2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Consensus in Hierarchical Multi-agent Dynamical Systems with Low-rank Interconnections: Analysis of stability and convergence rates

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Abstract— **In this paper, we propose a fairly general model for hierarchical multi-agent dynamical systems (HMADSs) with fractal structure and investigate their stability or convergence condition for consensus. We first generalize a model introduced by Smith et al. [1] to represent the weak cross-layer interconnection properly and explain the significance of focusing on the low-rank property instead of sparseness or small gain property. We then derive the analytical expression of eigenvalue distribution of the system matrix with rank 1 interconnection for cyclic pursuit. This provides us the stability or consensus condition for the whole system where each agent has a certain dynamics. We also clarify the relation between the property of interconnection and stability degree of multi-agent systems, which is confirmed by numerical examples. Further, we investigate the rank 2 case of the interconnection structure.**

I. INTRODUCTION

In recent years, systems to be handled in various fields of engineering including control have become large and complex due to the tremendous progress of computer, communication and network technologies. In addition, more high-level control such as an adaptation against changes of environments for open systems is required, where a lot of subsystems interact with each other via no centralized control function. This motivates us to develop a new framework for investigating such a large-scale system with decentralized information structure. From the above viewpoint, it has been paid much attention to analysis of multi-agent systems and its decentralized control scheme. In this line of research, one of the research topics that are nowadays receiving a lot of interest is the consensus control problem, also known as agreement or rendezvous problem, for multi-agent systems (see [2], [3], [4] and the references therein).

Most of the current literatures have focused on the simple local control schemes using decentralized information structures for achieving a desired global behavior; e.g., consensus behavior of a group of agents which means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. Nonetheless, there is another essential feature in large-scale multi-agent systems. That is the *hierarchical structure*, because large-scale systems normally include interactions of subsystems which have different scales in various aspects such as space and time. The class of such systems is often called as multi-scale systems, and one of the typical examples is bio-systems: they have different space scales at least from 10^{-9} in gene level to 10^{-3} in tissue level. The situation with small size of multi-agent systems ($n_1 = 4, n_2 = 3, n_3 = 2$) is illustrated in Fig. 1(a) to show the hierarchical structure, although we have many practical applications with quite large number of n_i.

For the above problem, Smith et al. [1] have recently proposed a class of hierarchical multi-agent system structures, and then have investigated the stability properties in consensus problem. However, the class of networks in multi-agent systems treated in their paper is restricted, and hence the distinctive features of hierarchical systems, which depend on the cross-layer interconnection structures of a group of agents, have not been thoroughly investigated. Further, no discussion for the case where each agent has a certain dynamics has been presented in their paper. Note that if agent's dynamics is not explicitly considered in developing control strategies, it may suffer from the potential problem that the global stability cannot be achieved (refer to [5], [6]).

In this paper, following the research direction initiated by Smith et al. [1], we first propose a fairly general model for hierarchical multi-agent *dynamical* systems (HMADSs) with fractal structure, and then investigate its stability analysis methods and global convergence properties. The key point of this research is to study a low-rank property of the interconnection structure beyond different layers. Although small gain property and sparseness have been broadly treated to handle the weak interactions between agents in the conventional researches, the low rank property of interconnection networks has not been studied yet. The low-rank property captures a kind of information aggregation which may work effectively for rapid convergence in the consensus, and hence we have to pay attention to the low-rank property. To this end, we first generalize a hierarchical model introduced in [1] to represent the weak cross-layer interconnection network properly, and explain the significance of focusing on the low-rank property instead of sparseness or small gain property. We then derive the analytical expression of eigenvalue distributions of the system matrix for cyclic pursuit. This provides us a simple stability criterion for the overall multi-agent system where each agent has a certain dynamics. Finally, we clarify the relation between the property of interconnection networks and the stability degree of multi-agent systems, which is confirmed by several numerical examples. These show that a certain aggregation process in information acquisition plays a key role for the consensus of

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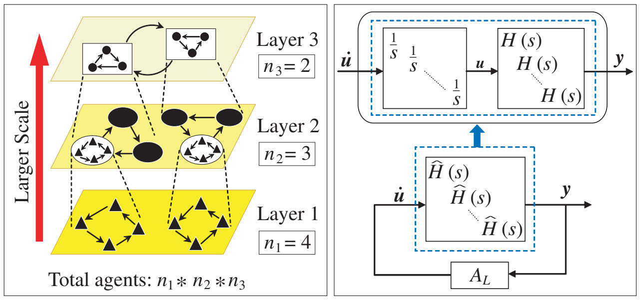


(a) (b)

Fig. 1. (a) Hierarchical multi-agent system, (b) The closed-loop hierarchical multi-agent dynamical system

large-scale multi-agent dynamical systems.

