilbert spaces with an appropriate class of disturbance terms and obtained a result on tracking and disturbance rejection.

By analyzing the existing literature, one notes that the PI-controller problem seems to be not well investigated for well-posed linear systems in Banach space due to the difficulty in using spectral theory to prove stabilty. The object of this paper is to study the PI–controller for well-posed linear systems in Banach spaces. Here, the semigroup generator of the system is not necessarily analytic semigroup and the control and observation operators are possibly unbounded. We are interested in the class of infinite dimensional regular linear systems in the Salamon–Weiss sense, [15], [16], [20]. As examples of such systems one includes boundary control problems and input-output delay systems. In fact, as shown in [6] a linear system with state, input and output delays can be reformulated as infinite dimensional regular linear system in product state Banach spaces. Our aim is to introduce an unified approach to PI-Controller for the general class of regular linear systems in state Banach spaces. The approach is mainly based on infinite-dimensional closed–loop systems, their spectral theory and the spectral mapping theorem for semigroups [4, Chap.VI]. Some robustness results are also investigated in this paper.

The organization of the paper is as follows: As we deal with infinite dimensional regular linear systems, Section 11 is devoted to a background on such systems. In Section IV we study the I–controller for regular linear systems. In Section IV we investigate the PI-controller for regular linear systems. In the last section we summarize the results obtained in this paper.

Notation. Let A be the generator of a C_0–semigroup $T := (T(t))_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$. We denote by $\rho(A)$ the resolvent set of A, i.e., the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is invertible. The spectrum of A is by definition $\sigma(A) = \mathbb{C}\backslash\rho(A)$. The domain $\mathcal{D}(A)$ endowed with the graph norm $\|x\|_1 = \|(\lambda - A)x\|$, for $\lambda \in \rho(A)$, is a Banach space. We define the resolvent operator of A as $R(\lambda, A) := (\lambda - A)^{-1},\ \lambda \in \rho(A)$. We also define the norm $\|x\|_{-1} = \|R(\lambda, A)x\|$ for some $\lambda \in \rho(A)$. The completion of X with respect to the norm $\|\cdot\|_{-1}$ is a Banach space denoted by X_{-1}, which is called the *extrapolation space* associated with X and A. Moreover, the continuous injection

$$X_1 \hookrightarrow X \hookrightarrow X_{-1}$$

holds. The semigroup T can be naturally extended to a strongly continuous semigroup $T_{-1} = (T_{-1}(t))_{t\geq 0}$ on X_{-1}, of which the generator $A_{-1} : X \to X_{-1}$ is the extension of A to X. The type of the semigroup T is defined as

$$\omega_0(A) = \inf_{t>0} \frac{1}{t} \log(\|T(t)\|).$$

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The spectral bound of the generator A is given by

$$s(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}.$$

II. An overview of Salamon–Weiss systems

For the reader's convenience, we briefly recall the concept of infinite dimensional well-posed and regular linear systems in the Salamon–Weiss sense. See [15], [16], [19], [20] for more details.

In this section A is the generator of a C_0–semigroup $T := (T(t))_{t\geq 0}$ on a Banach space X.

Let U be a Banach space. We say that $B \in \mathcal{L}(U, X_{-1})$ is *an admissible control operator* for A if

$$\int_0^t T_{-1}(t-s)Bu(s)\, ds \in X$$

for some $t \geq 0$ and $u \in L^p([0,t], U)$. This means that if z is the solution of

$$\dot{x}(t) = Ax(t) + Bu(t)$$

which is an equation in X_{-1} then $x(t) \in X$ for all $t \geq 0$, initial condition $x(0) = z \in X$ and control function $u \in L^p([0,\infty), U)$. On the other hand, if $\omega > \omega_0(A)$, then there exists a positive constant β such that

$$\|R(s, A_{-1})B\|_{\mathcal{L}(U,X)} \leq \frac{\beta}{\sqrt{\mathrm{Re}\, s}} \quad \text{for } \mathrm{Re}s > \omega, \tag{1}$$

