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Recursive Robust Regulator for Discrete-time State-space Systems

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***Abstract*— This paper deals with robust regulation problem for discrete-time linear systems subject to uncertainties. The uncertainties are assumed bounded. A new functional based on the combination of penalty functions and weighted game-type cost function is defined to deal with this problem. The solution provided is based on recursive Riccati equations. An interesting feature of this approach is that the recursiveness can be performed without the need of adjusting auxiliary parameters.**

***Index Terms*— Discrete-time systems, game theory, minimum squares, penalty functions, robust regulators.**

I. Introduction

In the last thirty years, the control community has researched on the robustness of regulators when the systems are subject to uncertainties in the parameter matrices of the state and input variables ([2], [4], [8], [9], [10], [11], [14], [15], [16] [17], [18]). An important tool to solve this class of problem that has been used is based on linear matrix inequalities (LMIs). However, the occurrence of possible unfeasible solutions of the robust regulators, inherent in this kind of approach, is an important and decisive limitation for on-line applications. Futhermore, the sufficient conditions to assure the existence of this kind of regulator are, in general, very conservative when the system is not subject to uncertainties.

A second major approach is to extend the standard Riccati-based design techniques from nominal regulators to robust regulators. Recursiveness and existence of solution are some of the useful characteristics provided by Riccati equations. In adition, when the disturbances are set to zero, the standard solution for the nominal regulator is attained.

In order to obtain robust regulators based on structural features of classical quadratic-cost designs, [13] proposed a framework to design regulators for systems with bounded data uncertainties. The problem solved is formulated in terms of a classical constrained two-player game. The solution provided in this reference depends on the state and on a certain regularization parameter which appears in the solution of the optimization problem defined. This solution however is very hard to implement in practical simulations, since it depends on the future solution of the Riccati recursion.

This paper proposes a new robust control design based on penalty functions and game theory. The solution provided has the advantage of not depending on the future solution of the Riccati recursion as occurs in the solution developed in [13]. Other important feature of this robust regulator is that it does not depend on any parameter to be adjusted, only on the parameters and weighting matrices which are known a-priori.

The notation used in this paper is standard: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the set of n-dimensional vectors whose elements are in $\mathbb{R}$, $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices, A^T is the transpose of the matrix A, $P\succ 0$ ($P\succeq 0$) denotes a positive definite (semi definite) matrix, $a>0$ denotes positive scalar, $\|x\|$ is the Euclidean norm of x, $\|x\|_P$ is the weighted norm of x defined by $(x^TPx)^{\frac{1}{2}}$, $A\bigoplus B$ denotes a diagonal matrix with entries A and B and $A^\dagger$ is a pseudo-inverse of A.

II. Problem Formulation

Consider the following discrete-time linear system subject to uncertainties

$$x_{i+1} = (F_i+\delta F_i)x_i + (G_i+\delta G_i)u_i \tag{1}$$

for all $i=0,\dots,N$, where $F_i\in\mathbb{R}^{n\times n}$ and $G_i\in\mathbb{R}^{n\times m}$ are nominal parameter matrices, $x_i\in\mathbb{R}^n$ is the state vector and $u_i\in\mathbb{R}^m$ is the control input. It is assumed that the initial state x_0 is known. Uncertain matrices $\delta F_i\in\mathbb{R}^{n\times n}$ and $\delta G_i\in\mathbb{R}^{n\times m}$ are unknown matrices modeled as

$$\begin{bmatrix}\delta F_i & \delta G_i\end{bmatrix} = H_i\Delta_i\begin{bmatrix}E_{F_i} & E_{G_i}\end{bmatrix} \tag{2}$$

for all $i=0,\dots,N$, where $H_i\in\mathbb{R}^{n\times k}$, $E_{F_i}\in\mathbb{R}^{l\times n}$, $E_{G_i}\in\mathbb{R}^{l\times m}$ are assumed known and $\Delta_i\in\mathbb{R}^{k\times l}$ with $\|\Delta_i\|\leq 1$.

We consider for each $i=N,\dots,0$, the problem of obtaining the optimal solution (x^*_{i+1},u^*_i) that solves the following min-max optimization problem

$$\min_{x_{i+1},u_i}\max_{\delta F_i,\delta G_i}\left\{J^\mu_i(x_{i+1},u_i)\right\} \tag{3}$$

for all $i=0,\dots,N$, where $J^\mu_i(x_{i+1},u_i)$ is the functional cost given by the following expression

