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Control Variation as a Source of Uncertainty: Single Input Case

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Abstract— **This paper presents a theoretical framework and the control strategy for single input discrete-time stochastic systems for which the control variations increase state uncertainty (CVIU systems). This type of system model can be useful in many practical situations, such as in monetary policy problems, medicine and biology, and, in general, in problems for which a complete dynamic model is too complex to be feasible. The optimal control strategy for a single-input CVIU system associated with a convex cost functional is devised using dynamic programming and tools from nonsmooth analysis. Furthermore, this strategy points to a region in the state space in which the optimal action is of no variation, as expected from the cautionary nature of controlling underdetermined systems. In addition, a specific result for the case when the cost functional is differentiable is obtained and discussed. These results are illustrated through a numerical example in economics.**

I. INTRODUCTION

In the last decade, many results from control theory have been used in non-traditional application areas, such as economics [1], [2], medicine [3], and biology [4], [5]. These areas provide new and challenging scenarios, where decision making and control occur in uncertain and non-linear environments. Traditionally, in stochastic control problems, the imbedded system uncertainty is modeled mainly by means of additive or multiplicative disturbances [6] and of parameter uncertainty—such as in Markov Jump Linear Systems [7]. However, these models may not suffice to describe uncertainty in many situations.

Consider, for instance, the problem faced by a National Central Bank (NCB) when defining the monetary policy [8], [9]. As it increases or decreases the interest rate, the NCB has an uncertainty about the expectations of the economic agents. Significant course change in the monetary policy may induce unexpected and undesired consequences such as an increase in inflation or reduction of the GDP. On the other hand, if the variation of the interest rate is too small, the NCB objectives may not be accomplished. This is an example of a system for which any change of policy by the decision maker (in this case, the NCB), leads to an increase in system uncertainty.

This situation is also present in other types of systems. For example, in medicine, the dynamics of a patient's response to the variation of the dosage of a medication is nonlinear and uncertain [10], [11], [12]. Large dosage variations may lead to unpredictable consequences, while small dosage variations may have no effect on the patient's health. Both the monetary policy and the control problem encountered in medicine are examples of *systems where control variations increase state uncertainty (CVIU)*. From a control theory perspective, CVIU systems can be used to control systems with complex, underdetermined dynamics, for which the behavior near a given point and for a given control policy can be fairly well described by a linear model. However, large variations of the control action can drift the system to regions where the linear approximation error is too large. In this case, the approximation error is in correspondence to the uncertainty generated by policy variations.

The aim of this paper is to provide a theoretical framework for CVIU systems and to characterize the optimal control strategy for these systems. Section II presents a formal definition of CVIU systems with a single input and the construction of a CVIU model. Section III introduces auxiliary results from non-smooth analysis, convex optimization theory and probability that will be used to characterize the optimal control policy for CVIU systems. Section IV deals with the one-stage optimal policy, applying the convexity assumption and the tools from non-smooth analysis in order to guarantee that the value function inherits convexity. Section V delves on the characterization of the optimal policy and identifies three regions on the state space for which the sign of the optimal policy is known. The optimal action for one of these regions is of no variation of the control, as expected from the cautionary nature of controlling under-determined systems. Taking advantage of the decision regions, an algorithm for obtaining the optimal control policy is presented in Section VI. An application in economics is presented in Section VII, and the paper is finalized with a summary of the results in Section VIII.

II. SINGLE INPUT MODEL

Consider a discrete-time system described by the state equation:

$$x_{k+1} = A_k x_k + b_k u_k + \omega_k, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$. Furthermore, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$ and $\omega_k \in \mathbb{R}^n$ are respectively, the state, the input and the noise, which is a stochastic process. We consider that the magnitude of the control action acts as a source of system uncertainty in such a way that the noise sequence ω_k is modulated by the *absolute value of the control* $|u_k|$, as follows:

$$\omega_k = (\overline{\sigma}_k + \sigma_k |u_k|)\,\varepsilon_k,$$

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where $\overline{\sigma}_k > 0, \sigma_k > 0$, and ε_k is an i.i.d. random vector with a normalized covariance matrix $\text{cov}(\tilde{\varepsilon}_k) = I_{n\times n}$. In connection, suppose that in a given time window $0 \le k \le N$, the system performance is evaluated by means of the cost functional,

$$J(x_0,\mu) = E\left[\sum_{k=0}^{N-1} C_k(x_k,u_k) + C_N(x_N)\right], \tag{2}$$

where $E[\cdot]$ stands for the expected value of the corresponding random variable, C_k is non-negative for each $0 \le k < N-1$ and convex in both arguments. Also, the terminal cost C_N is non-negative and convex. We assume that at each time instant k the policy maker can determine the input control, u_k, having perfect state information, and $\mu = \{u_0, u_1, \dots, u_{N-1}\}$ stands for an admissible policy.