Let Y be another Banach space. An operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is called an *admissible observation operator* for A (or T) if the estimate

$$\int_0^\tau \|CT(t)x\|^p\, dt \leq \gamma_\tau^p\, \|x\|^p \tag{2}$$

holds for any $x \in \mathcal{D}(A)$ and for some constants $\tau > 0$ and $\gamma_\tau > 0$. This means that there exists a linear bounded operator $\Psi : X \to L^p_{loc}([0,\infty), Y)$ such that

$$(\Psi x)(t) = CT(t)x, \qquad \forall \in \mathcal{D}(A). \tag{3}$$

As $\mathcal{D}(A)$ is dense in X, the operator Ψ is completely determined by (3). On the other hand, for every $\omega > \omega_0(A)$, there exists a positive constant $c > 0$ such that

$$\|CR(s, A)\|_{\mathcal{L}(X,Y)} \leq \frac{c}{\sqrt{\mathrm{Re}\, s}} \quad \text{for } \mathrm{Re}s > \omega, \tag{4}$$

Let Σ be a time-invariant linear system with state space X, control space U, observation space Y, state trajectory $z : [0,\infty) \to X$, input u and output y. Then Σ is called *well-posed linear system* in X, U, Y if for every $t > 0$ there is $\gamma_t > 0$ (independent of u and the initial state) such that

$$\|z(t)\|^p + \int_0^t \|y(\tau)\|_Y^p d\tau \leq c_t^p \Big[\|z(0)\|^p + \int_0^t \|u(\tau)\|_U^p d\tau\Big].$$

The reader is refereed to the standard references [15], [16], [19], [20] for more details.

To any well-posed linear system Σ on X, U, Y we can associate operators A, B, C satisfying the assumptions earlier in this section. In this case, $(T(t))_{t\geq 0}$ is called the *semigroup* of Σ, A is called its *semigroup generator*, B is called the *control operator* of Σ, and C is called *the observation operator* of Σ. The relationship between the input and the output of Σ is given by

$$y = \Psi z^0 + \mathbb{F}u,$$

where z^0 is the initial condition of Σ, Ψ is the operator defined by (3) (called the extended output map of Σ), and $\mathbb{F} : L^p_{loc}([0,\infty), U) \to L^p_{loc}([0,\infty), Y)$ is a linear continuous operator (called the extended input–output map of Σ). The operator $\mathbb{F}$ is determined as follows: if $y = \mathbb{F}u$, then y has a Laplace transform $\hat{y}$, and for $\mathrm{Re}\lambda > \max\{\omega_0(A), 0\}$ we have

$$\hat{y}(\lambda) = \mathbf{G}(\lambda)\hat{u}(\lambda),$$

where $\mathbf{G}$ is an $\mathcal{L}(U,Y)$-valued analytic function satisfying, for any $\omega > \omega_0(A)$,

$$\sup_{\mathrm{Re}\lambda > \omega} \|G(\lambda)\| < +\infty. \tag{5}$$

The function $\mathbf{G}$ is called *transfer function* of Σ. This function satisfies

$$\mathbf{G}(\lambda) - \mathbf{G}(\mu) = C(R(\lambda, A) - R(\mu, A))B \tag{6}$$

for $\mathrm{Re}\lambda, \mathrm{Re}\mu > \omega_0(A)$.

Let Σ be a well-posed linear system with extended input–output operator $\mathbb{F}$. We say that Σ is *regular* (with zero feedthrough) if the limit

$$\lim_{t\to 0} \frac{1}{t} \int_0^t (\mathbb{F}u_0)(\tau)\, d\tau = 0$$

exists in Y for the constant input $u_0(t) = z,\ z \in U,\ t \geq 0$.

The class of regular linear systems is very useful for the feedback theory of control systems. To recall this, we need some notation. The *Yosida extension* of an operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is defined as

$$\begin{aligned} C_\Lambda z &:= \lim_{\lambda\to+\infty} C\lambda R(\lambda, A)z \\ \mathcal{D}(C_\Lambda) &:= \{z \in X : \text{the above limit exists in } Y\}. \end{aligned} \tag{7}$$

By using the graph norm associated with A and Lemma 3.4 in [4, p.73] one can see that $\mathcal{D}(A) \subset \mathcal{D}(C_\Lambda)$ and that $C_\Lambda z = Cz$ for any $z \in \mathcal{D}(A)$.