$$\begin{aligned}J^\mu_i(x_{i+1},u_i) = &\begin{bmatrix}x_{i+1}\\u_i\end{bmatrix}^T\begin{bmatrix}P_{i+1} & 0\\0 & R_i\end{bmatrix}\begin{bmatrix}x_{i+1}\\u_i\end{bmatrix}+\\&+\left\{\left(\begin{bmatrix}0 & 0\\I & -G_i\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & -\delta G_i\end{bmatrix}\right)\begin{bmatrix}x_{i+1}\\u_i\end{bmatrix}-\left(\begin{bmatrix}-I\\F_i\end{bmatrix}x_i+\begin{bmatrix}0\\\delta F_i\end{bmatrix}x_i\right)\right\}^T\times\\&\times\begin{bmatrix}Q_i & 0\\0 & \mu^{-2}\Theta^{-1}\end{bmatrix}\left\{\bullet\right\}\end{aligned} \tag{4}$$

with $P_{i+1}\succ 0$, $R_i\succ 0$, $Q_i\succ 0$, $\mu>0$ fixed and $\Theta\succ 0$ with appropriate dimension.

Remark 2.1: The optimization problem (3)-(4) is a particular case of the basic robust least-squares problem to be established in the following section.

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III. The Basic Robust Least-Squares Problem

The preliminary results presented in this section deal with general optimization problems defined for systems subject to uncertainties in the data. Consider the following constrained two-player game problem

$$\min_{x}\max_{\delta A,\,\delta b}\left\{\|x\|_Q^2+\|(A+\delta A)x-(b+\delta b)\|_W^2\right\}; \qquad (5)$$

where $A\in\mathbb{R}^{N\times n}$ is a nominal matrix, $b\in\mathbb{R}^{N\times 1}$ is a measurement vector, x is an n-dimensional unknown vector, $Q\succ 0$ and $W\succ 0$ are weighting matrices, and $\{\delta A,\delta b\}$ are perturbations modeled by

$$\begin{bmatrix}\delta A & \delta b\end{bmatrix} \;=\; H\Delta\begin{bmatrix}E_A & E_b\end{bmatrix},\ \|\Delta\|\le 1. \qquad (6)$$

Assume known A, b, W, Q, E_A, E_b and H.

An interesting solution for this problem was presented in [13]. In virtue of the quantity of applications in which this approach can be useful, we denominate this result as fundamental lemma.

Lemma 3.1: The optimization problem (5)-(6) has a unique solution $\hat{x}$ given by

$$\hat{x} \;=\; \left(\hat{Q}+A^T\hat{W}A\right)^{-1}\left(A^T\hat{W}b+\hat{\lambda}E_A^TE_b\right) \qquad (7)$$

where the modified weighting matrices $\hat{Q}$ and $\hat{W}$ are defined as

$$\hat{Q} \;:=\; Q+\hat{\lambda}E_A^TE_A \qquad (8)$$

$$\hat{W} \;:=\; W+WH\left(\hat{\lambda}I-H^TWH\right)^{-1}H^TW \qquad (9)$$

and $\hat{\lambda}$ is a nonnegative scalar parameter obtained by the following optimization problem

$$\hat{\lambda}:=\arg\min_{\lambda>\|H^TWH\|}G(\lambda) \qquad (10)$$

with

$$G(\lambda):=\|x(\lambda)\|_Q^2+\lambda\,\|E_Ax(\lambda)-E_b\|^2+\|Ax(\lambda)-b\|_{W(\lambda)}^2 \qquad (11)$$

and the auxiliary functions are defined by

$$\begin{aligned} x(\lambda) &:= \left[Q(\lambda)+A^TW(\lambda)A\right]^{-1}\left[A^TW(\lambda)b+\lambda E_A^TE_b\right]\\ Q(\lambda) &:= Q+\lambda E_A^TE_A\\ W(\lambda) &:= W+WH\left(\lambda I-H^TWH\right)^{-1}H^TW. \end{aligned}$$

Proof: See [13].

◇

For the kind of problem we are interested in solving in this paper, it is useful to redefine this fundamental lemma in terms of an array of matrices.

Lemma 3.2: Let R be positive semidefinite and H a full-column rank matrix. Then, if $\begin{bmatrix}R & H\end{bmatrix}$ has full-row rank, the matrix

$$\begin{bmatrix}R & H\\ H^T & 0\end{bmatrix} \qquad (12)$$

is invertible.

Proof: See [12].

◇

In order to redefine the fundamental lemma aforementioned, note that the quadratic functional of the optimization problem defined in (5) can also be rewritten in terms of an array of matrices.