A. Construction of a CVIU model

A traditional discrete-time state space equation can be rearranged into a CVIU model through a simple change of variables. In order to show how this can be done, consider the discrete-time system given by (1). As discussed in the previous section, since in CVIU systems control variations act as a source of system uncertainty, ω_k will depend on successive control variations. Let us denote,

$$v_k = u_k - u_{k-1}.$$

We assume that the noise sequence ω_k is modulated by the *absolute value of control variations* $|v_k|$, as follows:

$$\omega_k = (\overline{\sigma}_k + \sigma_k|v_k|)\,\tilde{\varepsilon}_k,$$

where $\overline{\sigma}_k > 0, \sigma_k > 0$, and $\tilde{\varepsilon}_k$ is an i.i.d. random vector with a normalized covariance matrix $\text{cov}(\tilde{\varepsilon}_k) = I_{n\times n}$.

This system can be described in the state space form by defining the augmented system:

$$\tilde{x}_{k+1} = \tilde{A}_k\tilde{x}_k + \tilde{b}_k v_k + (\overline{\sigma}_k + \sigma_k|v_k|)\,\varepsilon_k, \tag{3}$$

where

$$\tilde{x}_k := \begin{bmatrix} x_k \\ u_{k-1}\end{bmatrix}, \quad \tilde{A}_k := \begin{bmatrix} A_k & b_k \\ 0 & 1\end{bmatrix}, \quad \tilde{b}_k := \begin{bmatrix} b_k \\ 1\end{bmatrix}, \varepsilon_k := \begin{bmatrix} \tilde{\varepsilon}_k \\ 0\end{bmatrix}.$$

Since the systems described by (1) and (3) are equivalent, in the remaining of this paper, the control input at a given time instant k will be denoted as u_k.

III. Useful Results

We assume throughout that the cost functions $C_k(\cdot,\cdot)$ for each k and $C_N(\cdot)$ are convex functions, and we will draw from the properties of such a class of functions.

The proof of the next propositions are mostly quoted from the literature.

Proposition 1 ([13]): Suppose f is differentiable. Then f is convex if and only if $\text{dom}(f)$ is convex and

$$f(x) - f(x_0) \le \nabla f(x)^T(x - x_0) \tag{4}$$

Definition 1 ([14]): Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^n$. Let Ω be any set of zero measure in $\mathbb{R}^n$, and let Ω_f be the set of points in $\mathbb{R}^n$ where f fails to be differentiable. Then the Generalized Gradient at x, denoted by $\partial f(x)$, will be the set

$$\partial f(x) = \text{co}\left\{\lim_{x_i\to x} \nabla f(x_i) : x_i \notin \Omega, x_i \notin \Omega_f\right\}$$

where "co" means "closed convex hull".

Proposition 2 ([14]): Let f be Lipschitz near each point of an open convex subset $U \subset \mathbb{R}^n$. Then f is convex in U if and only if ∂f is monotone in U, i.e.,

$$\langle \zeta - \tilde{\zeta}, x - \tilde{x}\rangle \ge 0, \quad \forall x,\ \tilde{x} \in U, \zeta \in \partial f(x),\ \tilde{\zeta} \in \partial f(\tilde{x})$$

Proposition 3 ([13]): Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty and convex sets. Then, if $f : U \times V \to \mathbb{R}$ is convex in V, the function g given by:

$$g(x) = \inf_{y\in V} f(x,y)$$

is convex in x, provided $g(x) > -\infty$ for all x.

Proposition 4 (Chain Rule [14]): Let $f = g \circ h$, where $h : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. One has

$$\partial f(x) \subset \text{co}\{\partial g(h(x))\partial h(x)\},$$

If f is convex, the inclusion becomes an equality. Note that $\partial h(x) \in \mathbb{R}^{n\times m}$ and $\partial g(h(x)) \in \mathbb{R}^{1\times n}$.

Proposition 5 (Local Extrema [14]): If $f : \mathbb{R} \to \mathbb{R}^n$ attains a local minimum or maximum at x_0, then $0 \in \partial f(x_0)$.