Now, if Σ is a regular linear system with control the we have

$$R(s, A_{-1})B \subset \mathcal{D}(C_\Lambda) \quad \text{for some (hence all) } s \in \rho(A). \tag{8}$$

Moreover, the transfer function of G satisfies

$$\mathbf{G}(s) = C_\Lambda R(s, A_{-1})B \quad \text{for } s \in \rho(A). \tag{9}$$

In the rest of this section we recall the feedback theory of regular systems. Let Σ be a regular system with extended input–output operator $\mathbb{F}$. For any $\tau > 0$ we define an operator $\mathbb{F}(\tau) : L^p([0,\tau],U) \to L^p([0,\tau],Y)$ by setting $(\mathbb{F}(\tau)u)(t) = (\mathbb{F}u)(t)$ for any $t \in [0,\tau]$ and $u \in L^p_{loc}([0,\infty),U)$.

An operator $K \in \mathcal{L}(Y,U)$ is called *an admissible feedback* for the system Σ if $I-\mathbb{F}(\cdot)K$ has uniformly bounded inverse. Note that in the case of Hilbert spaces and $p=2$ one can use transfer functions instead of input-output operators for the definition of admissible feedback (see [19] for more details).

We now state a very general perturbation theorem due to Weiss in Hilbert spaces [20] and to Staffans in general Banach spaces [16, Chap.7].

Theorem 2.1: Let Σ be a regular linear system on X,U,Y with semigroup $(T(t))_{t\geq 0}$, semigroup generator A, control operator B, observation operator C and admissible feedback K. Then the operator defined by

$$R = A_{-1} + BKC_\Lambda$$
$$\mathcal{D}(R) := \big\{z \in \mathcal{D}(C_\Lambda) : (A_{-1} + BKC_\Lambda)z \in X\big\}$$

with the sum defined in X_{-1} generates a C_0–semigroup T_K on X satisfying $T_K(\sigma)z \in \mathcal{D}(C_\Lambda)$ for a. e. $\sigma \geq 0$ and

$$T_K(t)\eta = T(t)\eta + \int_0^t T_{-1}(t-\sigma)BKC_\Lambda T_K(\sigma)\eta\, d\sigma \quad (10)$$

for $\eta \in X,\ t \geq 0$.

III. Design of the I-controller

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + w, & t>0,\ x(0)=x \\ y(t) = Cx(t), & t>0, \end{cases}, \quad (11)$$

where $A : \mathcal{D}(A)$ is the generator of a C_0–semigroup $(T(t))_{t\geq 0}$ on a Banach space X, the control operator $B \in \mathcal{L}(U,X_{-1})$ and the observation operator $C \in \mathcal{L}(\mathcal{D}(A),Y)$. We are looking for parameters $k_I \in \mathbb{R}$ for which the feedback law

$$u(t) = k_I K_I \int_0^t (y(\tau) - y_r)\, d\tau := k_I K_I z(t)$$

stabilizes, where y_r is suitable reference output and K_I is an appropriate feedback operator. As our setting is general, extra conditions on the plant (A,B,C) is needed.

We assume that A,B,C are the semigroup generator, the control operator and the observation operator of a regular linear system Σ, respectively. Moreover, we assume that K_I is an admissible feedback for Σ. By invoking the feedback law $u(t) = k_I K_I z(t)$ and introducing the new state

$$\xi : [0,\infty) \to \mathcal{X} := X \times Y, \quad \xi = \left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right), \quad (12)$$

the system (11) can be reformulated as

$$\begin{cases} \dot{\xi}(t) = \mathcal{A}_I\xi(t) + \left(\begin{smallmatrix} w \\ -y_r \end{smallmatrix}\right), & t>0, \\ y(t) = \mathcal{M}\xi(t), & t>0, \end{cases}, \quad (13)$$

with

$$\mathcal{A}_I := \begin{pmatrix} A_{-1} & k_I BK_I \\ C_\Lambda & 0 \end{pmatrix},$$
$$\mathcal{D}(\mathcal{A}_I) = \left\{\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \in \mathcal{D}(C_\Lambda) \times Y : \mathcal{A}_I\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \in \mathcal{X}\right\} \quad (14)$$

and

$$\mathcal{M} := \mathcal{D}(C_\Lambda) \times Y \to Y, \quad \mathcal{M} = (C_\Lambda\ \ 0).$$

We are interested in showing the following three items

(a) The solution of the system (11) approaches zero exponentially.
(b) The regulation of the output. This means that y approaches y_r asymptotically
(c) The aforementioned stabilization and regulation are independent of the initial condition $x(0)$ and the perturbation term w.

