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Stochastic Extremum Seeking With Applications to Mobile Sensor Networks

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Abstract— **In this paper the extremum seeking algorithm with sinusoidal perturbation has been modified and extended in two ways: a) the amplitudes of the perturbation signals, as well as the gain of the integrator block, are time varying and tend to zero at a pre-specified rate; b) the output of the system is corrupted with measurement noise. Local convergence to the extremal point, with probability one and in the mean square sense, has been proved. Also, it has been shown how the proposed algorithm can be applied to mobile sensor networks as a tool for achieving the optimal observation positions. The proposed algorithms have been illustrated through several simulations.**

I. Introduction

The extremum seeking methodology application to diverse problems in control and communications has a long history. It represents a nonmodel based method for adaptive control which deals with systems where the reference-to-output map is uncertain but is known to have an extremum. In 1950s and 1960s this approach was popular as "extremum control" or "self-organizing control" (see *e.g.* [1], [2], [3]). Following the emergence of the main theoretical breakthroughs in the field of adaptive control, Åström and Wittenmark stated in [4] that extremum control belongs to the most promising future areas for adaptive control. A significant contribution to this field has been made in the last years by Krstić and his co-workers, who succeeded both to clarify the main conceptual aspects of this methodology and to present interesting and useful applications (see [5], [6], [7], [1], [8], [9], [10], [11]). They presented stability analysis for extremum seeking systems in both continuous and discrete-time case using averaging and singular perturbations providing sufficient conditions for the plant output to converge to a neighborhood of the extremum value.

However, all these approaches have been related to the case of deterministic systems and perturbing sinusoidal signals with constant amplitudes. This paper represents an attempt to clarify properties of the main discrete-time extremum seeking scheme presented in [6] in which the following new assumptions are introduced: a) the amplitudes of the sinusoidal perturbation signals, as well as the gain of the integrator block, are time varying and tend to zero at a pre-specified rate; b) the output of the system is corrupted with measurement noise. In general, the first assumption opens up a possibility to obtain convergence of the whole scheme to a unique extremum point and not to its neighborhood which depends on the perturbation amplitude even in the deterministic context. The second assumption, *i.e.* the inclusion of the additive stochastic component in the extremum seeking loop, allows important generalizations and applications of the extremum seeking methodology to a large number of real adaptation problems in control and signal processing. The main contribution of the paper is a theoretical analysis of the extremum seeking scheme which starts from the above assumptions and formulates conditions for the local convergence to the extremum point in the mean-square sense and with probability one. In order to illustrate new application areas for the presented theoretical results, it is also shown how the extremum seeking scheme can be applied to an adaptive state estimation problem which minimizes the observation noise influence and, thus, can be used for the optimal positioning of mobile sensors.

Section II contains the problem definition. Section III is devoted to the main theoretical results, formulated as Theorem 1, whose proof consists of two main parts. In the first part it is proved that the extremum seeking scheme under time varying perturbations and time varying integrator gain is locally stable at the extremum point and that asymptotically the system model can be linearized and represented by a time varying gain and a constant coefficient. This coefficient is found to be equal to the linearization coefficient obtained in the time invariant case by the averaging methodology [6], [12]. In the second part the stochastic aspects are treated using the methodology of stochastic approximation with colored noise, *e.g.* [13], [14], [15], [16], [17]. It is proved that the system converges under the specified conditions to the extremum point in the mean square sense and with probability one. In Section IV a new application of the extremum seeking scheme is presented. It is shown that it can be used for mobile sensor networks as a tool for achieving the best observation points for distributed state and parameter estimation schemes based on Kalman filtering. The proposed algorithms have been illustrated through several simulations.

II. Discrete-Time Extremum Seeking Algorithm with Time-Varying Gains

We will consider a discrete-time extremum seeking algorithm with sinusoidal perturbation, as shown in Figure 1. The basic idea is as follows. Since we cannot measure the gradient of the function whose extremum we are seeking, a slow sinusoidal perturbation (compared to the dynamics of the stable systems $F_i(z)$ and $F_o(z)$), with frequency $\omega = a\pi$, $0 < a < 1$, a is a rational number, is added to the system input in order to observe its effects to the output $y(k)$.

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Without loss of generality, we will assume that $f(\theta)$ has a minimum at $\theta = \theta^*$ and that it has the quadratic form:

$$f(\theta) = f^* + (\theta - \theta^*)^2 \quad (1)$$

where f^* is a constant. Possible higher order terms can be neglected in the local convergence analysis; hence we are omitting them here. The high pass filter $\frac{z-1}{z+h}$, $0 < h < 1$ filters out a DC component of the measurement $y(k)$ which is corrupted by noise $\zeta(k)$. Then, the resulting signal is being demodulated (by the multiplication with the same frequency sinusoid, followed by integration). Hence, the input to the integrator $-\frac{1}{z-1}$ is proportional to the slope of the function $f(\theta)$ and it will drive θ to the extremal value (for which the slope of the function $f(\theta)$ is zero).