Lemma 3.3: Let $f:\mathbb{R}^n\mapsto\mathbb{R}$ defined by

$$f(x):=\|x\|_Q^2+\|(A+\delta A)x-(b+\delta b)\|_W^2.$$

Consider the constrained two-player game problem

$$\min_x\max_{\delta A,\,\delta b}\{f(x)\};$$

$$\begin{bmatrix}\delta A & \delta b\end{bmatrix}=H\Delta\begin{bmatrix}E_A & E_b\end{bmatrix},\ \|\Delta\|\le 1.$$

The following statements are equivalent:

(i)

$$\begin{array}{c}\hat{x}\in\arg\min_x\max_{\delta A,\,\delta b}\{f(x)\}\\ \begin{bmatrix}\delta A & \delta b\end{bmatrix}=H\Delta\begin{bmatrix}E_A & E_b\end{bmatrix},\ \|\Delta\|\le 1\end{array}; \qquad (13)$$

(ii)

$$\hat{x}\in\arg\min_x\left\{\left(\begin{bmatrix}I\\A\\E_A\end{bmatrix}x-\begin{bmatrix}0\\b\\E_b\end{bmatrix}\right)^T\begin{bmatrix}Q&0&0\\0&\hat{W}&0\\0&0&\hat{\lambda}I\end{bmatrix}\left(\bullet\right)\right\}; \qquad (14)$$

$(iii)\,(\alpha,\beta,\gamma,x)=(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{x})$ is a solution for

$$\begin{bmatrix}Q^{-1}&0&0&I\\0&\hat{W}^{-1}&0&A\\0&0&\hat{\lambda}^{-1}I&E_A\\I&A^T&E_A^T&0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\\x\end{bmatrix}=\begin{bmatrix}0\\b\\E_b\\0\end{bmatrix}, \qquad (15)$$

where, for the items $(ii)-(iii)$, $\hat{W}$ is given by (9) and $\hat{\lambda}$ is a non-negative scalar parameter given by (10).

Furthermore, the unique solution $\hat{x}$ can be given, alternatively, by

$$\hat{x}=\begin{bmatrix}0\\0\\0\\I\end{bmatrix}^T\begin{bmatrix}Q^{-1}&0&0&I\\0&\hat{W}^{-1}&0&A\\0&0&\hat{\lambda}^{-1}I&E_A\\I&A^T&E_A^T&0\end{bmatrix}^{-1}\begin{bmatrix}0\\b\\E_b\\0\end{bmatrix}. \qquad (16)$$

Proof: Omitted.

◇

IV. Penalty Function

Based on [3], penalty functions transform constrained in unconstrained optimization problems. The constraints are placed into the objective function via a penalty parameter in such way that penalizes any violation of the constraints.

Consider the following constrained optimization problem

$$\begin{array}{l}\min_x\{f(x)\}\\ s.t.\ h(x)=0\end{array}, \qquad (17)$$

with optimal solution x^o. Suppose that this problem is replaced by,

$$\min_x\{f(x)+\mu^{-2}h(x)^Th(x)\}, \qquad (18)$$

where μ is a positive real number. For each $\mu > 0$, let $\hat{x}(\mu)$ be the optimal solution to the problem (18). Then,

$$x^o = \lim_{\mu \to 0} \hat{x}(\mu). \tag{19}$$

The term $\mu^{-2} h(x)^T h(x)$ is referred as penalty function. More details on penalty functions can be seen in [1], [3], [7] and [19].

Lemma 4.1: Let $V \in \mathbb{R}^{n\times n}$ positive definite, $G \in \mathbb{R}^{k\times m}$ and $H \in \mathbb{R}^{n\times m}$. Consider the following optimization problem with constraint

$$\begin{array}{c} \min_x \left\{ (Hx - z)^T V (Hx - z) \right\} \\ s.t.\ Gx = u \end{array}, \tag{20}$$

where $z \in \mathbb{R}^n$, $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^k$. Associated with (20) we have the following optimization problem without constraint

$$\hat{x}(\mu) := \arg\min_x \left\{ (\mathscr{G}x - \mathscr{B})^T \mathscr{V}(\mu) (\mathscr{G}x - \mathscr{B}) \right\} \tag{21}$$

where

$$\mathscr{G} = \begin{bmatrix} H \\ G \end{bmatrix},\ \mathscr{V}(\mu) = \begin{bmatrix} V & 0 \\ 0 & \mu^{-2}\Theta^{-1} \end{bmatrix},\ \mathscr{B} = \begin{bmatrix} z \\ u \end{bmatrix},$$

$\mu > 0$ and $\Theta \succ 0$ has an appropriated dimension. Suppose that the matrix $\mathscr{G}$ is full column rank then, the following statements are valid

(i) For each $\mu > 0$, the optimal solution $\hat{x}(\mu)$ with the unconstraint optimization problem (21) is given by

$$\hat{x}(\mu) = \begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} \mathscr{V}^{-1}(\mu) & \mathscr{G} \\ \mathscr{G}^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathscr{B} \\ 0 \end{bmatrix}. \tag{22}$$

(ii) When $\lim_{\mu\to 0} \hat{x}(\mu) = x^o$, where x^o is the optimal solution for (20) given by

$$x^o = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}^T \begin{bmatrix} V^{-1} & 0 & H \\ 0 & 0 & G \\ H^T & G^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} z \\ u \\ 0 \end{bmatrix}. \tag{23}$$

Furthermore,

$$\begin{aligned} &\lim_{\mu\to 0} (\mathscr{G}\hat{x}(\mu) - \mathscr{B})^T \mathscr{V}(\mu) (\mathscr{G}\hat{x}(\mu) - \mathscr{B}) = \\ &= (Hx^o - z)^T V (Hx^o - z). \end{aligned} \tag{24}$$

Proof: Omitted.