Proposition 6 (Integral Functionals [14]): Let $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a Lipschitz measurable function such that

$$|g(x_1,y) - g(x_0,y)| \le K(y)||x_1 - x_0||, \forall x_1, x_0 \in \mathbb{R}^n; y \in \mathbb{R}^m$$

Also, let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(x) = \int_U g(x,y)dy, U \subset \mathbb{R}^m$$

Then

$$\partial f(x) = \partial \int_U g(x,y)dy \subset \int_U \partial_x g(x,y)dy$$

Proposition 7 (Danskin's Theorem [15]): For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ let $(x,v) \to V(x,v)$ be a convex differentiable function. Furthermore, let $u^*(x)$ defined as

$$u^*(x) = \arg\min_{u\in\mathbb{R}} V(x,u) \tag{5}$$

be unique for every x. Then, the function $x \to V^*(x)$ defined as

$$V^*(x) = \inf_{u\in\mathbb{R}} V(x,u) \tag{6}$$

is differentiable for every x.

Proposition 8 ([16]): For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, let $(x,u) \to V(x,u)$ be a continuous convex function and let V^* and u^* be as in (6) and (5). Then,

$$\partial V^*(x) = \text{co}\{\partial_x V(x,u) : u \in u^*(x)\}$$

Lemma 1: Consider $f : \mathbb{R}^n \to \mathbb{R}$ a differentiable convex function and ε a random variable with zero mean, and $g : \mathbb{R} \to \mathbb{R}$ given by:

$$g(t) = E[f(x_0 + \varepsilon t)],$$

where $E[\cdot]$ stands for the expected value of the random variable.

(i) $t \to g(t)$ is non-decreasing for all $t \geq 0$.

(ii) If ε is also symmetrically distributed, then $g(t_1) \leq g(t_2)$ for $|t_1| \leq |t_2|$.

Proof: For part *(i)*, we evaluate

$$\begin{aligned}\frac{dg(t)}{dt} &= E[\nabla^T f(x_0+\varepsilon t)\varepsilon] \\ &= \frac{1}{t}E[\nabla^T f(x_0+\varepsilon t)(t\varepsilon + x_0 - x_0)]\end{aligned}$$

Since f is convex, one can use (4) with $x = x_0 + \varepsilon t$ and Jensen's Inequality to obtain:

$$\begin{aligned}\frac{1}{t}E[\nabla^T f(x_0+\varepsilon t)(t\varepsilon + x_0 - x_0)] &\geq \frac{1}{t}E[\, f(x_0+\varepsilon t) - f(x_0)] \\ &\geq \frac{1}{t}(f(x_0) - f(x_0)) \geq 0\end{aligned}$$

Consequently, $g'(t) \geq 0$ and g is non-decreasing. For part *(ii)*, note that if ε has zero mean and is symmetrically distributed, $\varepsilon \sim -\varepsilon$ holds and:

$$\begin{aligned}E[f(x_0+\varepsilon t)] &= E[f(x_0-\varepsilon t)] \\ &= E[f(x_0+\varepsilon|t|)], \forall t \in \mathbb{R}.\end{aligned}$$

In this situation, $g(t)$ is even non-decreasing for $t \geq 0$ and non-increasing for $t \leq 0$, which completes the proof. ■

IV. Convexity Characterization

We aim at the dynamic programming method, and in a preliminary step we are interested in characterizing the function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $V^* : \mathbb{R}^n \to \mathbb{R}$, defined as

$$V(x,u) = C(x,u) + E\left[F(x_1)\right], \tag{7}$$

and

$$V^*(x) = \inf_{u\in\mathbb{R}^m} V(x,u), \tag{8}$$

where $C : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$ are both convex, non-negative and Lipschitz functions. The random vector x_1 is determined by (1) with $x_0 = x$ and $u_0 = u$. Note that the system is time homogeneous in the sense that if $x_k = x$ and $u_k = u$, one is evaluating equivalently the expected value in (7) of x_{k+1}. The following lemma is important for the characterization of V^*.

Lemma 2: The functions $V(x,u)$ and $V^*(x)$ given by (7) and (8) , respectively, are convex.