Notation: $\boldsymbol{I}_n$ denotes the identity matrix with size n, and $\otimes$ represents the Kronecker product of two matrices. The transpose of the matrix M is denoted by M^T. The real and imaginary parts of a complex variable z is represented by $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$, respectively.

II. Hierarchical Multi-Agent Dynamical System with Fractal Structure

A. A general model

We consider a class of HMADSs with L layers, where all the agents have the identical dynamics represented by

$$H(s) = c_h\,(sI_k - A_h)^{-1}\,b_h + d_h, \tag{1}$$

or its state-space realization:

$$\dot{z}(t) = A_h z(t) + b_h u(t), \quad y(t) = c_h z(t) + d_h u(t). \tag{2}$$

The most typical situation is that $H(s)$ is the closed-loop transfer function represented by

$$H(s) = P(s)K(s)/(1 + P(s)K(s)), \tag{3}$$

where $K(s)$ is an identical feedback controller which locally stabilizes the common plant $P(s)$ of each agent.

Let the interconnection among all the agents be expressed as

$$\dot{\boldsymbol{u}} = \boldsymbol{A}_L \boldsymbol{y}\,, \tag{4}$$

where $\boldsymbol{u}$ and $\boldsymbol{y}$, respectively, denote all the collections of input and output signals of each agent. See Fig. 1(b) for the block diagram of the whole system, where $\hat{H}(s) := H(s)/s$ and it has the following state-space realization:

$$\hat{A}_h = \begin{pmatrix} A_h & b_h \\ 0 & 0 \end{pmatrix},\ \hat{b}_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \hat{c}_h = \begin{pmatrix} c_h \\ d_h \end{pmatrix}^T,\ \hat{d}_h = 0.$$

It is clear that the feedback system combined with (2) and (4) belongs to a class of LTI systems with a generalized frequency variable proposed by Hara et al. [7]. Hence, the system matrix of the feedback system is given by

$$\mathcal{A}_L = I_n \otimes \hat{A}_h + \boldsymbol{A}_L \otimes (\hat{b}_h \hat{c}_h) = \hat{A}_h \otimes I_n + (\hat{b}_h \hat{c}_h) \otimes \boldsymbol{A}_L, \tag{5}$$

of which the set of eigenvalues determines the stability of the whole feedback system. The dynamic part of our feedback system interested in is uniformly expressed as $\hat{H}(s)\boldsymbol{I}$, while we have a fractal structure in the interconnection represented by $\boldsymbol{A}_L$ as investigated in [1]. In other words, we here suppose that the system has a fractal structure and the system matrix $\boldsymbol{A}_L$ is recursively defined as follows:

(i) For the lowest layer with the gain $g_1 > 0$,

$$\boldsymbol{A}_1 = g_1(\boldsymbol{P}_{n_1} - \boldsymbol{I}_{n_1}). \tag{6}$$

(ii) For the higher layers with gains $g_\ell > 0$ ($\ell = 2, \cdots, L$),

$$\begin{aligned} \boldsymbol{A}_\ell &= \boldsymbol{I}_{n_\ell} \otimes \boldsymbol{A}_{\ell-1} + g_\ell(\boldsymbol{P}_{n_\ell} \otimes \boldsymbol{\Delta}_{N_{\ell-1}} - \boldsymbol{I}_\ell) \\ &= \boldsymbol{I}_{n_\ell} \otimes (\boldsymbol{A}_{\ell-1} - g_\ell \boldsymbol{I}_{N_{\ell-1}}) + \boldsymbol{P}_{n_\ell} \otimes (g_\ell \boldsymbol{\Delta}_{N_{\ell-1}}), \end{aligned} \tag{7}$$

where N_ℓ is the total number of agents in the ℓ-th layer system, $N_\ell := \prod_{i=1}^{\ell} n_i$, and

$$\boldsymbol{\Delta}_{N_{\ell-1}} := \prod_{i=1}^{\ell-1} g_i \cdot \boldsymbol{\Delta}_{n_{\ell-1}} \otimes \boldsymbol{\Delta}_{n_{\ell-2}} \otimes \cdots \otimes \boldsymbol{\Delta}_{n_1} \tag{8}$$

with $\sum_{j=1}^{n_\ell} \boldsymbol{\Delta}_{n_\ell}(i,j) = 1$.

Here, the diagonal elements of $\boldsymbol{P}$ are assumed to be all zero, and the suffix represents the dimension of a square matrix. Eq. (7) implies the i-th subsystem can obtain the information of the j-th subsystem when $\boldsymbol{P}(i,j)$ is nonzero, where $\boldsymbol{\Delta}$ is referred as the incidence matrix since it represents the property of information acquisition. We set the dynamics of the system so that $\boldsymbol{A}_\ell$ becomes a Laplacian matrix, thus on the ℓ-th layer each agent is subjected to the force which is obtained by multiplying each relative coordinate by gain g_ℓ. Hence, $-\boldsymbol{I}$ in (7) involves that each agent can obtain a relative coordinate for each layer, and the overall system is constructed by feedback of these relative coordinates.