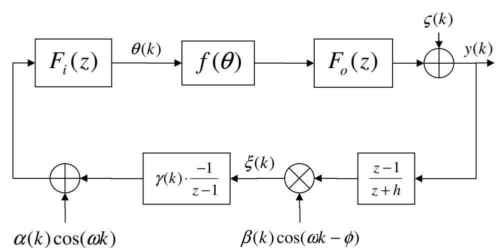

Fig. 1. Discrete-time extremum seeking scheme

In the next section we will prove convergence of $\theta(k)$ to the extremal point θ^* (with probability one and in the mean square sense) in the presence of the measurement noise $\zeta(k)$. What makes this possible is, similarly like in stochastic approximation algorithms (*e.g.* [18], [19], [20]), the introduction of the time varying, vanishing gains $\beta(k)$, $\gamma(k)$ and $\alpha(k)$ which make the system capable of eliminating noise. Note that in the case of similar algorithm whose local stability has been analyzed in ([6], [7]) noisy measurements and time-varying gains have not been assumed; hence $\theta(k)$ in their case was proved to converge only to some $O(\alpha)$ neighborhood of the extremal point. Also, because of the time varying gains, the averaging theory can not be applied directly, like in ([6], [7]), which makes the analysis of the above system much more complicated. For the clarity of presentation, we will assume that dynamics of the systems $F_i(z)$ and $F_o(z)$ are fast enough so that they can be neglected in the convergence analysis.

The following equations model the behavior of the above described system:

$$y(k) = f^* + (\theta(k) - \theta^*)^2 + \zeta(k) \quad (2)$$

$$\theta(k) = \alpha(k)\cos(\omega k) - \gamma(k)\tfrac{1}{z-1}[\xi(k)] \quad (3)$$

$$\xi(k) = \beta(k)\cos(\omega k - \phi)\tfrac{z-1}{z+h}[y(k)] \quad (4)$$

where $\zeta(k)$ is the measurement noise, and the expression $H(z)[x(k)]$ denotes a time domain signal obtained as the output of the transfer function $H(z)$ when the input is $x(k)$.

Similarly like in [6] we define the tracking error as:

$$\tilde{\theta}(k) = \theta^* - \theta(k) + \alpha(k)\cos(\omega k) \quad (5)$$

and obtain the following equation (for details of the derivation see [6] Sections 3 and 4):

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon(k)\{L(z)[\alpha(k)\tilde{\theta}(k)] + \\ +\Phi_1(k) + \Phi_2(k) + \Phi_3(k) + u(k)\} \quad (6)$$

where $\epsilon(k) = \gamma(k)\beta(k)$ and

$$L(z) = -\tfrac{1}{2}[e^{j\phi}M(z, e^{j\omega}) + e^{-j\phi}M(z, e^{-j\omega})], \quad (7)$$

$$\Phi_1(k) = s(2\omega k)\mathrm{Im}\{M(z, e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\}, \quad (8)$$

$$\Phi_2(k) = -c(2\omega k)\mathrm{Re}\{M(z, e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\}, \quad (9)$$

$$\Phi_3(k) = c(\omega k)\tfrac{z-1}{z+h}[\tilde{\theta}(k)^2], \quad (10)$$

$$u(k) = d(k) + c(\omega k)\tfrac{z-1}{z+h}[\zeta(k)] \quad (11)$$

$$d(k) = c(\omega k)\tfrac{z-1}{z+h}[f^* + \alpha(k)^2\cos^2(\omega k)] + \varepsilon^{-k}. \quad (12)$$

In addition, $s(2\omega k) = \sin(2\omega k - \phi)$, $c(2\omega k) = \cos(2\omega k - \phi)$, $c(\omega k) = \cos(\omega k - \phi)$, ε^{-k} denotes exponentially decaying terms, and $M(z, e^{j\omega}) = (e^{j\omega}z - 1)/(e^{j\omega}z + h)$. Hence, all the terms in equation (6) are time-varying; the first four terms depend on $\tilde{\theta}$ ($\Phi_3(k)$ is nonlinear), while the input term $u(k)$ is composed of the deterministic part $d(k)$ and the stochastic part $n(k) = c(\omega k)\frac{z-1}{z+h}[\zeta(k)]$.

III. Convergence Analysis

In the convergence analysis we will assume that the following basic assumptions are satisfied:

(A.1) The sequence $\{\zeta(k)\}$ is a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with a specified sequence of σ-algebras $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$, such that the variables $\zeta(k)$ are measurable with respect to $\mathcal{F}_k$ and they satisfy

$$E\{\zeta(k)^2\} = \sigma(k)^2 < \infty, \ k = 1, 2, ... \quad (13)$$

(A.2) The sequence $\epsilon(k)$ is decreasing, $\epsilon(k) > 0$, $k = 1, 2, ...$, $\lim_{k\to\infty}\epsilon(k) = 0$

(A.3) The sequence $\alpha(k)$ is decreasing, $\alpha(k) > 0$, $k = 1, 2, ...$, $\lim_{k\to\infty}\alpha(k) = 0$

(A.4) $\sum_{k=1}^{\infty}\epsilon(k)\alpha(k) = \infty$

(A.5) $\sum_{k=1}^{\infty}\epsilon(k)^2 < \infty$

(A.6) $\sum_{k=2}^{\infty}\epsilon(k)[\alpha(k-1) - \alpha(k)] < \infty$

(A.7) $\sum_{k=2}^{\infty}[\epsilon(k-1) - \epsilon(k)] < \infty$

(A.8) $\sum_{k=1}^{\infty}\epsilon(k)\alpha(k)^2 < \infty$

The following theorem deals with the asymptotic behavior of the above algorithm.