Proof: To simplify the notation, let us write $\mathscr{E}_F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, the expected value in (7)

$$\mathscr{E}_F(x,u) = E\left[F(x_1(x,u))\right] \tag{9}$$

with

$$x_1(x,u) = Ax + bu + (\overline{\sigma} + \sigma|u|)\varepsilon.$$

Using Proposition 1 we have that

$$\partial x_1(x,u) = \begin{bmatrix} A & b + \sigma\,\mathbb{S}_u\,\varepsilon \end{bmatrix} \tag{10}$$

where $\mathbb{S}_u$ is a set given by:

$$\mathbb{S}_u = \begin{cases} +1, & \text{if } u > 0 \\ -1, & \text{if } u < 0\,, \\ [-1,1], & \text{if } u = 0 \end{cases}$$

The first step is to prove that $\mathscr{E}_F(x,u)$ is convex. Let $\zeta \in \partial\mathscr{E}_F(x,u)$ and $\tilde{\zeta} \in \partial\mathscr{E}_F(x,\tilde{u})$. From Proposition 4, we have

$$\begin{aligned}\zeta &\in E[\beta^T\gamma], \quad \beta \in \partial F(x_1(x,u)),\ \gamma \in \partial x_1(x,u) \\ \tilde{\zeta} &\in E[\tilde{\beta}^T\tilde{\gamma}], \quad \tilde{\beta} \in \partial F(x_1(x,\tilde{u})),\ \tilde{\gamma} \in \partial x_1(x,\tilde{u}).\end{aligned}$$

with

$$\begin{aligned}\gamma &= \begin{bmatrix} A & b + \sigma\, s\,\varepsilon \end{bmatrix}, \quad s \in \mathbb{S}_u \\ \tilde{\gamma} &= \begin{bmatrix} A & b + \sigma\,\tilde{s}\,\varepsilon \end{bmatrix} \quad \tilde{s} \in \mathbb{S}_{\tilde{u}}.\end{aligned}$$

For simplicity, we define

$$\Delta = \begin{bmatrix} x - \tilde{x} \\ u - \tilde{u} \end{bmatrix}, \quad x, \tilde{x} \in \mathbb{R}^n;\, u, \tilde{u} \in \mathbb{R}$$

In view of Proposition 2, to prove that $\mathscr{E}_F(x,u)$ is convex is equivalent to showing that

$$E[\beta^T\gamma - \tilde{\beta}^T\tilde{\gamma}]\Delta \geq 0, \quad \beta \in \partial F(x_1(x,u)),\ \tilde{\beta} \in \partial F(x_1(x,\tilde{u})). \tag{11}$$

Note that

$$\begin{aligned}\begin{bmatrix} A & b + \sigma\, s\,\varepsilon \end{bmatrix}\Delta =& Ax + bu + (\overline{\sigma} + \sigma\, s\, u)\varepsilon \\ &- [A\tilde{x} + b\tilde{u} + (\overline{\sigma} + \sigma\, s\,\tilde{u})\varepsilon]\end{aligned}$$

and similarly,

$$\begin{aligned}-\begin{bmatrix} A & b + \sigma\,\tilde{s}\,\varepsilon \end{bmatrix}\Delta =& A\tilde{x} + b\tilde{u} + (\overline{\sigma} + \sigma\,\tilde{s}\tilde{u})\varepsilon \\ &- [Ax + bu + (\overline{\sigma} + \sigma\,\tilde{s}\, u)\varepsilon]\end{aligned}$$

Based on these equations and noting that F is convex, we have that

$$\begin{aligned}E[\beta^T\gamma]\Delta \geq& E\left[F(Ax + bu + (\overline{\sigma} + \sigma s u)\varepsilon)\right] \\ &- E\left[F(A\tilde{x} + b\tilde{u} + (\overline{\sigma} + \sigma s\tilde{u})\varepsilon)\right]\end{aligned} \tag{12}$$

and similarly,

$$\begin{aligned}E[\tilde{\beta}^T\tilde{\gamma}^T]\Delta \geq& E\left[F\left(A\tilde{x} + b\tilde{u} + (\overline{\sigma} + \sigma\,\tilde{s}\,\tilde{u})\varepsilon\right)\right] \\ &- E\left[F\left(Ax + bu + (\overline{\sigma} + \sigma\,\tilde{s}\, u)\varepsilon\right)\right]\end{aligned} \tag{13}$$

Since $\overline{\sigma} > 0$ and from the definition of s and $\tilde{s}$, we observe that

$$\begin{aligned}\overline{\sigma} + \sigma\tilde{s}u &\leq \overline{\sigma} + \sigma|u| = \overline{\sigma} + \sigma s u \\ \overline{\sigma} + \sigma\, s\,\tilde{u} &\leq \overline{\sigma} + \sigma|\tilde{u}| = \overline{\sigma} + \sigma\,\tilde{s}\,\tilde{u}\end{aligned}$$

Adding (12) and (13) we obtain:

$$E[\beta^T[A \quad b + \sigma\, s\,\varepsilon] - \tilde{\beta}^T[A \quad b + \sigma\,\tilde{s}\,\varepsilon]]\Delta \geq 0$$

Therefore, $x, u \to \mathscr{E}_F(x,u)$ is convex. Finally, note that C is convex, and thus, $x, u \to V(x,u)$ and $x \to V^*(x)$ (due to Proposition 3) will also be convex, completing the proof. ■

Even though V^* is defined as a piecewise function, it will be differentiable if V is strictly convex and differentiable. Moreover, based on V, we can characterize the generalized gradient of V^*. These facts are stated in the following lemma.