$\diamond$

Remark 4.1: Note that the optimal cost (24) obtained depends on x^o and the quadratic term

$$(G\hat{x}(\mu) - u)^T \mu^{-2}\Theta^{-1} (G\hat{x}(\mu) - u) \tag{25}$$

goes to zero when $\mu \to 0$.

V. Nominal Regulator Problem

In this section, we revise the standard regulator for nominal discrete-time state-space systems (without uncertainties). We first show that the same classical solution can be obtained if, in the optimization problem, the minimizing variable is (x_{i+1}, u_i) rather than only u_i.

Consider the following discrete-time linear system

$$x_{i+1} = F_i x_i + G_i u_i \tag{26}$$

for all $i = 0,...,N$, where $x_i \in \mathbb{R}^n$ is an n-dimensional state vector; $u_i \in \mathbb{R}^m$ is an m-dimensional control input; $F_i \in \mathbb{R}^{n\times n}$ and $G_i \in \mathbb{R}^{n\times m}$ are matrices of known nominal parameters; the initial state vector x_0 is considered known and $\{u_i\}_{i=0}^N$ is defined as a sequence of inputs without constraint. Consider also the following standard quadratic functional

$$J = x_{N+1}^T P_{N+1} x_{N+1} + \sum_{j=0}^{N} L_j(x_j, u_j); \tag{27}$$

where

$$L_j(x_j, u_j) = x_j^T Q_j x_j + u_j^T R_j u_j;\ j = 0,...,N \tag{28}$$

and the weighting matrices satisfy the conditions $Q_j \succ 0$, $R_j \succ 0$ and $R_{N+1} = P_{N+1} \succ 0$. The optimization problem widely used to deduce classical regulators is defined in order to determine an optimal control sequence $\{u_i^*\}_{i=0}^N$ which is a solution of

$$\begin{array}{c} \min_{u_i} \{J\}, \\ s.t.\ x_{i+1} = F_i x_i + G_i u_i,\ i = 0,...,N. \end{array} \tag{29}$$

The optimization problem we propose in the following provides an optimal control sequence $\{(x_{i+1}^*, u_i^*)\}_{i=0}^N$ which is a solution for

$$\begin{array}{c} \min_{x_{i+1}, u_i} \{J\}, \\ s.t.\ x_{i+1} = F_i x_i + G_i u_i,\ i = 0,...,N. \end{array} \tag{30}$$

where J is given by (27)-(28). It is important to emphasize that the minimization problem is formulated in terms of u_i and x_{i+1} whose solution is given by the following lemma.

Lemma 5.1: The minimization problem (30) can be solved through the following recursive procedure

$$\begin{aligned} &\min_{x_1, u_0} \Bigg\{ L_0(x_0, u_0) + \min_{x_2, u_1} \Bigg\{ L_1(x_1, u_1) + \\ &+ \ldots + \min_{x_k, u_{k-1}} \Bigg\{ L_{k-1}(x_{k-1}, u_{k-1}) + \ldots + \\ &+ \min_{x_{N+1}, u_N} \Bigg\{ L_N(x_N, u_N) + x_{N+1}^T P_{N+1} x_{N+1} \Bigg\} \Bigg\} \ldots \Bigg\}, \end{aligned} \tag{31}$$

subject to $x_{i+1} = F_i x_i + G_i u_i,\ i = 0,...,N$.

$\diamond$

The proof of this lemma is easily obtained applying dynamic programming [5]. The result of the application of this approach is the classical optimal linear quadratic regulator (LQR) which provides an elegant recursive algorithm, useful to be implemented digitally.

The redefinition of this problem we are proposing is based on the concept of penalty functions and weighted least-squares problem. This concept also follows the idea of quadratic optimization subject to equality constraints.

With lemmas 4.1 and 5.1 in mind, we can define the following alternative unconstrained minimization problem for each step i

$$\min_{x_{i+1}, u_i} \left\{ \begin{bmatrix} x_{i+1} \\ u_i \end{bmatrix}^T \begin{bmatrix} P_{i+1} & 0 \\ 0 & R_i \end{bmatrix} \begin{bmatrix} x_{i+1} \\ u_i \end{bmatrix} + \right. \tag{32}$$

$$+\left(\begin{bmatrix}0 & 0\\ I & -G_i\end{bmatrix}\begin{bmatrix}x_{i+1}\\ u_i\end{bmatrix}-\begin{bmatrix}-I\\ F_i\end{bmatrix}x_i\right)^T\begin{bmatrix}Q_i & 0\\ 0 & \Xi^{-1}\end{bmatrix}\left(\bullet\right)\Bigg\},$$

where the constraints are incorporated in the quadratic form, through of weighting matrix $\Xi = \mu^2\Theta$.

