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Iterative Design of Suboptimal Feedback Control for Bilinear Parabolic PDE Systems

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***Abstract*— Optimal control of infinite dimensional systems is one of the central problems in the control of distributed parameter systems. With the development of high performance computers, numerical methods for optimal control design have regained attention and achieved significant progress, mostly in the form of open-loop solutions. We consider in this work an optimal control problem for a bilinear parabolic partial differential equation (PDE) system. Based on the optimality conditions derived from Pontryagin's maximum principle for a reduced-order model, and stated as a two-boundary-value problem, we propose an iterative scheme for suboptimal closed-loop control design. In each iteration step, we take advantage of linear synthesis methods to construct a sequence of controllers. The convergence of the controller sequence is proved in appropriate functional spaces. When compared with previous iterative schemes, the proposed scheme avoids repeated numerical computation of the Riccati equation and therefore reduces significantly the number of ODEs that must be solved at each iteration step. A numerical simulation study shows the effectiveness of this new approach.**

I. Introduction

Physical actuation can appear in parabolic partial differential equations (PDEs) in three different ways: source terms (interior control), boundary conditions (boundary control) and diffusivity coefficient (diffusivity control). Interior and boundary controls have been studied extensively, and many approaches to PDE control have been proposed (e.g., [1], [2] and references therein). Studies on diffusivity control of PDEs are however more scarce (e.g., [3], [4]). In this paper we consider an optimal control problem for a parabolic system with diffusivity and interior actuation mechanisms. We consider a 1D parabolic system over $\Omega = \{(x,t) : 0 \leq x \leq L, t_0 \leq t \leq t_f\}$, which is governed by

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(\zeta \frac{\partial z}{\partial x}\right) + \lambda(x) z + \xi(x) u(t) + v(t) \frac{\partial}{\partial x}\left(\zeta \frac{\partial z}{\partial x}\right), \quad (1)$$

$$z(0,t) = z(L,t) = 0, \; z(x,0) = \varphi(x), \quad (2)$$

where $z(x,t)$ represents the system state, $u(t)$ and $v(t)$ the interior and diffusivity controls respectively, and $\varphi(x)$ the initial distribution. For sake of compatibility, it is necessary to assume that $\varphi(0) = \varphi(L) = 0$. We assume that $\zeta(x)$, $\lambda(x)$ and $\xi(x)$ are positive functions.

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We state an optimal control problem for the parabolic system (1) with the following cost functional

$$\begin{aligned}\min_{u,v} J &= \frac{1}{2}\int_0^L \mathcal{S}(x) z^2(x,t_f) dx \\ &+ \frac{1}{2}\int_\Omega \mathcal{Q}(x) z^2(x,t) dx dt + \frac{1}{2}\int_{t_0}^{t_f} \left(r_u u^2 + r_v v^2\right) dt,\end{aligned} \quad (3)$$

where $\mathcal{S}(x)$ and $\mathcal{Q}(x)$ are positive weighting functions; r_u and r_v are positive definite control weighting factors. In [5], we have demonstrated the existence of solution for this optimal control problem, and obtained open-loop controllers using the Sequential Quadratic Programming (SQP) optimization algorithm. However, the uniqueness of optimal control solution of an arbitrary bilinear infinite dimensional system can not be guaranteed in general because of the convexity limitations due to the bilinearity of the problem. Uniqueness of solution can only be proved under special conditions. For instance, in [6] the authors have proved uniqueness of solution for the optimal control problem of a bilinear distributed parameter system (DPS) only when the initial state satisfies specific smallness conditions. In terms of controllability, it has been demonstrated that bilinear controls can always improve the controllability obtained by just using either interior or boundary controls (see, e.g., [7] and references therein).