Remark 1. *In the previous paper [8], we studied a model for a class of HMADSs with L layers. However, only the case that the gain g_ℓ in $\boldsymbol{A}_\ell$ is assumed to be as $g_\ell = 1$ for $\ell = 1, 2, \cdots, L$ is considered in that paper. Further, no general agent dynamics such as $H(s)$ in (1) was considered; i.e., $H(s)$ was tacitly assumed as $H(s) = 1/s$.*

B. Low-rank interconnection

We here assume that $\boldsymbol{P}$ in (7) is a circulant matrix of the form $\boldsymbol{P} = \mathrm{circ}(0, 1, 0, \cdots, 0)$ for investigating a cyclic pursuit strategy [5], [9], [10]. The fractal structure means that the agents in the i-th subsystem can use information on the $(i+1)$-th subsystem only.

In this paper, we focus on the low-rank property of incidence matrix $\boldsymbol{\Delta}$, which has not been investigated yet. However, it clearly captures the situation where weak couplings between subsystems are expressed by rank-deficiency or aggregation via information-sharing. In particular, we propose the following rank one incidence matrix:

$$\boldsymbol{\Delta} = \boldsymbol{1} \cdot \boldsymbol{\zeta}^T, \tag{9}$$

where $\boldsymbol{1} := (1, 1, \cdots, 1)^T$, and $\boldsymbol{\zeta}^T := (\zeta_1, \zeta_2, \cdots, \zeta_n)$ is a row probability vector which satisfies $\zeta_i \geq 0$ and $\sum_{i=1}^{n} \zeta_i = 1$. For example, if we set $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} = \boldsymbol{1} \cdot \boldsymbol{\zeta}^T, \quad \boldsymbol{\zeta}^T = (1, 0, \cdots, 0), \tag{10}$$

TABLE I

PROPERTY OF INCIDENCE MATRIX

Δ	Low-rank	High-rank
Sparse	$\Delta = 1 \cdot \zeta^T$	$\Delta = I$
Dense		

the only the first state is available, and hence $\boldsymbol{\Delta}$ is sparse. On the other hand, if we set $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} = \frac{1}{n}\mathbf{1}\cdot\mathbf{1}^T, \quad \mathbf{1}^T = (1, 1, \cdots, 1), \tag{11}$$

any agent can obtain no information about individual agents of the super group, but the rough information or the center of gravity coordinate is available. Note that the incidence matrix $\boldsymbol{\Delta}$ in this case is dense but rank one.

In general, the row vector $\boldsymbol{\zeta}^T$ implies that only a weighted average or a certain aggregated information of the super group is available, and the column vector $\mathbf{1}$ means that all the agents in a subgroup share the information and use it for control in the same manner. This situation happens when each agent can see the approximate position of one of the other groups rather than one of individual agents in the group. The following implementation is fairly reasonable for the case where the costs for local communication are quite low in comparison with those for inter groups. Then, each subgroup has one special agent which can only communicate with other group, and any other agents can communicate only inside the subgroup and get the information about outside subgroups from the special agent. There are physical models which adopt the situation: Consider a heat system, where a bunch of molecules exist in the microscopic level. If we want to see the macro behavior, we have to investigate the heat behavior, which is reflected by a certain average motion of independent molecules.

The model proposed in this paper is a generalization of the model in [1]. Contrary to our setting, Smith et al. [1] proposed a class of HMADSs and investigated its rate of convergence and number of links for the case of $\boldsymbol{\Delta} = \boldsymbol{I}$, which is sparse but has a maximal rank. Those three situations are summarized in Table 1. In the following sections, comparing these incidence matrices, we study the relation between the property of interconnection structure and the stability degree of multi-agent system.

III. ANALYSIS FOR EIGENVALUE AND EIGENSTRUCTURE

A. Eigenvalue distribution

We can derive the eigenvalue distribution of system matrix $\boldsymbol{A}_\ell$ for the rank one incidence matrix similarly to the case of $\boldsymbol{\Delta} = \boldsymbol{I}$ in [1]. Here the results in [11] on the eigenvalue distribution of circulant matrices play an important role.