Theorem 1. Let the assumptions (A.1-8) be satisfied. Then $\theta(k)$ converges to θ^* almost surely (a.s) and in the mean square sense under the condition that $\sup_k(|\tilde{\theta}(k)|) < K$ (a.s.), $0 < K < \infty$.

Proof. We shall analyze the right hand side of (6) term by term.

We start by writing

$$\epsilon(k)L(z)[\alpha(k)\tilde{\theta}(k)] = \rho(k)L(z)[\tilde{\theta}(k)] + \epsilon(k)\delta l(k), \quad (14)$$

where $\delta l(k) = L(z)[\alpha(k)\tilde{\theta}(k)] - \alpha(k)L(z)[\tilde{\theta}(k)]$ and $\rho(k) = \epsilon(k)\alpha(k)$. If $l(k), k = 0, 1, ...$ is the impulse response of the system **S** with transfer function $L(z)$, we have

$$\delta l(k) = l(0)[\alpha(k) - \alpha(k)]\tilde{\theta}(k) + l(1)[\alpha(k-1) - \\ -\alpha(k)]\tilde{\theta}(k-1) + \cdots + l(k-1)[\alpha(1) - \alpha(k)]\tilde{\theta}(1) \quad (15)$$

so that

$$\delta l(k) = [\alpha(k-1) - \alpha(k)]y_1(k) \quad (16)$$

where $y_1(k)$ can be considered as the output of a time varying system $\mathbf{S}_1$ with the impulse response $h_1(k,j) = l(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$ and input $\tilde{\theta}(k)$, i.e. $y_1(k) = \sum_{j=0}^{k-1} h_1(k,j)\tilde{\theta}(k-j)$. System $\mathbf{S}_1$ is b.i.b.o. stable, having in mind that $h_1(k,j)$ is absolutely summable under the adopted assumptions ($\mathbf{S}$ is asymptotically stable and $\alpha(k)$ satisfies (A.3)).

Now, we will use the result of [6] and apply the averaging operator

$$\mathrm{AVG}\{L(z)[\tilde{\theta}]\} = \lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} L(z)[\tilde{\theta}] \quad (17)$$

so that we obtain

$$\mathrm{AVG}\{L(z)[\tilde{\theta}]\} = -\kappa\tilde{\theta} \quad (18)$$

where $\kappa = \mathrm{Re}\{e^{j\phi}\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$. Notice that we also have $\kappa = -\sum_{j=0}^{\infty} l(j)$, according to the above notation.

Writing now $L(z)[\tilde{\theta}(k)] = -\kappa\tilde{\theta}(k) + \delta l_\kappa(k)$, we have

$$\delta l_\kappa(k) = L(z)[\tilde{\theta}(k)] + \kappa\tilde{\theta}(k) = \quad (19)$$
$$= \sum_{j=0}^{k-1} l(j)[\tilde{\theta}(k-j) - \tilde{\theta}(k)] + [\sum_{i=0}^{k-1} l(i) + \kappa]\tilde{\theta}(k)$$

where the last term is equal to $\lambda(k)\tilde{\theta}(k)$, with $\lambda(k) = \sum_{i=k}^{\infty} l(i)$. After iterating (6) back to the initial condition and using (14), we obtain

$$\begin{aligned}\delta l_\kappa(k) = &-l(1)\{\epsilon(k-1)[\alpha(k-1)L(z)[\tilde{\theta}(k-1)]+ \\ &+\delta l(k-1) + \Phi(k-1) + u(k-1)]\} - l(2)\{\epsilon(k-2)\cdot \\ &[\alpha(k-2)L(z)[\tilde{\theta}(k-2)] + \delta l(k-2) + \Phi(k-2)+ \\ &u(k-2)] + \epsilon(k-1)[\alpha(k-1)L(z)[\tilde{\theta}(k-1)]+ \\ &\delta l(k-1) + \Phi(k-1) + u(k-1)]\} + ... + \lambda(k)\tilde{\theta}(k)\end{aligned} \quad (20)$$

where $\Phi(k) = \Phi_1(k) + \Phi_2(k) + \Phi_3(k)$. After regrouping the terms in (20), we obtain

$$\begin{aligned}\delta l_\kappa(k) = &\sum_{j=1}^{k-1}[-\sum_{i=j}^{k-1} l(i)]\epsilon(k-j)\cdot \\ &\cdot\{\alpha(k-j)L(z)[\tilde{\theta}(k-j)] + \delta l(k-j) + \Phi(k-j)+ \\ &u(k-j)\} + \lambda(k)\tilde{\theta}(k)\end{aligned} \quad (21)$$

Defining two time-varying systems $\mathbf{S}_2$ and $\mathbf{S}_3$ with impulse responses $h_2(k,j) = \bar{l}(k,j)\frac{\rho(k-j)}{\rho(k-1)}$ and $h_3(k,j) = \bar{l}(k,j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, respectively, where $\bar{l}(k,j) = -\sum_{i=j}^{k-1} l(i)$, we can write

$$\delta l_\kappa(k) = \rho(k-1)y_2(k) + \epsilon(k-1)y_3(k) + \lambda(k)\tilde{\theta}(k) \quad (22)$$

where $y_2(k) = \sum_{j=0}^{k-1} h_2(k,j)L(z)[\tilde{\theta}(k-j)]$ and $y_3(k) = \sum_{j=0}^{k-1} h_3(k,j)[\delta l(k-j)+\Phi(k-j)+u(k-j)]$ are the outputs of $\mathbf{S}_2$ and $\mathbf{S}_3$, respectively. One can easily verify that both $\mathbf{S}_2$ and $\mathbf{S}_3$ are b.i.b.o. stable under the adopted assumptions, while $\lambda(k)$ is exponentially decaying.