Lemma 3: Let V be defined as in (7) and V^* as in (8). Also, let $u^*(x)$ be defined as

$$u^*(x) = \arg\min_{u\in\mathbb{R}} V(x,u). \tag{14}$$

then

1) $\partial V^*(x) = \mathrm{co}\{\partial_x V(x,u) : u \in u^*(x)\}$;
2) $V^*(x)$ will be differentiable if V is a strictly convex differentiable function.

Proof: The proof follows from Propositions 7 and 8. ■

V. When No Variation is Optimal

With the fact that the value function is convex at hand, we proceed further with the purpose of determining the sign of u^* based solely on the value of the state x. Assume that a function $(x,u) \to f(x,v)$ is differentiable. Provided that f is also convex in u, one can obtain the sign of the minimum in u by analyzing $\nabla_u f|_{u=0}$ for each x. If $\nabla_u f|_{u=0} > 0\ (< 0)$, then the function is increasing (decreasing) at the origin and, consequently, the minimum will be in the negative (positive) half-plane. Of course, if $\nabla_u f|_{u=0} = 0$ the optimal solution is $u^* = 0$. Note that this analysis can not be applied to V in (7) since, even though V is convex, it will not necessarily be differentiable at $u = 0$ (in fact, this will never be the case). The following Lemma presents a result regarding this issue and points out that there will be a region in the state space where $u^* = 0$ if the cost function is Lipschitz. Furthermore, for the case where C and F are differentiable, we will show that a region where $u^* = 0$ will always exist.

Lemma 4: For the function V described in (7) and u^* given by (14) we have that

$$\begin{cases} u^*(x) > 0, & \text{if } x \in \mathscr{R}_1(V), \\ u^*(x) < 0, & \text{if } x \in \mathscr{R}_2(V), \\ u^*(x) = 0, & \text{if } x \in \mathscr{R}_3(V). \end{cases} \tag{15}$$

where

$$\mathscr{R}_1(V) = \left\{x : x \in \mathbb{R}^n, \lim_{u_i \downarrow 0} \nabla V(x,u_i) < 0\right\}, \tag{16}$$

$$\mathscr{R}_2(V) = \left\{x : x \in \mathbb{R}^n, \lim_{u_i \uparrow 0} \nabla V(x,u_i) > 0\right\}, \tag{17}$$

$$\mathscr{R}_3(V) = \overline{\mathscr{R}_1 \cup \mathscr{R}_2}. \tag{18}$$

Proof: Since $\partial_u V$ is either a point or a closed connected interval of the line, we get from convexity a non-decreasing notion in u for $\partial_u V$, in the sense that

$$u_1 \le u_2 \iff \exists \gamma_1 \le \gamma_2, \quad \gamma_1 \in \partial_u V(u_1,x), \gamma_2 \in \partial_u V(u_2,x).$$

Therefore, we can determine the sign of $u^*(x)$ observing only $\partial_u V|_{u=0}$. If $\gamma > 0, \forall \gamma \in \partial_u V|_{u=0}$, then V is increasing at the origin and $u^* < 0$. Conversely, if $\gamma < 0, \forall \gamma \in \partial_u V|_{u=0}$ then V is decreasing at the origin and $u^* > 0$; clearly, if $0 \in \gamma \partial_u V|_{u=0}$ then $u^* = 0$, e.g., see Proposition 5.

Note that for the verification above we do not need to examine all elements of $\partial_u V|_{u=0}$ to determine the sign of u^*. Based on the fact that $\partial_u V$ is increasing in u and using the definition of generalized gradient we have that,

$$\lim_{u_i \downarrow 0} \nabla V(x,u_i) = \max \partial_u V|_{u=0},$$

$$\lim_{u_i \uparrow 0} \nabla V(x,u_i) = \min \partial_u V|_{u=0},$$

where the limits are calculated avoiding sets of zero measure and the points where $u \to V(x,u)$ fails to be differentiable.