Theorem 1. *If the incidence matrix $\boldsymbol{\Delta}$ in (9) of the L-layer system is rank one, then the set of eigenvalues of $\boldsymbol{A}_L$ is recursively given by*

$$\mathrm{eig}(\boldsymbol{A}_L) = \begin{cases} \mathrm{eig}(\boldsymbol{A}_{L-1})\setminus\{0\} - g_L \\ g_L(\omega_L^{k_L} - 1) \end{cases} \tag{12}$$

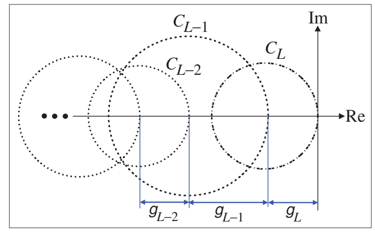
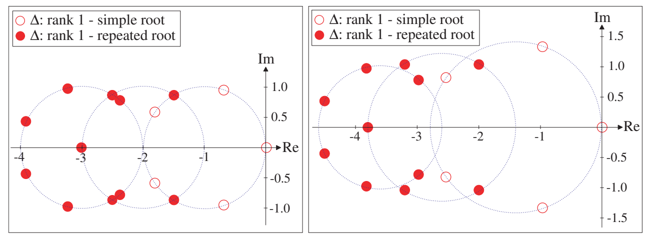


Fig. 2. Centers of circles corresponding to the each layer



(a) Unit gain case (b) Non-unit gain case

Fig. 3. Eigenvalue distributions for unit and non-unit gain cases.

where ω_ℓ ($\ell = 1, 2, \cdots, L$) is defined as $\omega_\ell := \exp\left(\frac{2\pi}{n_\ell}\mathrm{j}\right)$. Moreover, the explicit representation of a set of eigenvalues of $\boldsymbol{A}_L$ is given by the union of

$$\lambda = \begin{cases} -\sum_{i=1}^{L} g_i + g_1\omega_1^{k_1} & (k_1 = 1, 2, \cdots, n_1) \\ -\sum_{i=2}^{L} g_i + g_2\omega_2^{k_2} & (k_2 = 1, 2, \cdots, n_2) \\ \vdots & \\ -\sum_{i=\ell}^{L} g_i + g_\ell\omega_\ell^{k_\ell} & (k_\ell = 1, 2, \cdots, n_\ell) \\ \vdots & \\ -g_L + g_L\omega_L^{k_L} & (k_L = 0, 1, \cdots, n_L) \end{cases} \tag{13}$$

Although we omit the proof due to the space limitation, it should be emphasized that there are two features on the eigenvalue distribution:

- There is a simple eigenvalue at the origin in the complex plane and other eigenvalues are all in the open left-half complex plane. This is due to the property of graph Laplacian and guarantees the stability of whole system provided that each agent has no dynamics.
- A set of eigenvalues is irrelevant to the choice of $\boldsymbol{\zeta}$, and hence it is valid even for the sparse case of (10) which has the same number of links with the case of $\boldsymbol{\Delta} = \boldsymbol{I}$.

Fig. 2 show that L set of circle eigenvalues of $\boldsymbol{A}_L$ should lie on C_ℓ in the complex plane, where C_ℓ denotes the circle with center at $-\sum_{t=\ell}^{L} g_t$ and radius g_ℓ. Figs. 3(a) and 3(b), respectively, show the eigenvalue distributions for HMADS with $n_1 = 7$, $n_2 = 6$, $n_3 = 5$ for the unit gains ($g_1 = g_2 = g_3 = 1$) and the non-unit gain cases.

B. Convergence properties

We here investigate the convergence properties based on the eigenvalue distribution for the rank one case derived in the previous sub-section. It is completely different from one for the full rank case, $\boldsymbol{\Delta} = \boldsymbol{I}$, investigated in [1]. The eigenvalues in the latter case are scattered to whole directions

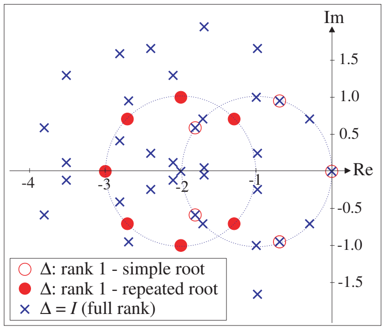


Fig. 4. Eigenvalue distribution for $n_1 = 8, n_2 = 5$

in the left-half complex plane, while the eigenvalues in the former case just shift to the left without scattering to above or below. These can be confirmed by comparing the eigenvalue distributions for the case of $n_1 = 8$ and $n_2 = 5$ illustrated in Fig. 4 .

Since all the eigenvalues except one at the origin lie in the left-half complex plane, the system is stable if each agent has no dynamics. However, the transient behaviors may depend on the locations of eigenvalues, especially nearest one to the imaginary axis. In order to investigate this fact, we introduce the following two indices for evaluating the stability degree:

- Rate of convergence: eigenvalue λ_γ with minimum absolute value of real part γ (except for zero eigenvalue)
- Rate of damping: eigenvalue λ_θ with maximum argument θ from negative direction of real-axis

For notational simplicity, we only show the results for the unit gain case.