We focus now on the terms $\Phi_i(k)$ from (6), $i = 1, 2, 3$.

Considering first $\Phi_1(k)$ defined by (8), we form, similarly as above, the difference

$$\begin{aligned}\delta l_1(k) = &\alpha(k)s(2\omega k)\mathrm{Im}\{M(z,e^{j\omega})[\tilde{\theta}(k)]\}- \\ &-s(2\omega k)\mathrm{Im}\{M(z,e^{j\omega})[\alpha(k)\tilde{\theta}(k)]\}\end{aligned} \quad (23)$$

and obtain that

$$\delta l_1(k) = \alpha(k)s(2\omega k)[\alpha(k-1) - \alpha(k)]y_4(k) \quad (24)$$

where $y_4(k)$ is the output of the b.i.b.o. stable system $\mathbf{S}_4$ with the input $\tilde{\theta}(k)$ and with the impulse response sequence $h_4(k,j) = m_1(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$, where $\{m_1(j)\}$ is the impulse response of $\mathrm{Im}\{M(z,e^{j\omega})\}$ which is stable.

Further, we write $\mathrm{Im}\{M(z,e^{j\omega})[\tilde{\theta}(k)]\} = \kappa_1\tilde{\theta}(k) + \delta_\kappa^1(k)$, where $\kappa_1 = \mathrm{Im}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$, and, following the methodology of deriving (20) and (21), we obtain

$$\begin{aligned}\delta_\kappa^1(k) = &\sum_{j=1}^{k-1}[-\sum_{i=j}^{k-1} m_1(i)]\epsilon(k-j)\cdot \\ &\cdot\{\alpha(k-j)L(z)[\tilde{\theta}(k-j)] + \delta l(k-j) + \Phi(k-j) \\ &+u(k-j)\} + \mu_1(k)\tilde{\theta}(k)\end{aligned} \quad (25)$$

where $\mu_1(k) = \sum_{i=k}^{\infty} m_1(i)$. Following further an analogous reasoning as above, we obtain

$$\delta_\kappa^1(k) = \rho(k-1)y_5(k) + \epsilon(k-1)y_6(k) + \mu_1(k)\tilde{\theta}(k) \quad (26)$$

where $y_5(k)$ and $y_6(k)$ are the outputs of b.i.b.o. stable systems $\mathbf{S}_5$ and $\mathbf{S}_6$ with impulse responses $h_5(k,j) = \bar{m}_1(k,j)\frac{\rho(k-j)}{\rho(k-1)}$ and $h_6(k,j) = \bar{m}_1(k,j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, where $\bar{m}_1(k,j) = -\sum_{i=j}^{k-1} m_1(i)$, and with the inputs $L(z)[\tilde{\theta}(k-j)]$ and $\delta l(k-j) + \Phi(k-j) + u(k-j)$, respectively. Consequently, we have

$$\Phi_1(k) = \alpha(k)s(2\omega k)[\kappa_1\tilde{\theta}(k) + \delta_\kappa^1(k)] + \delta l_1(k) \quad (27)$$

Using completely analogous arguments we obtain that

$$\Phi_2(k) = -\{\alpha(k)c(2\omega k)[\kappa_2\tilde{\theta}(k) + \delta_\kappa^2(k)] + \delta l_2(k)\} \quad (28)$$

where $\kappa_2 = \mathrm{Re}\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$, while

$$\delta l_2(k) = \alpha(k)c(2\omega k)[\alpha(k-1) - \alpha(k)]y_7(k) \quad (29)$$

and

$$\delta_\kappa^2(k) = \rho(k-1)y_8(k) + \epsilon(k-1)y_9(k) + \mu_2(k)\tilde{\theta}(k) \quad (30)$$

where $y_7(k)$ is the output of the b.i.b.o. stable system $\mathbf{S}_7$ with the input $\tilde{\theta}(k)$ and with the impulse response sequence $h_7(k,j) = m_2(j)\frac{\alpha(k-j)-\alpha(k)}{\alpha(k-1)-\alpha(k)}$. $\{m_2(j)\}$ is the impulse response of $\mathrm{Re}\{M(z,e^{j\omega})\}$ which is stable, $\mu_2(k) = \sum_{i=k}^{\infty} m_2(i)$, while $y_8(k)$ and $y_9(k)$ are the outputs of b.i.b.o. stable systems $\mathbf{S}_8$ and $\mathbf{S}_9$ with impulse responses $h_8(k,j) = \bar{m}_2(k,j)\frac{\rho(k-j)}{\rho(k-1)}$ and $h_9(k,j) = \bar{m}_2(k,j)\frac{\epsilon(k-j)}{\epsilon(k-1)}$, where $\bar{m}_2(k,j) = -\sum_{i=j}^{k-1} m_2(i)$, and with the inputs $L(z)[\tilde{\theta}(k-j)]$ and $\delta l(k-j)+\Phi(k-j)+u(k-j)$, respectively.