Theorem 5.1: [6] Considering the optimization problem (30), the optimal recursive solution is given through the following linear regulator

$$\begin{bmatrix}x^*_{i+1}\\ u^*_i\end{bmatrix}=\begin{bmatrix}L_i\\ K_i\end{bmatrix}x_i,\ i=0,\dots,N, \tag{33}$$

where

$$\begin{bmatrix}L_i\\ K_i\end{bmatrix}=\begin{bmatrix}0&0&0&0&I&0\\0&0&0&0&0&I\end{bmatrix}\times$$

$$\begin{bmatrix}P_{i+1}^{-1}&0&0&0&I&0\\0&R_i^{-1}&0&0&0&I\\0&0&Q_i^{-1}&0&0&0\\0&0&0&0&I&-G_i\\I&0&0&I&0&0\\0&I&0&-G_i^T&0&0\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\-I\\F_i\\0\\0\end{bmatrix}, \tag{34}$$

$$P_i = L_i^T P_{i+1}L_i + K_i^T R_i K_i + Q_i,\ i=N,\dots,0. \tag{35}$$

The optimal total cost is given by $J^* = x_0^T P_0 x_0$.

◇

As it was pointed out, this section intends only to formulate an alternative procedure to solve the classical optimal linear quadratic regulator. More details on this approach can be seen in [6]. The framework we will provide in the next section to design the robust quadratic regulator for the system subject to uncertainties includes also the fundamental solution for the nominal problem.

VI. Recursive Robust Regulator

The unconstrained minimization problem (32) can be redefined to deduce the robust controller to regulate the System (1). The following identifications are assumed

$$F_i \to F_i + \delta F_i \text{ and } G_i \to G_i + \delta G_i. \tag{36}$$

Remark 6.1: Notice that (3) deals with a particular case of the general optimization problem (5) when are performed the following identifications

$$Q \leftarrow \begin{bmatrix}P_{i+1}&0\\0&R_i\end{bmatrix},\ x\leftarrow\begin{bmatrix}x_{i+1}\\u_i\end{bmatrix},\ W\leftarrow\begin{bmatrix}Q_i&0\\0&\mu^{-2}\Theta^{-1}\end{bmatrix},$$

$$A\leftarrow\begin{bmatrix}0&0\\I&-G_i\end{bmatrix},\ \delta A\leftarrow\begin{bmatrix}0&0\\0&-\delta G_i\end{bmatrix},$$

$$b\leftarrow\begin{bmatrix}-I\\F_i\end{bmatrix}x_i,\ \delta b\leftarrow\begin{bmatrix}0\\ \delta F_i\end{bmatrix}x_i,$$

$$H\leftarrow\begin{bmatrix}0\\H_i\end{bmatrix},\ \Delta\leftarrow\Delta_i,\ E_A\leftarrow\begin{bmatrix}0 & -E_{G_i}\end{bmatrix},\ E_b\leftarrow E_{F_i}x_i. \tag{37}$$

The next lemmas are useful to obtain a recursive robust solution to regulate the System (1).

Lemma 6.1: Let $\mu>0$ and $\Theta\succ 0$. Consider the optimization problem (3). The following statements are equivalent

(i)

$$(\hat{x}_{i+1},\hat{u}_i)\in\arg\min_{x_{i+1},u_i}\left\{(\mathscr{A}_i\mathscr{X}_i-\mathscr{B}_i)^T\mathscr{W}_i(\mathscr{A}_i\mathscr{X}_i-\mathscr{B}_i)\right\} \tag{38}$$

where

$$\mathscr{A}_i=\begin{bmatrix}I&0\\0&I\\0&0\\ \widehat{I}&-\widehat{G}_i\end{bmatrix},\ \mathscr{X}_i=\begin{bmatrix}x_{i+1}\\u_i\end{bmatrix},\ \mathscr{B}_i=\begin{bmatrix}0\\0\\-I\\ \widehat{F}_i\end{bmatrix}x_i,$$

$$\mathscr{W}_i=\begin{bmatrix}P_{i+1}&0&0&0\\0&R_i&0&0\\0&0&\widehat{W}&0\\0&0&0&\widehat{\lambda}_iI\end{bmatrix},\ \widehat{G}_i=\begin{bmatrix}G_i\\E_{G_i}\end{bmatrix},\ \widehat{I}=\begin{bmatrix}I\\0\end{bmatrix},$$

$$\widehat{F}_i=\begin{bmatrix}F_i\\E_{F_i}\end{bmatrix},\ \widehat{W}=\begin{bmatrix}Q_i&0\\0&(\mu^2\Theta-\widehat{\lambda}_i^{-1}HH^T)^{-1}\end{bmatrix}. \tag{39}$$