Proposition 1. *If the incidence matrix $\boldsymbol{\Delta}$ in (9) of the L-layer system is rank one, then we have*

$$\lambda_\gamma = -1 + \omega_2 \quad \textit{for} \quad \begin{cases} n_2 \geq 4 & \\ n_2 = 3, & n_1 \leq 6 \\ n_2 = 2, & n_1 \leq 4 \end{cases} \tag{14}$$

$$\lambda_\gamma = -2 + \omega_1 \quad \textit{for} \quad \begin{cases} n_2 = 3, & n_1 \geq 6 \\ n_2 = 2, & n_1 \geq 4 \end{cases} \tag{15}$$

Proposition 2. *If the incidence matrix $\boldsymbol{\Delta}$ in (9) of the L-layer system is rank one, then we have*[1]

$$\lambda_\theta = -1 + \omega_2, \quad \theta = (\pi/2) - (\pi/n_2). \tag{16}$$

We can readily see that the values of λ_γ and θ shown above are greater than or equal to those for the case $\boldsymbol{\Delta} = \boldsymbol{I}$ derived in [1], which can also be confirmed in Fig. 4 . It should be noted that this difference affects more seriously when each agent has a certain dynamics (see Section IV).

C. Averaged model

We here investigate the (weighted) average behavior in the subgroup to make the hierarchical structure more clear.

Consider the simplest case where we have only two layers and all the gains are unit ($L = 2$ and $g_1 = g_2 = 1$) to avoid the notational complexity. Let T be defined as $T := \mathbf{1} \cdot \boldsymbol{\zeta}^T + e_1 \cdot e_1^T - \boldsymbol{I}$, which transforms the original state $\boldsymbol{x}$ to $\hat{\boldsymbol{x}} = T\boldsymbol{x} = (\boldsymbol{\zeta}^T\boldsymbol{x},\ \boldsymbol{\zeta}^T\boldsymbol{x} - x_2,\ \cdots,\ \boldsymbol{\zeta}^T\boldsymbol{x} - x_n)^T$ where the first component of $\hat{\boldsymbol{x}}$ is the weighted average and the remainders are the differences between the average and the values. We can show that T^{-1} can be written as $T^{-1} = \mathbf{1} \cdot e_1^T + e_1 \cdot \boldsymbol{\zeta}^T/(e_1^T\boldsymbol{\zeta}) - \boldsymbol{I}$ and then verify that

$$\begin{aligned}\hat{\boldsymbol{P}} := T\boldsymbol{P}T^{-1} &= \boldsymbol{P} + e_1 \cdot (e_1 - e_2)^T - e_1 \cdot \boldsymbol{\zeta}^T/(e_1^T\boldsymbol{\zeta}) \\ &\quad + \{(e_n^T\boldsymbol{\zeta})\mathbf{1} \cdot \boldsymbol{\zeta}^T - (e_1^T\boldsymbol{\zeta})\mathbf{1} \cdot \boldsymbol{\zeta}^T\boldsymbol{P}\}/(e_1^T\boldsymbol{\zeta})\end{aligned}$$

where $\hat{\boldsymbol{\Delta}} := T\boldsymbol{\Delta}T^{-1} = e_1 \cdot e_1^T$. It is clear that $\hat{\boldsymbol{\Delta}}$ is block diagonal. Also, we see that $\hat{\boldsymbol{P}}$ has the following upper triangular form: $\hat{\boldsymbol{P}} = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & * \end{array}\right)$. This upper triangular structure implies that the average behaviors of the subgroups can be governed by $\boldsymbol{A}_2 \sim \left(\begin{array}{c|c} \boldsymbol{P} & * \\ \hline 0 & * \end{array}\right)$. In other words, the collection of the weighted average of each subgroup is a nice way for aggregating the system to get a good reduced order model. Note that the eigenvalues of the matrix in the top/left corner, $\boldsymbol{P}$, correspond those lie on the most right circle or the closest one to the imaginary-axis. Hence, they are dominant to represent the slow response after quicker convergence inside subgroups. This phenomena actually can be seen in Fig. 6(a) in the next section.

IV. Stability analysis: General case

In order to guarantee that a group of dynamic agents achieves convergence to a common point through the hierarchical control scheme introduced in Sections II and III, gains g_ℓ in (6)-(7) should be set carefully. Hence, in the following sub-sections, we will present a simple diagrammatic stability analysis method which clearly shows how to determine g_ℓ in relation to agent's dynamics $H(s)$ given in (1).

A. Necessary and sufficient condition for consensus

In this subsection, we will present a necessary and sufficient condition for stability or consensus achievement for hierarchical multi-agent dynamical systems where each agent has a general dynamics rather than just a point mass. The investigation is mainly based on recent researches in [5]-[7].