Therefore, after replacing the obtained expressions for $L(z)[\alpha(k)\tilde{\theta}(k)] + \Phi_1(k) + \Phi_2(k) + \Phi_3(k)$ in (6), we obtain

$$\tilde{\theta}(k+1) = [1 - \kappa\rho(k) + \eta(k)]\tilde{\theta}(k) + \pi(k) + \epsilon(k)u(k) \quad (31)$$

where $\eta(k) = [\kappa_1 s(2\omega k) - \kappa_2 c(2\omega k)]\rho(k)$ and

$$\begin{aligned}\pi(k) = &\epsilon(k)\delta l(k) + \rho(k)\delta l_\kappa(k) + \epsilon(k)c(2\omega k)\delta_\kappa^1(k) + \\ &\epsilon(k)\delta l_1(k) + \epsilon(k)s(2\omega k)\delta_\kappa^2(k) + \epsilon(k)\delta l_2(k) + \epsilon(k)\Phi_3(k)\end{aligned} \quad (32)$$

Considering first the term $\eta(k)$, we can easily derive that $\eta(k) = \rho(k)\sin(2\omega k+\psi)$, where ψ depends on ϕ and $\phi_M =$

$Arg\{\frac{e^{j\omega}-1}{e^{j\omega}+h}\}$. If N is the integer period of $\sigma(k)$, we have further that

$$|\sum_{k=1}^{\infty}\eta(k)| = \sum_{j=1}^{\lfloor \frac{N}{2}\rfloor} b_j \sum_{k=0}^{\infty}[\rho(j+kN)-\rho(j+\lfloor\frac{N}{2}\rfloor+kN)] < \infty \tag{33}$$

where $b_j \geq 0$, $j = 1, \ldots, \lfloor\frac{N}{2}\rfloor$, having in mind that ω is rational and that (A.7) holds. Therefore, having in mind that $\sum_{k=1}^{\infty}\rho(k) = \infty$, from (31) we obtain for k large enough that

$$\tilde{\theta}(k+1) = \prod_{j=1}^{k}(1-\kappa'\rho(j))\tilde{\theta}(1)+ \\ + \sum_{j=1}^{k}[\pi(j)+\epsilon(j)u(j)]\prod_{i=j+1}^{k}(1-\kappa'\rho(i)) \tag{34}$$

where $0 < \kappa' < \kappa$. Now, using the inequality $1 - x \leq e^{-x}$ it is easy to see that $\prod_{j=1}^{k}(1-\kappa'\rho(j)) \to 0$ when $k \to \infty$, having in mind the condition A.4. Furthermore, after applying the Kronecker's lemma to the second term at the right hand side of (34), we conclude that $\tilde{\theta}(k)$ converges to zero almost surely if $\sum_{j=1}^{\infty}[\pi(j)+\epsilon(j)u(j)]$ converges (a.s.).

In order to show that the last condition holds, we shall decompose $\pi(j)$ as $\pi(j) = \sum_{i=1}^{3}\pi_i(j)$, where $\pi_2(j)$ and $\pi_3(j)$ contain only those components of $y_3(j)$, $y_6(j)$ and $y_9(j)$ (outputs of b.i.b.o. stable linear systems $\mathbf{S}_3$, $\mathbf{S}_6$ and $\mathbf{S}_9$) that are responses to the inputs $\epsilon(j)d(j)$ and $\epsilon(j)n(j)$, respectively; $\pi_1(j)$ conatins all the remaining terms of $\pi(j)$.

According to the assumptions (A.2) - (A.7), boundedness of $\tilde{\theta}(k)$ guarantees the property that $\sum_{k=1}^{\infty}\pi_1(k)$ converges. This statement is evident for all the terms in $\pi_1(k)$ except the last one, where the conclusion that $\sum_{k=1}^{\infty}\epsilon(k)\Phi_3(k)$ converges follows from similar arguments as those used in deriving (33).

The analysis goes now to the terms in (31), depending on $d(k)$. Using the identity $\cos^2(\omega k) = \frac{1}{2}(1+\cos(2\omega k))$, we obtain that $d(k) = d_1(k) + d_2(k) + \varepsilon^{-k}$, where

$$d_1(k) = c(\omega k)\tfrac{z-1}{z+h}[f^* + \tfrac{1}{2}\alpha(k)^2] \tag{35}$$

$$d_2(k) = \tfrac{1}{2}c(\omega k)\tfrac{z-1}{z+h}[\alpha^2(k)\cos(2\omega k)] \tag{36}$$