To that purpose we need some preparations.

Proposition 3.1: Assume that A,B,C are the semigroup generator, the control operator and the observation operator of a regular linear system Σ, respectively. Moreover, we assume that K_I is an admissible feedback for Σ. Then the operator $\mathcal{A}_I$ coincides with the generator of an appropriate closed–loop system, so it is a generator.

Proof: Define

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times Y.$$

Then $\mathcal{A}$ generates the following diagonal semigroup

$$\mathcal{T}(t) = \begin{pmatrix} T(t) & 0 \\ 0 & I_Y \end{pmatrix}, \quad t \geq 0.$$

On the other hand, define

$$\mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & I_Y \end{pmatrix}, \mathcal{C} = \begin{pmatrix} 0 & I_Y \\ C & 0 \end{pmatrix} \text{ and } \mathcal{K} = \begin{pmatrix} k_I K_I & 0 \\ 0 & I_Y \end{pmatrix}.$$

A straightforward arguments shows that $(\mathcal{A},\mathcal{B},\mathcal{C})$ generates a regular linear system on $\mathcal{X}, U \times Y, Y \times Y$ with $\mathcal{K}$ as admissible feedback operator. Then by Theorem 2.1 the following operator

$$\mathcal{R} = \mathcal{A}_{-1} + \mathcal{B}\mathcal{K}\mathcal{C}_\Lambda,$$
$$\mathcal{D}(\mathcal{R}) = \left\{\left(\begin{smallmatrix} x \\ \zeta \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{C}_\Lambda) : (\mathcal{A}_{-1} + \mathcal{B}\mathcal{K}\mathcal{C}_\Lambda)\left(\begin{smallmatrix} x \\ \zeta \end{smallmatrix}\right) \in \mathcal{X}\right\} \quad (15)$$

generates a C_0–semigroup on $\mathcal{X}$, where $\mathcal{C}_\Lambda$ is the Yosida extension of $\mathcal{C}$ with respect to $\mathcal{A}$. Now we will prove that $\mathcal{R} = \mathcal{A}_I$. Let $\lambda \in \rho(\mathcal{A}) = \rho(A)\backslash\{0\}$. Then

$$\mathcal{C}\lambda R(\lambda,\mathcal{A}) = \begin{pmatrix} 0 & I \\ C\lambda R(\lambda,A) & 0 \end{pmatrix}.$$

Then $\mathcal{D}(\mathcal{C}_\Lambda) = \mathcal{D}(C_\Lambda) \times Y$ and

$$\mathcal{C}_\Lambda = \begin{pmatrix} 0 & I \\ C_\Lambda & 0 \end{pmatrix}.$$

The proof follows then from the fact that

$$\mathcal{A}_{-1} = \begin{pmatrix} A_{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\mathcal{A}_L$ is a generator of a C_0–semigroup on $\mathcal{X}$. ■

Proposition 3.2: Assume that A,B,C are the semigroup generator, the control operator and the observation operator of a regular linear system Σ, respectively. Moreover, we assume that K_I is an admissible feedback for Σ. Then, for

$\lambda \in \rho(A)\backslash\{0\}$ we have $\lambda \in \sigma(\mathcal{A}_I)$ if and only if $1 \in \sigma(\frac{k_I}{\lambda} C_\Lambda R(\lambda, A_{-1}) B K_I)$. On the other hand, if A is invertible, then $\mathcal{A}_I$ is invertible if and only if $C_\Lambda(-A_{-1})^{-1} B K_I$ is so.

Proof: Let $\mathcal{R}$ be the generator of the closed loop system constructed in the proof of Proposition 3.1 (see (15)). As $\mathcal{A}_I = \mathcal{R}$ then by [23, Theorem 1.2], for $\lambda \in \rho(\mathcal{A}) = \rho(A)\backslash\{0\}$ we have $\lambda \in \sigma(\mathcal{A}_I)$ if and only if $I - \mathcal{G}(\lambda)\mathcal{K}$ is not invertible. Now for $\mathrm{Re}\lambda > w_0(A)$ we have

$$\mathcal{G}(\lambda) = \mathcal{C}_\Lambda R(\lambda, \mathcal{A}_{-1})\mathcal{B} = \begin{pmatrix} 0 & \frac{1}{\lambda} \\ C_\Lambda R(\lambda, A_{-1})B & 0 \end{pmatrix}.$$

Then

$$I - \mathcal{G}(\lambda)\mathcal{K} = \begin{pmatrix} I & -\frac{1}{\lambda} \\ -k_I C_\Lambda R(\lambda, A_{-1})BK & I \end{pmatrix}$$

Hence $I - \mathcal{G}(\lambda)\mathcal{K}$ is not invertible if and only if $1 \in \sigma(\frac{k_I}{\lambda} C_\Lambda R(\lambda, A_{-1}) B K_I)$.