(ii) $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,x_{i+1},u_i)=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\alpha}_4,\hat{x}_{i+1},\hat{u}_i)$ is the solution for the following linear system

$$\begin{bmatrix}P_{i+1}^{-1}&0&0&0&I&0\\0&R_i^{-1}&0&0&0&I\\0&0&Q_i^{-1}&0&0&0\\0&0&0&\Sigma(\mu,\widehat{\lambda}_i)&\widehat{I}&-\widehat{G}_i\\I&0&0&\widehat{I}^T&0&0\\0&I&0&-\widehat{G}_i^T&0&0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ x_{i+1}\\ u_i\end{bmatrix}=\begin{bmatrix}0\\0\\-I\\ \widehat{F}_i\\0\\0\end{bmatrix}x_i, \tag{40}$$

where $\Sigma(\mu,\widehat{\lambda}_i)=\begin{bmatrix}\mu^2\Theta-\widehat{\lambda}_i^{-1}HH^T & 0\\ 0 & \widehat{\lambda}_i^{-1}I\end{bmatrix}$.

For the items $(i)-(ii)$, $\hat{\lambda}_i$ is a non-negative scalar parameter obtained through the following optimization problem

$$\hat{\lambda}_i := \arg\min_{\lambda_i>\|H^TWH\|} G_i(\lambda_i),$$

with $G_i(\lambda_i)$ given by (11).

Proof: It follows from Lemma 3.3 and Remark 6.1.

◇

Remark 6.2: If we suppose that $(\mu^2\Theta-\widehat{\lambda}_i^{-1}HH^T)\succeq 0$ and E_{G_i}, $i=0,\dots,N$ is full row rank, then the System (40) has a unique solution, according to Lemma 3.2.

Now we are in a position to state the main result of this paper.

Theorem 6.1: Consider the min-max optimization problem (3) with $\mu>0$ fixed. Suppose that $(\mu^2\Theta-\widehat{\lambda}_i^{-1}HH^T)\succeq 0$ and that E_{G_i}, $i=0,\dots,N$ is full row rank. Then, a recursive solution for (3) is given by

$$\begin{bmatrix}x^*_{i+1}\\ u^*_i\end{bmatrix}=\begin{bmatrix}L_i^r\\ K_i^r\end{bmatrix}x_i,\ i=0,\dots,N; \tag{41}$$

$$\begin{bmatrix}L_i^r\\ K_i^r\end{bmatrix}:=\begin{bmatrix}0&0&0&0&I&0\\0&0&0&0&0&I\end{bmatrix}^T\times$$

$$\times \begin{bmatrix} P_{i+1}^{-1} & 0 & 0 & 0 & I & 0 \\ 0 & R_i^{-1} & 0 & 0 & 0 & I \\ 0 & 0 & Q_i^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma_i & \widehat{I} & -\widehat{G}_i \\ I & 0 & 0 & \widehat{I}^T & 0 & 0 \\ 0 & I & 0 & -\widehat{G}_i^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I \\ \widehat{F}_i \\ 0 \\ 0 \end{bmatrix}, \quad (42)$$

$$P_i = L_i^{rT} P_{i+1} L_i^r + K_i^{rT} R_i K_i^r + Q_i + (\widehat{I}L_i^r - \widehat{G}_i K_i^r - \widehat{F}_i)^T \Sigma_i^{-1}(\bullet), \quad (43)$$

for $i = N, \ldots, 0$, where

$$\Sigma_i \equiv \Sigma(\mu, \widehat{\lambda}_i), \ \widehat{G}_i = \begin{bmatrix} G_i \\ E_{G_i} \end{bmatrix}, \ \widehat{F}_i = \begin{bmatrix} F_i \\ E_{F_i} \end{bmatrix}, \ \widehat{I} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (44)$$

and $\widehat{\lambda}_i$ is obtained by the minimization of $G_i(\lambda_i)$ over the interval $(\|H_i^T \mu^{-2} \Theta^{-1} H_i\|, +\infty)$.

Proof: It follows from Lemma 6.1.

◇

Remark 6.3: Notice that in the recursive solution provided in Theorem 6.1, for each step $i = 0, \ldots, N$ the parameter $\widehat{\lambda}_i$ is determined over the interval $(\|H_i^T \mu^{-2} \Theta^{-1} H_i\|, +\infty)$ where the bounds are always well defined. In the solution provided by [13] the optimal parameter $\widehat{\lambda}_i$ should be determined over an interval given by $(a, +\infty)$ where a depends on the solution of the Riccati equation P_{i+1}, which is not useful for on-line applications.