Considering the term $d_1(k)$ we first conclude that $\frac{z-1}{z+h}[f^*] = 0$ (high pass filter). Furthermore,

$$\tfrac{z-1}{z+h}[\alpha(k)^2] = \alpha(k)^2\sum_{j=0}^{k-1}l^*(j)\tfrac{\alpha(k-j)^2}{\alpha(k)^2} \tag{37}$$

where sequence $\{l^*(j)\}$ is the impulse response of the system $\frac{z-1}{z+h}$. Reasoning as above, we conclude that $|d_1(k)| \leq k_2\alpha(k)^2$, where $k_2 > 0$ is a constant. Similarly, for $d_2(k)$ we have

$$d_2(k) = \tfrac{1}{2}c(\omega k)\alpha^2(k)\cdot \\ \cdot\sum_{j=0}^{k-1}l^*(j)\cos(2\omega(k-j))\tfrac{\alpha^2(k-j)}{\alpha(k)^2} \tag{38}$$

which leads, as above, to the conclusion that $|d_2(k)| \leq k_3\alpha(k)^2$, where $k_3 > 0$ is a constant. Therefore, we have

$$|d(k)| \leq k_4\alpha(k)^2 \tag{39}$$

for some constant $k_4 > 0$.

Consequently, it follows clearly that, under the adopted assumption (A.8), $\sum_{j=1}^{\infty}[\epsilon(j)d(j)+\pi_2(j)]$ converges.

The last part of (31) to be analyzed is the stochastic component, obtained as a consequence of $\epsilon(k)n(k)$. We shall first demonstrate that

$$\textstyle\sum_{k=1}^{\infty}\epsilon(k)n(k) \text{ converges a.s.} \tag{40}$$

We shall use the results from [15] (Theorem 1) which state that the sufficient conditions for (40) to be satisfied, both with probability 1 and in the mean square sense, are

$$\text{(a)}\ \ r(k) = \textstyle\sum_{j=k+1}^{\infty}\epsilon(j)\Psi_{j,k} \to 0,\ k \to \infty \tag{41}$$

$$\text{(b)}\ \ \textstyle\sum_{k=1}^{\infty}\epsilon(k)^2\sigma(k)^2 < \infty \tag{42}$$

$$\text{(c)}\ \ \textstyle\sum_{k=1}^{\infty}\epsilon(k)\sigma(k)r(k) < \infty \tag{43}$$

where $\Psi_{k,m} = \|E\{n(k)|\mathcal{F}_m\}\|_2$ with $k > m$, $\|\cdot\|_2 = (E\{|\cdot|^2\})^{\frac{1}{2}}$, $\mathcal{F}_k$ is a sequence of σ-algebras such that the variables $n(k)$ are measurable with respect to $\mathcal{F}_k$. These conditions specify a class of noise with a sufficiently slowly increasing second moment and a sufficiently fast decreasing correlation.

For condition (a) we have

$$\begin{aligned} E\{n(j)|\mathcal{F}_k\} &= E\{\textstyle\sum_{i=1}^{j}l^*(j-i)\zeta(i)|\mathcal{F}_k\}c(\omega j) \\ &= \textstyle\sum_{i=1}^{j}l^*(j-i)E\{\zeta(i)|\mathcal{F}_k\}c(\omega j) \\ &= c(\omega j)[\textstyle\sum_{i=1}^{k}l^*(j-i)\zeta(i) \\ &\quad + \textstyle\sum_{s=k+1}^{j}l^*(j-i)E\{\zeta(s)|\mathcal{F}_k\}] \\ &= c(\omega j)\textstyle\sum_{i=1}^{k}l^*(j-i)\zeta(i) \end{aligned} \tag{44}$$

where we used the fact that $E\{\zeta(s)|\mathcal{F}_k\} = 0$ for $s > k$ and $E\{\zeta(s)|\mathcal{F}_k\} = \zeta(s)$ for $s \leq k$ (since $\zeta(i)$ is a martingale difference sequence), $\{l^*(i)\}$ is the impulse response sequence of $\frac{z-1}{z+h}$. Furthermore, from (41) and (44), we have

$$\begin{aligned} r(k) &= \textstyle\sum_{j=k+1}^{\infty}\epsilon(j)|c(\omega j)|\cdot \\ &\cdot E\{(\textstyle\sum_{i=1}^{k}l^*(j-i)\zeta(i))^2\}^{\frac{1}{2}} \leq \\ &\leq K'\textstyle\sum_{j=k+1}^{\infty}\epsilon(j)\sum_{i=1}^{k}l^*(j-i)^2 \end{aligned} \tag{45}$$

for some positive constant K', where we used the fact that $E\{\zeta(i)\zeta(j)\} = 0$ for $i \neq j$ and $E\{\zeta(i)\zeta(j)\} = \sigma(i)^2$ for $i = j$. The last term in (45) goes to zero when $k \to \infty$ having in mind that $\epsilon(k) \to 0$ and $l^*(k) \to 0$ when $k \to \infty$. Therefore, the condition (41) is satisfied. Condition (42) follows directly from the assumptions (A.1) and (A.5). To prove condition (43) we have

$$\begin{aligned} &\textstyle\sum_{k=1}^{\infty}\epsilon(k)\sigma(k)r(k) \leq \\ &\leq K'\textstyle\sum_{k=1}^{\infty}\epsilon(k)\sigma(k)\sum_{j=k+1}^{\infty}\epsilon(j)\sum_{i=1}^{k}l^*(j-i)^2 = \\ &= \textstyle\sum_{k=1}^{\infty}\epsilon(k)^2\sigma(k)\sum_{j=k+1}^{\infty}\frac{\epsilon(j)}{\epsilon(k)}\sum_{i=1}^{k}l^*(j-i)^2 \end{aligned} \tag{46}$$

The last term converges having in mind conditions (A.2) and (A.5). Therefore, the property (40) holds.