We show the last assertion. Assume that A is invertible. We have $(-A_{-1})^{-1}BK \in \mathcal{L}(Y, X)$, so we can decompose $\mathcal{A}_I$ as follows

$$\mathcal{A}_I = \begin{pmatrix} A_{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & k_I(A_{-1})^{-1}BK_I \\ C_\Lambda & 0 \end{pmatrix}.$$

Thus the result follows by Schur complement. ∎

Proposition 3.3: Assume that A, B, C are the semigroup generator, the control operator and the observation operator of a regular linear system Σ, respectively. Moreover, we assume that K_I is an admissible feedback for Σ. Then

(i) If $\omega_0(A) \geq 0$, then for all $\omega > \omega_0(A)$ there exists $\kappa_\omega > 0$ such that $s(\mathcal{A}_I) \leq \omega$ for all $k_I \in (0, \kappa_\omega)$.

(ii) If $\omega_0(A) < 0$ and $\sigma(C_\Lambda R(0, A_{-1})BK_I) \subset \mathbb{C}^+$, there exists $\kappa_\omega > 0$ such that $s(\mathcal{A}_I) < 0$ for all $k_I \in (0, \kappa_\omega)$.

Proof: We show (i). Let $\omega > \omega_0(A)$. Then by [19] we have

$$a_\omega := \sup_{\mathrm{Re}\lambda \geq \omega} \|C_\Lambda R(\lambda, A_{-1})B\| < +\infty. \tag{16}$$

If we put $\kappa_\omega := \omega/(a_\omega\|K\|)$ then for all $k_I \in (0, \kappa_\omega)$ we have

$$\big\|\frac{k_I}{\lambda} C_\Lambda R(\lambda, A_{-1})BK\big\| \leq \frac{k_I a_\omega \|K\|}{\omega} < 1.$$

The assertion then follows by Proposition 3.1.

We show (ii). This is much more difficult. This needs some decomposition on the spectrum of A. Let $\omega \in (\omega_0, 0)$ and define

$$\Omega_\omega := \big\{\lambda \in \mathbb{C} : \mathrm{Re} > \omega, \quad |\lambda| \geq |\omega|\big\}.$$

In view of Proposition 3.1 one sees that for $\lambda \in \Omega_\omega$, $\lambda - \mathcal{A}_I$ is invertible if and only if $I - k_I\lambda^{-1}C_\Lambda R(\lambda, A_{-1})BK_I$ is invertible. As we have seen before, there exists $k_0 > 0$ such that $I - k_I\lambda^{-1}C_\Lambda R(\lambda, A_{-1})BK_I$ is invertible for all $k_I \in (0, k_0)$. Hence

$$\Omega_\omega \subset \rho(\mathcal{A}_I). \tag{17}$$

For $\beta > 0$, define

$$\overline{D}(0, \beta) := \big\{\lambda \in \mathbb{C} : |\lambda| \leq \beta\big\}.$$

Using the exponential stability of the semigroup generated by A and the equation (6) one can see that there exist $\beta > 0$ (independent of k_I) such that

$$\overline{D}(0, \beta)^+ := \overline{D}(0, \beta) \cap \mathbb{C}^+ \quad \text{for all } k_I > 0. \tag{18}$$

According to (17) and (18) we have

$$\overline{D}(0, \beta)^+ \cup \Omega_\omega \subset \rho(\mathcal{A}_I) \quad \text{for all } k_I \in (0, k_\omega). \tag{19}$$