Control of bilinear parabolic PDE systems arises in different application scenarios. In the control of the toroidal current density radial profile in magnetically confined fusion plasmas [8], the dynamics of the plasmas transport are governed by a singularly perturbed system (see, e.g., Chapter VI and Chapter VII in [9]). By exploiting the time scale separation in the evolution of the kinetic and magnetic variables, it is possible to obtain a magnetic diffusion equation describing the evolution of the current profile and admiting diffusivity, interior and boundary actuation. Physical actuators such as plasma total current, line-averaged density and non-inductive total power entering the diffusivity-interior-boundary control terms are used to steer the plasma current density to a desired profile in a designated time period [5]. In [10], a saturated flow through a one-dimensional idealized tube packed with soil is considered. The soil contains contaminant samples and a fluid is pumped through the tube (from left to right) to remove the contaminants. The velocity of the fluid pumped into the tube is considered as the control variable which appears as the convective coefficient in the convective-diffusive PDE system governing the contaminant concentration. In [3], the viscosity coefficient is considered as a control function for the Burgers' equation.

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In the design of optimal control strategies for infinite dimensional systems, reduced order modeling techniques play a crucial role. Different numerical methods of lines (MOL), based on the finite element, finite difference and spectral discretization for the spatial coordinate, have been used to compute the optimal open-loop controls for parabolic distributed parameter systems (see, e.g., [11]). In this paper, we use the proper orthogonal decomposition (POD) method to obtain a low dimensional dynamical system (LDDS) for a bilinear parabolic PDE system. The POD method is an efficient reduced order modeling (ROM) technique used to obtain LDDS's from data ensembles which arise from numerical simulation or experimental observation. The POD method has been widely used and proved successful to discover coherent structures from complex physical processes (see, e.g., [12], [13]). In [13], the POD approach is applied to derive a reduced-order model of the Burgers' equation, and then the associated optimal control is solved by using the sequential quadratic programming (SQP) method. Fundamental aspects of POD methods applied to parabolic problems, such as error estimates of Galerkin-POD for both linear and nonlinear parabolic systems, are discussed in [14].

By using the POD model reduction technique, the obtained reduced order system in this work is a bilinear system. Generally, for the numerical solution of the optimal control problem of a finite dimensional bilinear system, a convergent scheme based on quasi-linearization has been proposed in [15], and references therein, to solve the optimality conditions successively. The algorithm in [15], constructs linear systems by updating system and input matrices at each iteration step. The linear state-costate duality structure of the optimality conditions is preserved at each iteration step. Then, Riccati equations are derived to establish successive feedback laws. Similarly, instead of solving a Riccati equation iteratively, a Lyapunov equation is solved at each iteration step in [16]. In this paper, we present a new iteration scheme based on the optimality condition, which introduces an inhomogeneous term in the successive linear state-costate duality structure. In comparison to our previous work [17], the new proposed scheme avoids repeated computations of the Riccati equation at each iteration step by introducing an iterative scheme for the inhomogeneous term involved in the feedback law, and guarantees convergence to the solution of the two-boundary-value problem derived from Pontryagin's principle.

This work represents a novel effort to connect nonlinear parabolic PDE feedback controls and iterative control methodologies using model reduction. The paper is organized as follows. In Section II, we discuss the POD method to obtain reduced order models. In Section III, Galerkin projection is discussed based on a test function set composed by dominant POD modes. In Section IV, we propose an iterative convergent scheme based on the Picard approximation to compute the suboptimal control laws. The convergence of the iteration algorithm is demonstrated in Section V. The simulation studies are presented in Section VI. Section VII closes the paper by stating the conclusions.

II. POD Reduced Order Modeling

Given a collection of functions $\mathcal{V} = \{z(x,t_j)\} = \{z_j(x)\}$, $j = 1, 2, \ldots, n$ on the domain $0 \le x \le L$, the goal of the POD process is to produce an optimal orthogonal set of basis functions $V_{POD} = \{\psi_1(x), \psi_2(x), \ldots, \psi_l(x)\}, (l \le n)$ to approximate the space spanned by the given collection. We will refer to the set $\mathcal{V}$ as the data collection and the set V_{POD} as the POD basis. For any two functions $f_i(x)$ and $f_j(x)$ in either $\mathcal{V}$ or V_{POD}, we define their inner product as $\langle f_i, f_j\rangle = \int_0^L f_i f_j dx$, and the induced norm of any function $f_i(x)$ as $\|f_i\|_{L^2} = \langle f_i, f_i\rangle = \int_0^L f_i^2 dx$. Given any snapshot $z_j(x)$ from the collection set $\mathcal{V}$, we assume that it is possible to form an l-dimensional subspace $V_{POD} = \text{span}\{\psi_1, \psi_2, \ldots, \psi_l\}$ to span it, i.e., $z_j \approx \sum_{i=1}^{l} \langle z_j, \psi_i\rangle \psi_i$. The POD problem is to find the set V_{POD} minimizing the approximation error of $z_j \approx \sum_{i=1}^{l} \langle z_j, \psi_i\rangle \psi_i$, i.e.,