From these arguments, we obtain the following conditions

$$\lim_{u_i \downarrow 0} \nabla V(x,u_i) < 0 \implies u^*(x) > 0,$$

$$\lim_{u_i \uparrow 0} \nabla V(x,u_i) > 0 \implies u^*(x) < 0,$$

which results in three complementary regions covering the state space given by (16), (17) and (18) defined in the statement of the Lemma. ■

Note that the region $\mathscr{R}_3$ can be defined as

$$\mathscr{R}_3 = \{x : 0 \in \partial_u V(x,0)\}$$

or, equivalently as

$$\mathscr{R}_3 = \{x : \lim_{u \downarrow 0} \nabla V(x,u) > 0,\ \lim_{u \downarrow 0} \nabla V(x,u) < 0\} \tag{19}$$

In this manner, we can calculate

$$\lim_{u_i \downarrow 0} \nabla V(x,u_i) = \lim_{u_i \downarrow 0} \nabla_u C + E\left[\nabla F(Ax + bu_i + (\overline{\sigma} + \sigma u_i)\varepsilon)(b + \sigma\varepsilon)\right],$$

$$\lim_{u_j \uparrow 0} \nabla V(x,u_j) = \lim_{u_j \uparrow 0} \nabla_u C + E\left[\nabla F(Ax + bu_j + (\overline{\sigma} - \sigma u_j)\varepsilon)(b - \sigma\varepsilon)\right].$$

Therefore, if C and F are differentiable, these equations simplify to

$$\lim_{u_i \downarrow 0} \nabla V(x,u_i) = \nabla_u C(x,0) + E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)(b + \sigma\varepsilon)\right],$$

$$\lim_{u_j \uparrow 0} \nabla V(x,u_j) = \nabla_u C(x,0) + E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)(b - \sigma\varepsilon)\right],$$

and (19) can be rewritten as

$$\mathscr{R}_3 = \{x : |\nabla_u C(x,0) + E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)b\right]| < E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)\sigma\varepsilon\right]\}$$

In order to prove that $\mathscr{R}_3$ is not empty, it is necessary to show that $[\nabla F(Ax + \overline{\sigma}\varepsilon)\sigma\varepsilon] > 0$. Observing that F is convex, this can be done by noting that

$$E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)\sigma\varepsilon\right] = E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)(Ax + \overline{\sigma}\varepsilon - Ax)\frac{\sigma}{\overline{\sigma}}\right]$$

and using Proposition 2 and Lemma 1 directly to obtain

$$E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)\sigma\varepsilon\right] \ge \frac{\sigma}{\overline{\sigma}} E[F(Ax + \overline{\sigma}\varepsilon) - F(Ax)] \ge 0$$

Also, it is interesting to note that an increase in the value of σ leads to an increase of $E\left[\nabla F(Ax + \overline{\sigma}\varepsilon)\sigma\varepsilon\right]$ and, consequently, to a larger region where no variation is optimal.

VI. The Dynamic Programming

In view of Lemma 2, the Dynamic Programming method applied to system (1) reads as follows. Let us denote by J_k^* the optimal cost-to-go at instant k for each x as

$$J_k^*(x) = \inf_{\mu_k} E\left[\sum_{n=k}^{N-1} C_n(x_n,u_n) + C_N(x_N)\right]$$

where $\mu_k = \{u_k, u_{k+1}, \ldots, u_{N-1}\}$ is a sequence of admissible policy drawn from complete state observation. The following roposition gives a sufficient condition for the existence of the optimal feedback policies, $\mu_k^*(x)$.

Proposition 9 ([17]): Suppose that (2) is limited for all $x_0 \in \mathbb{R}^n$ and μ. Then, if the set

$$U_k(x,\lambda) = \{u \in \mathbb{R} : C_k(x,u) + E[J_{k+1}^*(A_k x + b_k u + (\overline{\sigma}_k + \sigma_k|u|)\varepsilon_k))] \le \lambda\}$$

is compact for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $k \in [0, N-1]$, an N-stage optimal policy exists.

Theorem 1: Suppose that for each $0 \leq k < N$ we have that $x \to C_k(x,u), u \to C_k(x,u)$ and $x \to C_N(x)$ are convex functions and that (2) is limited for all $x_0 \in \mathbb{R}^n$ and μ. For the system in (1) and evaluated by mean of the cost function (2), the optimal policy can be obtained recursively as follows.