Now if we set $\overline{D}(0, \beta)^- := \overline{D}(0, \beta) \cap \mathbb{C}^-$ then we have

$$d(\overline{D}(0, \beta)^-, \overline{D}(0, \beta)^+) > 0,$$

where $d(\overline{D}(0, \beta)^-, \overline{D}(0, \beta)^+)$ denotes the distance between the sets $\overline{D}(0, \beta)^-$ and $\overline{D}(0, \beta)^+$. This implies that $s(\mathcal{A}_I) < 0$. ∎

In what follows we assume that $U = Y = \mathbb{C}^m$. We will prove the exponential stability of the semigroup generated by $\mathcal{A}_I$. Before going into details, we recall some notions. We denote by $\omega_{ess}(A)$ the essential growth bound of the generator A (see [4]). From the Spectral Mapping Theorem (see [4, Chap.IV]) we have

$$\omega_0(A) = \max\{\omega_{ess}(A), s(A)\}. \tag{20}$$

Theorem 3.4: Assume that A, B, C are the semigroup generator, the control operator and the observation operator of a regular linear system Σ, respectively. Moreover, we assume that K_I is an admissible feedback for Σ. Assume that $\omega_0(A) < 0$ and let K_I a $m \times m$ matrix such that $\sigma(CA^{-1}BK_I) \subset \mathbb{C}^+$. Then there exists $\kappa_I > 0$ such that for all $k_I \in (0, \kappa_I)$ the operator $\mathcal{A}_I$ is exponentially stable. For all $x_0 \in \mathcal{D}(A)$ and every constant perturbation w, the output function of (13) satisfies

$$\lim_{t \to +\infty} \|y(t) - y_r\| = 0.$$

Proof: From the proof of proposition 3.1 the operator $\mathcal{A}_I$ coincide with the operator $\mathcal{R}$, which is the generator of a closed loop system. Since $\mathcal{K}$ is compact, then by [23, Proposition 2.5], the operators $\mathcal{A}_I$ and $\mathcal{A}$ have the same essential spectrum. Hence $\omega_{ess}(\mathcal{A}_I) = \omega_{ess}(\mathcal{A}) = \omega_{ess}(A)$. Since $\omega_0(A) < 0$ then by (20) we have $\omega_{ess}(\mathcal{A}_I) < 0$. Now in view of Proposition 3.2 we have $s(\mathcal{A}_I) < 0$. Again by (20) (with respect to $\mathcal{A}_I$) we conclude that $\omega_0(\mathcal{A}_I)$. We now prove the last assertion. The solution of the nonhomogeneous system (13) is given by

$$\xi(t) = \mathcal{T}_I(t)\big(\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\big) + \int_0^t \mathcal{T}_I(\tau)\big(\begin{smallmatrix} w \\ -y_r \end{smallmatrix}\big)d\tau \tag{21}$$

for all $\big(\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\big) \in \mathcal{X}$. As $y(t) = \mathcal{M}\xi(t)$ and $\mathcal{M}\mathcal{A}_I^{-1} = [0 \quad I]$ then

$$y(t) - y_r = \mathcal{M}\mathcal{T}_I(t)\big[\big(\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\big) + \mathcal{A}_I^{-1}\big(\begin{smallmatrix} w \\ -y_r \end{smallmatrix}\big)\big].$$

Since $\mathcal{M}$ is an admissible observation operator for the exponentially stable C_0–semigroup $(\mathcal{T}_I(t))_{t\geq 0}$ then we have

$$\int_0^\infty \|y(t)-y_r\|^p\,dt < \gamma^p \|\left(\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right) + \mathcal{A}_I^{-1}\left(\begin{smallmatrix} w \\ -y_r \end{smallmatrix}\right)\|^p$$

with constant $\gamma > 0$. This complete the proof. ■

IV. PI-controller

In this section we use the following feedback law (with is the feedback law used in the previous section plus a proportional term)

$$u(t) = k_P K_P(y(t)-y_r) + k_I K_I \int_0^t (y(\tau)-y_r)\,d\tau. \quad (22)$$

Throughout this section we assume that A, B, C are issued from a regular linear system Σ such that K_I and K_P are admissible feedback operators for Σ, and that $U = Y = \mathbb{C}^m$. Then from Theorem 2.1, the following operator

$$A_P := A + k_P B K_P C_\Lambda,$$
$$\mathcal{D}(A_P) = \{x \in \mathcal{D}(C_\Lambda) : (A + k_P B K_P C_\Lambda)x \in X\}$$