Remark 6.4: Consider that E_{G_i}, $i = 0, \ldots, N$ is full row rank. The optimal solution (x_{i+1}^*, u_i^*), for $\mu \to 0$, is given by

$$\begin{bmatrix} x_{i+1}^* \\ u_i^* \end{bmatrix} = \begin{bmatrix} L_i^r \\ K_i^r \end{bmatrix} x_i, \ i = 0, \ldots, N; \quad (45)$$

$$\begin{bmatrix} L_i^r \\ K_i^r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \times \quad (46)$$

$$\times \begin{bmatrix} P_{i+1}^{-1} & 0 & 0 & 0 & 0 & I & 0 \\ 0 & R_i^{-1} & 0 & 0 & 0 & 0 & I \\ 0 & 0 & Q_i^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & -G_i \\ 0 & 0 & 0 & 0 & 0 & 0 & -E_{G_i} \\ I & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & -G_i^T & -E_{G_i}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I \\ F_i \\ E_{F_i} \\ 0 \\ 0 \end{bmatrix},$$

$$P_i = L_i^{rT} P_{i+1} L_i^r + K_i^{rT} R_i K_i^r + Q_i, \ i = N, \ldots, 0. \quad (47)$$

Note that (45)-(47) do not depend on H_i.

The variable μ can be seem as a robustness parameter under which the regulator can actuate on the level of uncertainties of the System (1). Notice that $\widehat{\lambda}_i \in \left[\|H_i^T \mu^{-2} \Theta^{-1} H_i\|, +\infty\right)$ for each $\mu \in (0, +\infty)$. When $\mu \to 0$ we have in consequence that $\widehat{\lambda}_i \to +\infty$ and $\Sigma(\mu, \widehat{\lambda}_i) \to 0$ ($\lim_{\mu \to 0} \Sigma(\mu, \widehat{\lambda}_i) = 0$). Furthermore, similar to the Remark 4.1 we have that the quadratic term of (43), $(\widehat{I}L_i^r - \widehat{G}_i K_i^r - \widehat{F}_i)^T \Sigma_i^{-1} (\widehat{I}L_i^r - \widehat{G}_i K_i^r - \widehat{F}_i) \to 0$ when $\mu \to 0$. In consequence, the robust state feedback gain K_i^r is obtained such that the eigenvalues of $(F_i + G_i K_i^r)$ are located inside the unit circle and that $E_{F_i} + E_{G_i} K_i^r = 0$ (for future works we intend to formulate a general rule to explain this phenomenon).

Lemma 6.2: Define

$$\Sigma_{1,i} := (\mu^2 \Theta - \Sigma_{2,i} H H^T); \ \Sigma_{2,i} := \widehat{\lambda}_i^{-1} I; \quad (48)$$

$$\Gamma_i := (R_i + G_i^T (\Sigma_{1,i} + P_{i+1}^{-1})^{-1} G_i); \quad (49)$$

$$\Omega_i := (E_{F_i} - E_{G_i} \Gamma_i^{-1} G_i^T (\Sigma_{1,i} + P_{i+1}^{-1})^{-1} F_i). \quad (50)$$

Then, the expression (42) can be rewritten as:

$$\begin{aligned} K_i^r = & -\Gamma_i^{-1} G_i^T (\Sigma_{1,i} + P_{i+1}^{-1})^{-1} F_i - \\ & -\Gamma_i^{-1} E_{G_i}^T (\Sigma_{2,i} + E_{G_i} \Gamma_i^{-1} E_{G_i}^T)^{-1} \Omega_i; \end{aligned} \quad (51)$$

$$\begin{aligned} L_i^r = & P_{i+1}^{-1} (\Sigma_{1,i} + P_{i+1}^{-1} + G_i R_i^{-1} G_i^T)^{-1} F_i - \\ & - P_{i+1}^{-1} (\Sigma_{1,i} + P_{i+1}^{-1})^{-1} G_i \Gamma_i^{-1} E_{G_i}^T (\Sigma_{2,i} + E_{G_i} \Gamma_i^{-1} E_{G_i}^T)^{-1} \Omega_i. \end{aligned} \quad (52)$$