Using the above arguments, if follows directly that $\sum_{j=1}^{\infty}\pi_3(j)$ converges almost surely.

Therefore, $\sum_{j=1}^{\infty}[\pi(j)+\epsilon(j)u(j)]$ converges almost surely, and we have the result. ■

Remark 1. The results of Theorem 1 hold under the general condition that $|\tilde{\theta}(k)|$ is bounded w.p. 1. If we are, in general, interested in the probability $P(|\tilde{\theta}(k)| < K$ for all $k \geq k_0)$, where K is a preselected constant, we can follow the line of thought in [19] based on the Kolmogorov's inequality for martingales. Assuming that the noise amplitude is bounded w.p. 1 by a constant K_ζ, it is possible to show that $P(|\tilde{\theta}(k)| < K + M^* \text{ for all } k \geq k_0) \geq 1 - \frac{N^*}{K^2}$, where M^* is proportional to K_ζ and $N^* = E\{\tilde{\theta}(1)^2\} + N_\zeta$, where N_ζ is proportional to $\sup_k(\sigma(k)^2)$. Therefore, $\frac{N}{K^2}$ corresponds to the small probability that $\tilde{\theta}$ will get out of the predefined local region. A rigorous analysis of this issue is out of the scope of this paper.

IV. An Application to Mobile Sensor Networks and Experiments

In this section some direct applications of the above extremum seeking scheme to the optimal positioning of mobile sensors will be presented.

Assume that we have a random number generator which generates zero-mean white noise $\{\xi(k)\}$ with variance which depends on a parameter θ, *i. e.* $E\{\xi(k)^2\} = R(\theta)$, where $R(\theta)$ is assumed to be a convex function of θ. Our goal is to find θ minimizing $R(\theta)$ by measuring $\{\xi(k)\}$ generated for different values of θ. According to the above results, we can apply the extremum seeking (ES) scheme for this purpose. Assume that $\eta(k) = \xi(k)^2$ and write $\eta(k) = R(\theta) + \zeta(k)$, where $\zeta(k) = \xi(k)^2 - R(\theta)$. The sequence $\{\zeta(k)\}$ is white and zero-mean with finite variance, assuming that the fourth-order moment of $\xi(k)$ is finite. Therefore, we assume that in the ES scheme depicted in Fig. 1, $\eta(k)$ represents the noisy output, $R(\theta(k))$ represents the noiseless output and $\zeta(k)$ represents the measurement noise. It is easy to conclude that the above theorem can be applied, and that the ES under the conditions of Theorem 1 provides convergence of $\tilde{\theta}(k)$ to θ^* with probability 1.

Following the approach in [10], the scheme in Fig.1 can be generalized to two dimensional case, using orthogonal sinusoidal perturbations. Then, the ES scheme is able to find the position in the plane corresponding to the minimal noise variance, assuming that we can model the vehicle as a single integrator.

Assume now that we are faced with the problem of state estimation in which the Kalman filter is applied, and that it is necessary to find the best place in the plane for an observer, assuming that the measurement noise variance is coordinate dependent. This problem is fundamental in applications related to mobile sensor networks. Recall that in the optimal steady state regime of the estimator the innovation sequence $\{\nu(k)\} = \{z(k) - C\hat{x}(k|k)\}$ is white, where $z(k)$ is the system output, $\hat{x}(k|k)$ is the state estimate and C the output matrix of the system, assuming that we have a scalar output. Assume that $\{n(k)\}$ is the measurement noise, which is white, with variance depending on the position of the observer in a plane, *i.e.*, $E\{n(k)^2\} = R(x(k), y(k))$ ($x(k)$ and $y(k)$ are the coordinates). Then, we have

$$E\{\nu(k)^2\} = R_\nu(x, y) = CP(x, y)C^T + R(x, y) \quad (47)$$

where $P(x, y)$ is the steady state estimation error covariance matrix which satisfies the algebraic Riccati equation

$$P = \Phi P \Phi^T - \Phi P C^T[CPC^T + R]^{-1}CP\Phi^T + Q \quad (48)$$

where Φ is the state matrix of the system model and Q is the input driving noise covariance. We can calculate $p = CPC^T$ by assuming that $C\Phi P \Phi^T C^T = ap$ and $C\Phi P C^T = bp$, for some constants a and b. From (48) we obtain that p, which is scalar, is a solution of the quadratic equation

$$b^2p^2 + (1 - a)p(p + R) - q(p + R) = 0 \quad (49)$$

where $q = CQC^T$. It is easy to verify that for R small enough $p \approx p^* + a^*R$ where p^* and $a^* > 0$ are constants depending on the parameters a, b and q. Therefore, from (47) we derive

$$R_\nu(x, y) \approx R_\nu^* + a_x^*(x - x^*)^2 + a_y^*(y - y^*) \quad (50)$$

for some constants R_ν^*, a_x^* and b_x^*, assuming that $R(x, y)$ can be approximated by a quadratic function. From this result we conclude that the observer position can be asymptotically optimized by applying the ES scheme similarly like in the above case. Namely, we take the realizations $\nu^2(k)$ as measurements (instead of $\xi^2(k)$) and apply the ES scheme from Fig. 1 (for two dimensional case see [10]); the scheme asymptotically provides the optimal observer position.