generates a C_0–semigroup $(T_p(t))_{t\geq 0}$ on X. Similarly to Section III, we can reformulate the closed–loop system associated to the system (11) and the feedback law (22) as

$$\begin{cases} \dot{\xi}(t) = \mathcal{A}_{P,I}\xi(t) + \left(\begin{smallmatrix} w - k_p B K_P y_r \\ -y_r \end{smallmatrix}\right), & t > 0, \\ y(t) = \mathcal{M}\xi(t), & t > 0, \end{cases} \quad (23)$$

with

$$\mathcal{A}_{P,I} := \begin{pmatrix} A_P & k_I B K_I \\ C_\Lambda & 0 \end{pmatrix},$$
$$\mathcal{D}(\mathcal{A}_{P,I}) = \left\{ \left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \in \mathcal{D}(C_\Lambda) \times Y : \mathcal{A}_{P,I}\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \in \mathcal{X} \right\} \quad (24)$$

and

$$\mathcal{M} := \mathcal{D}(C_\Lambda) \times Y \to Y, \quad \mathcal{M} = (C_\Lambda \;\; 0).$$

To follow the results obtained in Section III, we need to show that the semigroup $(T_P(t))_{t\geq 0}$ is exponentially stable. We have the following result.

Lemma 4.1: If $\omega_0(A) < 0$ then there exists $\kappa_P > 0$ such that $\omega_0(A_P) < 0$ for all $k_P \in (0, \kappa_P)$.

Proof: Let $\delta \in (\omega_0(A), 0)$ and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda \geq \delta$. From (5) we have

$$\alpha := \sup_{\mathrm{Re}\lambda \geq \delta} \|C_\Lambda R(\lambda, A_{-1})B\| < +\infty.$$

we set $\kappa_P := 1/(\alpha\|K_P\|)$. Then for all $k_P \in (0, \kappa_P)$ we have $\|k_p K_P C_\Lambda R(\lambda, A_{-1})B\| < 1$. Then by [23, Theorem 1.2] we have $\lambda \in \rho(A_P)$, which means that $s(A_p) < 0$. On the other hand, by [23, Proposition 2.5], we have $\omega_{ess}(A_P) = \omega_{ess}(A) < 0$. Now by the Spectral Mapping Theorem for semigroups we have $\omega_0(A_P) = \max\{\omega_{ess}(A_P), s(A_P)\} < 0$. ■

Theorem 4.2: Assume that A generates an exponentially stable C_0–semigroup on X and that $\mathrm{Rank}(C_\Lambda A_{-1}^{-1}B) = m$. Then there exist $\kappa_P > 0$ such for all $k_p \in (0, \kappa_P)$ there exists $\kappa_I > 0$ such that for all $k_I \in (0, \kappa_I)$ we have $\omega_0(\mathcal{A}_{P,I}) < 0$ and $y(t) - y_r$ approaches zero in X as t approaches $+\infty$ for every disturbance $w \in X$ and initial condition $x^0 \in X$.

Proof: Set $G(\lambda) = C_\Lambda R(\lambda, A_{-1})B$ the transfer function of the regular linear system Σ. Then the transfer function of the closed loop system associated with Σ and the admissible feedback K_P is given by $G_P(\lambda) = (I - K_P G(\lambda))^{-1} G(\lambda)$ for $\lambda \in \rho(A) \cap \rho(A_P)$. Then $\mathrm{Rank}(G_P(0)) = \mathrm{Rank}(G(0)) = m$. The rest of the proof follows by Lemma 4.1 and a similar argument as in the proof of Theorem 3.4. ■

V. Conclusion

In this paper, we have addressed the question of PI-controller (proportional and integral output feedback) for regular linear systems with unbounded control and observation operators in Banach state space and a constant disturbance term. We have first solved the integral controller problem using an approach mainly based on regular linear systems in the Salamon–Weiss sense. In particular the spectral theory of the closed loop system developed in [23] together with Spectral Mapping Theorem for semigroups [4, Chap.IV]. In the second part of the paper, the PI–controller problem is regarded as the perturbation (in the closed loop sense [20]) of the I–controller. Thus we used the results of Section III and a robustness result to solve the PI–controller. The regulation is also obtained in this paper using properties of admissible observation operators. The abstract results of this paper will be applied in a forthcoming paper to both a large class retarded and neutral linear systems with state [6], [7], [8], [9].

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