1) Define $J_N^*(x) = C_N(x)$, $x \in \mathbb{R}^n$ and set $k = N-1$;
2) Define $J_k(x,u)$ for each $x \in \mathbb{R}^n$, as
$$J_k(x,u) = C_k(x,u) + E[J_{k+1}^*(A_k x + b_k u + (\overline{\sigma}_k + \sigma_k |u|)\varepsilon_k)],$$
3) For each $x \in \mathbb{R}^n$, determine if the optimal action u_k^* will be positive, negative or zero with
$$\begin{cases} u_k^*(x) > 0, & \text{if } x \in \mathscr{R}_1(J_k), \\ u_k^*(x) < 0, & \text{if } x \in \mathscr{R}_2(J_k), \\ u_k^*(x) = 0, & \text{if } x \in \mathscr{R}_3(J_k). \end{cases} \tag{20}$$
If $u_k^* \in \mathscr{R}_1(J_k)$ or $u_k^* \in \mathscr{R}_2(J_k)$, determine u_k^* such that $J_k(x,u_k^*) \leq J_k(x,u)$ for each $u \in \mathbb{R}$. This is equivalent to requiring that
$$0 \in \partial_u C_k(x,u_k^*) + \partial_u E[J_{k+1}^*(A_k x + b_k u_k^* + (\overline{\sigma}_k + \sigma_k |u_k^*|)\varepsilon_k)]$$
4) Define the function $J_k^*(x)$ by
$$J_k^*(x) = J_k(x, u_k^*(x)) = C(x, u_k^*(x)) + E[J_{k+1}^*(x_{k+1})]$$
with $x_{k+1} = A_k x + b_k u_k^*(x) + (\overline{\sigma}_k + \sigma_k |u_k^*(x)|)\varepsilon_k$. If $k = 0$, stop. If else, return to step 2.

The optimal policy u_k for each $0 \leq k < N$ is thus obtained, in such a way that
$$J_k^*(x) = J_k(x, u_k^*(x)) \leq J_k(x,u),$$
holds for each (x,u) and, in particular, $J^*(x) = J_0(x, u_0^*(x)) \leq J_0(x,u)$, $\forall (x,u)$.

Proof: The proof follows directly from Lemma 2, Lemma 4 and Bellman's Equation. From Proposition 9, observe that since C_k is convex for all $k \in [0,N]$, the feedback policies, $x \to u_k^*(x)$, will always exist. ∎

Note that at each instant k, the decision maker will either increase, decrease or maintain the previous input depending on the state value. A possible interpretation for the region where $u_k^* = 0$ is that, in face of the uncertainty generated by changing the input of the system, it may be better to keep the input constant. Intuitively, this strategy agrees with real world problems when a policy maker, in face of the uncertainty that his decision may generate, decides to maintain the same policy he had before.