The closed-loop transfer function $\mathcal{G}(s)$ of hierarchical multi-agent dynamical systems, which is derived from $H(s)$ in (1) and interconnection topology $\boldsymbol{A}_L$ in (4), is written as follows:

$$\mathcal{G}(s) = \Big((1/\hat{H}(s))I_n - \boldsymbol{A}_L\Big)^{-1}, \tag{17}$$

where $\hat{H}(s)$ is defined as $\hat{H}(s) := H(s)/s$ (see Fig. 1(b)). Then, by considering the transfer function

$$L(s) = (sI_n - \boldsymbol{A}_L)^{-1}, \tag{18}$$

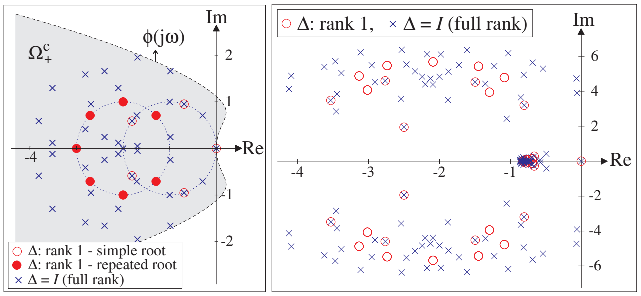
it follows from (17) that

$$\mathcal{G}(s) = L(\phi(s)), \quad \phi(s) := 1/\hat{H}(s), \tag{19}$$

where $\phi(s)$ is defined as a generalized frequency variable [5]-[7]. Next, we introduce the following notations which will be used throughout this paper:

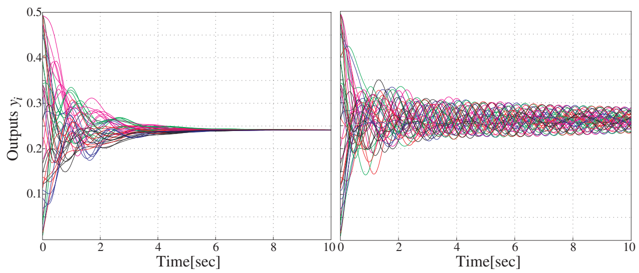
$$\Omega_+ := \phi(\mathbb{C}_+), \quad \Omega_+^c := \mathbb{C}\backslash\Omega_+, \tag{20}$$

[1] Here, we consider only the eigenvalues with positive imaginary part because of the symmetric distribution of eigenvalues about real axis.



(a) (b)

Fig. 5. For $g_1 = g_2 = 1$, (a) the domain Ω_+^c and eigenvalues of A_2, (b) the closed-loop poles of $\mathcal{G}(s)$



(a) $\boldsymbol{\Delta}$ is rank 1 (b) $\boldsymbol{\Delta} = \boldsymbol{I}$

Fig. 6. Time behaviors of all agents: $g_1 = g_2 = 1$

where $\mathbb{C}_+ = \{s \in \mathbb{C} : \mathrm{Re}[s] \geq 0\}$. Since $\Omega_+ = \{\lambda \in \mathbb{C} : \exists s \in \mathbb{C}_+ \text{ such that } \phi(s) = \lambda\}$, it follows that Ω_+^c can be alternatively expressed as $\Omega_+^c = \{\lambda \in \mathbb{C} : \forall s \in \mathbb{C}_+, \ \phi(s) \neq \lambda\}$. Note that the domain Ω_+^c includes the origin of the complex plane.

Then, the key theorem providing the global stability criterion for hierarchical multi-agent dynamical systems is as follows:

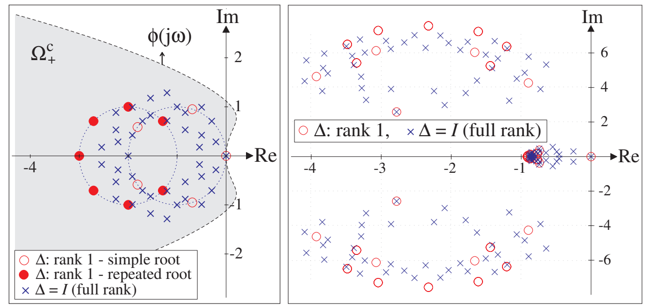
Theorem 2. *Consider the linear systems $\mathcal{G}(s)$ in (17) and $L(s)$ in (18). Also, assume that $\hat{H}(s)$ is strictly proper and stable. Then, all nonzero poles of $\mathcal{G}(s)$ depending on g_ℓ ($\ell = 1, 2, \cdots, L$) are located in the left-half complex plane, if and only if all nonzero poles of $L(s)$ (i.e., all nonzero eigenvalues of $\boldsymbol{A}_L$) belong to the domain Ω_+^c defined in (20).*

It means that the stability of hierarchical multi-agent dynamical systems $\mathcal{G}(s)$ can be judged by just looking at the locations of eigenvalues of interconnection topology $\boldsymbol{A}_L$ in relation to a domain Ω_+^c determined by a given $\hat{H}(s)$.