One practical modification of this scheme is to take $\frac{1}{T}\sum_{i=k-T}^{k-1} \nu(k)^2$ instead of $\nu^2(k)$ in order to reduce the equivalent noise variance (by the factor T).

In order to illustrate the proposed algorithms we provide two examples with simulation studies.

Example 1. In this example we demonstrate the results of Theorem 1 for the two dimensional case. The nonlinear function $f(x, y)$ is assumed to be quadratic $f(x, y) = 1 + \frac{1}{2}x^2 + \frac{1}{2}y^2$ and the measurement noise variance is $\sigma^2 = 0.4$. Other system parameters are $h = 0.1$ and $\omega = 0.6\pi$. In order to satisfy conditions (A.2-8) we assume that $\alpha(k) = \frac{1}{k^{0.25}}$ and $\epsilon(k) = \frac{1}{k^{0.75}}$. In Fig. 2, $x(k)$ and $y(k)$ coordinates are shown as a function of time, for the initial conditions $x(0) = 1.7$ and $y(0) = 1.7$. We can see that they both converge exactly to zero (the minimum of function $f(x, y)$), in accordance with the results of Theorem 1.

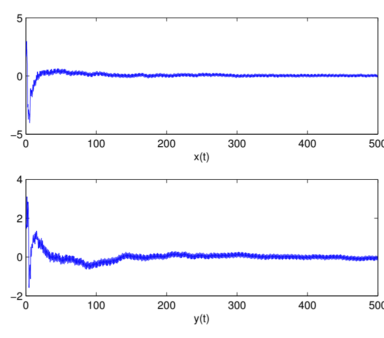

Fig. 2. $x(k)$ and $y(k)$ coordinates (Example 1)

Example 2. This example will illustrate the algorithm for the optimal positioning of the Kalman estimator in the plane. The vehicle, on which the Kalman state estimator is implemented, is modelled as a single integrator. We assume the following model for the Kalman state estimator $F =$

$\begin{bmatrix} 0.5 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}$, $G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $H = [0\ 1]$, where F is the system matrix, G is the input matrix, H is the output matrix, input noise covariance matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the measurement noise variance depends on the coordinates of the vehicle $x(k)$ and $y(k)$ as the quadratic function $R(x, y) = 0.5 + 5x^2 + 5y^2$. The other system parameters are $h = 0.1$ and $\omega = 0.6\pi$, and, like in the previous example, $\alpha(k) = \frac{1}{k^{0.25}}$ and $\epsilon(k) = \frac{1}{k^{0.75}}$. As described above, we take the filtered squared residuals $\frac{1}{T} \sum_{i=k-T}^{k} \nu(k)^2$ as the measurements in the ES scheme, where $T = 3$. The coordinates $x(k)$ and $y(k)$ as a function of time are shown in Fig. 3, for the initial conditions $x(0) = 1.5$ and $y(0) = 1$, respectively. The exact convergence to the minimum noise variance point (0,0) is evident. The trajectory of the vehicle is shown in Fig. 4.

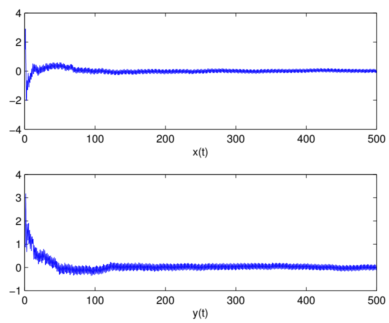

Fig. 3. $x(k)$ and $y(k)$ coordinates (Example 2)

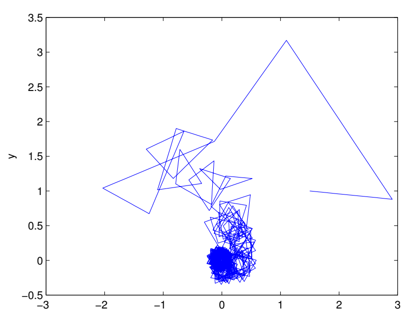

Fig. 4. Trajectory of the mobile observer (Example 2)

V. Conclusion

In this paper, new assumptions have been introduced into the extremum seeking algorithm with sinusoidal perturbation. It has been assumed that the integrator gain and the perturbation amplitude are time varying (decreasing in time with the proper rate) and that the output is corrupted with the measurement noise. The local convergence of the algorithm, with probability one and in the mean square sense, has been proved. Also, two direct applications to the optimal mobile sensor positioning have been proposed. These problems are fundamental in multi-agent systems and mobile sensor networks, which gives a great perspective for further applications of the proposed algorithms. The simulation studies illustrate the results of the main theorem - convergence to the extremal point with complete noise elimination.

A direct extension of this work would be to analyze an application of the proposed algorithm to the optimal placement in the plain for the case when vehicles are modeled as double integrators or unicycles.

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