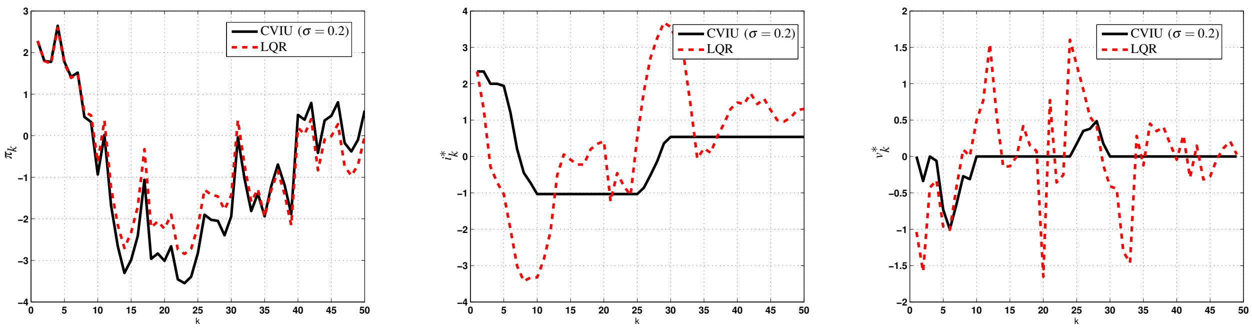
VII. A Monetary Application

In this section, a state-space form of a standard backward-looking model of the US economy is analyzed within a CVIU context. The model's parameters were taken from [18]. Quarterly data for the US economy was used, from the first quarter of 1960 to the fourth quarter of 2006. The interest rate , i_k, is a four-quarter average federal interest rate from the Board of Governors. Inflation, π_k, is the GDP chain-type price index, in percent, at an annual rate. The output gap (y_k) is built as $100(q_k - q_k^*)/q_k^*$, where q_k is the real GDP and q_k^* is the potential GDP. The data used is available from the Bureau of Economic Analysis. All variables were demeaned prior to estimation. Therefore, the state-space model used is
$$x_{k+1} = Ax_k + Bv_k + (\overline{\sigma} + \sigma |v_k|)\xi_k, \quad \xi_k \sim N(0, \Sigma) \tag{21}$$
where
$$x_k = \begin{bmatrix} \pi_k & \pi_{k-1} & \pi_{k-2} & y_k & i_{k-1} \end{bmatrix}^T, \qquad v_k = \begin{bmatrix} i_k - i_{k-1} \end{bmatrix},$$
$$A = \begin{bmatrix} 0.621 & 0.091 & 0.239 & 0.108 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.041 & -0.083 & 0.011 & 0.900 & (0.117 - 0.192) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 & 0 & 0.117 & 1 \end{bmatrix}^T, \Sigma = \begin{bmatrix} I_{4\times 4} & 0 \\ 0 & 0 \end{bmatrix}.$$
Also, $\overline{\sigma}_k$ and σ_k are diagonal matrices given by:
$$\overline{\sigma}_k = \mathrm{diag}(\overline{\sigma}_\pi, 0, 0, \overline{\sigma}_y, 0)$$
$$\sigma_k = \mathrm{diag}(\sigma_\pi, 0, 0, \sigma_y, 0)$$
System performance is evaluated by means of the the quadratic cost function
$$J(x) = E\left[x_N' Q x_N + \sum_{k=0}^{N-1} x_k' Q x_k + r v_k^2\right] \tag{22}$$
where
$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad r = 0.5.$$

Simulation results are shown in Figures 1 and 2. In Figure 1, note that the CVIU model generates a smooth monetary policy, while the LQR leads to large variations of the interest rate, even though in both cases the behavior of the inflation, π_k, is similar. Figure 2 contrasts the interest rate policy generated by the CVIU and LQR models with the policy conducted by the Federal Reserve. Note that the CVIU model describes more accurately the Federal Reserve Bank monetary policy than the traditional LQR model.

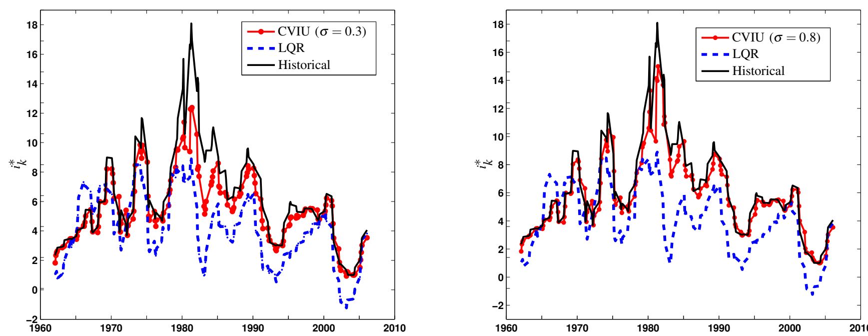
VIII. Conclusions

In this paper, we developed a theoretical framework and the optimal control strategy for CVIU systems associated with a convex cost functional. The convexity of the cost-to-go functions was asserted, making it simpler to obtain the optimal policy using dynamic programming. Since the state equation is not differentiable, an algorithm for determining the optimal policy was described using generalized gradients. The optimal strategy yields that the state-space will be divided into three disjoint regions, representing the regions where the optimal control policy is to increase, decrease, or maintain the previous input. Furthermore, for the case where the Cost Functional is differentiable, it was asserted that the region where no variation is optimal will always exist. This characteristic of the optimal policy is intuitively



(a) (b) (c)

Fig. 1: Behavior of (a) the inflation, π_k, (b) the optimal interest rate, i_k^*, and (c) the optimal interest rate variation, v_k^* for the CVIU model and the LQR. In the CVIU model, $\sigma_\pi = \sigma_y = 0.2$. In both cases, $\overline{\sigma}_\pi = \overline{\sigma}_y = 0.2$.



(a) (b)

Fig. 2: Curves of the historical interest rate values, the interest rate values generated from a standard LQR model ($\overline{\sigma} = 0.8$ and $\sigma = 0$), and the interest rate values obtained using a CVIU model with $\overline{\sigma} = 0.8$ and (a) $\sigma = 0.3$; (b) $\sigma = 0.8$.

sound, since in many real-world problems, in face of the uncertainty generated by changing the control policy, the best strategy is to maintain the same policy as before. The presented CVIU model and analysis can be applied to many practical scenarios, ranging from monetary policy problems to medicine and biology and, in general, to problems where a complete dynamic model is too complex to be feasible.

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