B. Stability test

The consensus condition in the above sub-section provides us a stability test method when the agent dynamics, $P(s)$ and $K(s)$, and the interconnection structure, L, n_i and g_i ($i = 1, 2, \cdots, L$), are specified. The procedure will be shown via a numerical example to avoid the notational complexity.

Consider a two-layer hierarchical control scheme where a group of $N_2 = 40$ agents is divided into $n_2 = 5$ subgroups, or each subgroup contains $n_1 = 8$ agents. The gains g_1 and g_2 are set as $g_1 = g_2 = 1$. Suppose that all agents have common dynamics $P(s) = \frac{\alpha}{s(s+\beta)}$ ($\alpha > 0$) and it is locally stabilized by an identical PD controller such as $K_{\mathrm{PD}}(s) = k_p(1 + t_d s)$ ($k_p > 0$, $t_d > 0$). Then, without loss of generality, the closed-loop transfer function of each agent can be written as $H(s) = \frac{s+1}{as^2+bs+1}$, $a := \frac{1}{\alpha k_p t_d^2}$, and $b := \frac{\beta}{\alpha k_p t_d} + 1$. Here, we set $\alpha = 1$, $\beta = -1$, $k_p = 0.4$ and $t_d = 5$. Thus, $a = 0.1$ and $b = 0.5$. In this case, the domain Ω_+^c is completely determined by the frequency property $\phi(j\omega)$ of $\phi(s) = \frac{s}{H(s)} = \frac{0.1s^3+0.5s^2+s}{s+1}$ as illustrated in Fig. 5(a). It also plots all the eigenvalues of $\boldsymbol{A}_2$ with $\boldsymbol{\Delta} = \boldsymbol{I} \in \mathbb{R}^{8\times 8}$ (full rank) and $\boldsymbol{\Delta} = \mathbf{1}\cdot\boldsymbol{\zeta}^T$ (rank 1) where $\boldsymbol{\zeta}^T = (1, 0, \cdots, 0) \in \mathbb{R}^8$. Note that all eigenvalues of both cases are belong to the domain Ω_+^c, and hence a group of dynamic agents can achieve convergence to a common point as shown in Theorem 2. We can, however, confirm in Figs. 6(a) and 6(b) that agents' behaviors when $\boldsymbol{\Delta}$ is rank 1 are more desirable than those when $\boldsymbol{\Delta} = \boldsymbol{I}$ from the viewpoint of convergence and damping rates. All the poles of closed-loop transfer function $\mathcal{G}(s)$ are depicted in Fig. 5(b), which verifies why the case $\boldsymbol{\Delta} = \boldsymbol{I}$ has low rates of convergence and damping.



(a) Eigenvalues of A_2 (b) The closed-loop poles of $\mathcal{G}(s)$

Fig. 7. Eigenvalues of A_2 and the closed-loop poles of $\mathcal{G}(s)$: (i) $\boldsymbol{\Delta} = \mathbf{1}\cdot\boldsymbol{\zeta}^T$ case, $g_1 = g_2 = 1$, (ii) $\boldsymbol{\Delta} = \boldsymbol{I}$ case, $g_1 = 1$ and $g_2 = 0.3$

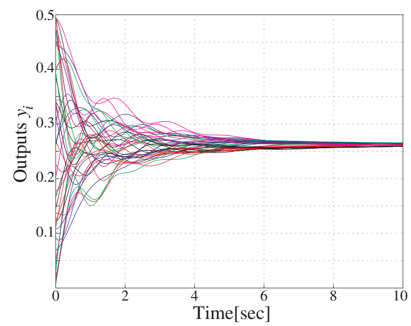
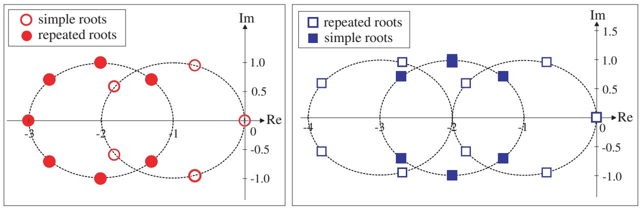


Fig. 8. Time behaviors of all agents when $\boldsymbol{\Delta} = \boldsymbol{I}$, and $g_1 = 1$, $g_2 = 0.3$

On the other hand, if we set $g_2 = 0.3$ for $\boldsymbol{\Delta} = \boldsymbol{I}$ case to improve the oscillating behaviors of agents, the eigenvalues of $\boldsymbol{A}_2$ and the closed-loop poles of $\mathcal{G}(s)$ are, respectively, illustrated in Figs. 7(a) and 7(b). In this case, the damping rate is clearly improved as shown in Fig. 8. However, its convergence rate is inferior to that achievable by $\boldsymbol{\Delta} = \mathbf{1}\cdot\boldsymbol{\zeta}^T$, which is easily confirmed from Fig. 8. The above facts verify the superiority of the proposed hierarchical control scheme where $\boldsymbol{\Delta}$ is rank one.