◇

Remark 6.5: Consider (51)-(52) with $E_{F_i} = 0, E_{G_i} = 0$, for all $i = 0, \ldots, N$. When $\mu \to 0$, we obtain the standard regulator for nominal discrete-time state-space systems:

$$K_i = -(R_i + G_i^T P_{i+1} G_i)^{-1} G_i^T P_{i+1} F_i; \quad (53)$$

$$L_i = F_i + G_i K_i; \quad (54)$$

$$P_i = L_i^T P_{i+1} L_i + K_i^T R_i K_i + Q_i. \quad (55)$$

Remark 6.6: Taking into account that K_i^r satisfies $E_{F_i} + E_{G_i} K_i^r = 0$ ($\delta F_i + \delta G_i K_i^r = 0$) when $\mu \to 0$

$$\begin{aligned} x_{i+1} &= [F_i + \delta F_i + (G_i + \delta G_i) K_i^r] x_i, \\ &= (F_i + G_i K_i^r) x_i = L_i^r x_i, \ \ i = 0, \ldots, N. \end{aligned} \quad (56)$$

Then the cost $J_i(.)$ in the interval N is computed as $J(N, x_{i+1}, u_i, \delta F_i, \delta G_i) = x_0^T P_0 x_0$, for all admissible uncertainties $\{\delta F_i, \delta G_i\}$ modeled according to (2).

VII. Numerical Example

Let the System (1) with the following parameter matrices

$$F_i = F = \begin{bmatrix} 1.91 & 0.75 & 0.52 \\ 0 & 1.20 & -0.25 \\ 0 & 0 & 1.4 \end{bmatrix}, \ G_i = G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$H_i = H = \begin{bmatrix} 0.90 \\ 0.25 \\ 1.00 \end{bmatrix}, \ E_{F_i} = E_F = \begin{bmatrix} 1.20 & 3.00 & -1.68 \end{bmatrix},$$

$$E_{G_i} = E_G = \begin{bmatrix} 0.84 & 1.40 & -2.16 \end{bmatrix}, \ -1 \le \Delta_i \le 1$$

and with the following weighting matrices

$$Q_i = Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ R_i = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The simulations we perform in this example are based on Remark 6.4. Firstly, the system subject to uncertainties was controlled through the standard nominal regulator, secondly the system was controlled by the robust regulator and thirdly the system without uncertainties was controlled through the standard nominal regulator.

For each instant i, each curve of the Figures 1 and 2 correspond to the mean of euclidean norms of the states and of the costs calculated over T experiments for N instants ($T = 1000, N = 30$). For each experiment j, the matrix Δ_i ($\|\Delta_i\| \leq 1$) was selected randomly and fixed for each instant i.

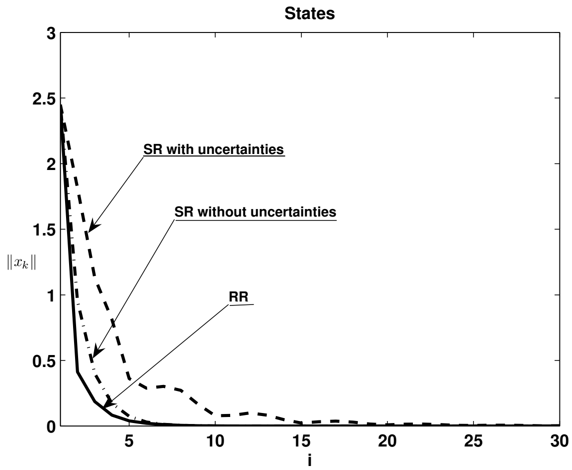

Fig. 1. Robust regulator (RR) proposed for the system with uncertainties ($-$), standard regulator (SR) applied to the system without uncertainties ($- \cdot - \cdot -$) and standard regulator (SR) applied to the system subject to uncertainties ($- - -$).

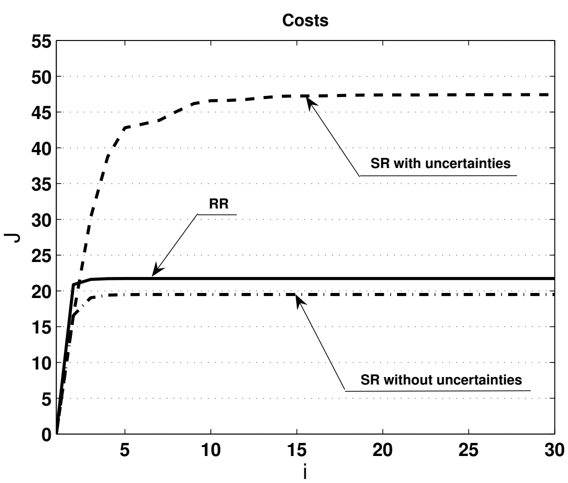

Fig. 2. Costs calculated for the RR (system with uncertainties) ($-$), for the SR (system without uncertainties) ($- \cdot - \cdot -$) and for the SR (system with uncertainties) ($- - -$).

VIII. Conclusion

This paper developed a robust regulator for linear discrete-time systems subject to uncertainties. The main feature of the approach proposed is the recursiveness of the algorithm whose Riccati equation is composed of independent matricial blocks. For each step, the solution depends only on the known parameter matrices of the system and on the weighting matrices.

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