(a) Rank 1 case (b) Rank 2 case

Fig. 9. Eigenvalue distributions: $n_1 = 8, n_2 = 5$

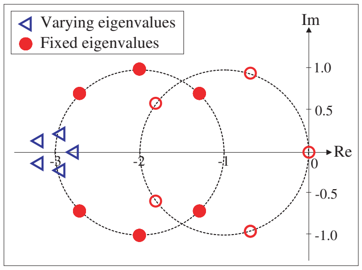


Fig. 10. Eigenvalue distribution for rank 2 general case: $n_1 = 8, n_2 = 5$

C. Further investigations: Rank two case

We here briefly sketch the main investigation results for the eignevalue distribution of a class of two-layer hierarchical multi-agent dynamical systems with rank m connection, which includes two special cases, $\Delta = \boldsymbol{I}$ and the rank 1 case treated in the previous sections. Note that the details can be found in Shimizu and Hara [12].

The form of Δ is expressed as follows: $\Delta = (\mathbf{1} \otimes \boldsymbol{I}_{n_1/m}) \cdot \boldsymbol{\Gamma} \in \boldsymbol{R}^{n_1 \times n_1}$ where n_1 is assumed to be an integer which is a multiple of m, and any row vector of $\boldsymbol{\Gamma} \in \boldsymbol{R}^{m \times n_1}$ is a probability vector; i.e., all the entries are non-negative and the sum of them is equal to 1. For example, the form of Δ for $m = 2$, or rank 2 case, is given as follows: for $a_i \geq 0$ and $b_i \geq 0$,

$$\Delta = \begin{pmatrix} \boldsymbol{I}_2 \\ \vdots \\ \boldsymbol{I}_2 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{a}^T \\ \boldsymbol{b}^T \end{pmatrix}, \quad \sum_{i=1}^{n_1} a_i = \sum_{i=1}^{n_1} b_i = 1. \qquad (21)$$

It is clear that the form is completely the same with one treated in the previous sections when $m = 1$, and that $m = n_1$ with $\boldsymbol{\Gamma} = \boldsymbol{I}$ provides the case of $\Delta = \boldsymbol{I}$.

Theorem 3. *Consider a two-layer HMADSs with unit gain, i.e., $g_1 = g_2 = 1$. For Δ with rank 2 represented by (21), the eigenvalue distribution of $\boldsymbol{A}_2$ is given as follows:*

$$\lambda = \begin{cases} -2 + \omega_1^m, \ m = 1, 2, \cdots, \frac{n_1 - 2}{2}, \frac{n_1 + 2}{2}, \cdots, n_1 - 1 \\ -1 + \omega_2^k, \ k = 1, \cdots, n_2 \\ -3 + \omega_2^k, \ \frac{1}{2} \sum_{i=1}^{n_1} (a_i - b_i)(-1)^{i+1}, \ k = 1, \cdots, n_2 \end{cases}$$

We have a couple of remarks on the result for the above rank 2 case:

- The eigenvalue distribution is slightly different from that for the rank 1 case. The difference only appears at the most left repeated eigenvalue at -3 as in Figs. 9(a)-9(b).
- The eigenvalue distribution depends on the choice of $\boldsymbol{\Gamma}$, or two probability vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. These determine the radius of the circle centered at -3 on which the eigenvalues spread. The largest radius is 1 as illustrated in Fig. 9(b), and the eigenvalue distribution in general is as seen in Fig. 10.

V. Conclusion

In this paper, we have proposed a fairly general model for hierarchical multi-agent dynamical systems (HMADSs) with fractal structure and investigated their stability or convergence condition. We have especially focused on the low-rank property instead of sparseness or small gain property, which clearly captures the aggregation process for large scale systems. We have derived the analytical expression of eigenvalue distribution of the system matrix for cyclic pursuit and shown a stability condition for the whole system where each agent has a certain dynamics. We then have clarified the relation between the property of cross-layer interconnection structure and the stability degree of hierarchical multi-agent system, which are confirmed by numerical examples. The results shows that a certain aggregation process in information acquisition plays a key role for the consensus of large scale multi-agent systems. This point is very essential in hierarchical dynamical systems, and our proposed model focusing on the low-rank property is a right way for the further investigations.

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