$$\min_{\psi_i} J_b = \sum_{j=1}^{n} \left\| z_j - \sum_{i=1}^{l} \langle z_j, \psi_i\rangle \psi_i \right\|_{L^2}^2 \tag{4}$$

subject to the orthogonality condition

$$\langle \psi_i, \psi_j\rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

We first simplify the cost functional $J_b(\psi_1, \ldots, \psi_l)$,

$$\begin{aligned} & J_b(\psi_1, \ldots, \psi_l) \\ = & \sum_{j=1}^{n} \left\langle z_j - \sum_{i=1}^{l} \langle z_j, \psi_i\rangle \psi_i, z_j - \sum_{i=1}^{l} \langle z_j, \psi_i\rangle \psi_i \right\rangle \\ = & \sum_{j=1}^{n} \left[\langle z_j, z_j\rangle - 2\sum_{i=1}^{l} \langle z_j, \psi_i\rangle^2 + \sum_{i=1}^{l} \langle z_j, \psi_i\rangle^2 \right] \\ = & \sum_{j=1}^{n} \left[\langle z_j, z_j\rangle - \sum_{i=1}^{l} \langle z_j, \psi_i\rangle^2 \right]. \end{aligned} \tag{6}$$

Therefore, to solve the minimization problem (4), it is equivalent to solve the following maximization problem

$$\max_{\psi_i} J_B = \sum_{j=1}^{n} \sum_{i=1}^{l} \langle z_j, \psi_i\rangle^2, \text{ subject to} : \langle \psi_i, \psi_j\rangle = \delta_{ij}. \tag{7}$$

By introducing the operators $\mathcal{K}(x, x') = \sum_{j=1}^{n} z_j(x) z_j(x')$ and $\mathcal{R}\psi = \int_0^L \mathcal{K}(x, x')\psi(x')dx'$, we can rewrite $J_B = \sum_{i=1}^{l} \langle \mathcal{R}\psi_i, \psi_i\rangle$. Therefore, for any POD basis function $\psi \in V_{POD}$, we formulate the following optimization problem

$$\max_{\psi} J_{POD} = \langle \mathcal{R}\psi, \psi\rangle, \quad \text{subject to} : \langle \psi, \psi\rangle = 1. \tag{8}$$

We define the associate Lagrange functional $L_{POD} = \langle \mathcal{R}\psi, \psi\rangle - \widetilde{\lambda}\langle \psi, \psi\rangle$, where $\widetilde{\lambda}$ is an Lagrange multiplier, and assume that $\psi = \psi^* + \eta\psi'$. Then we can compute $L_{POD}(\eta)$, where η is an arbitrary real number and ψ' is an arbitrary variation with respect to the optimal solution $\psi^* \in V_{POD}$. The optimality condition then becomes $\left.\frac{dL_{POD}(\eta)}{d\eta}\right|_{\eta=0} = 2\langle \mathcal{R}\psi^* - \widetilde{\lambda}\psi^*, \psi'\rangle = 0$. We note that ψ' is arbitrary, then the optimality condition becomes the following eigenvalue problem

$$\mathcal{R}\psi = \widetilde{\lambda}\psi, \quad \text{or} \quad \int_0^L \mathcal{K}(x,x')\,\psi(x')dx' = \widetilde{\lambda}\psi(x). \quad (9)$$

For each POD basis function ψ, we assume that it can be expressed by the observations (or snapshots) z_j, $j = 1, 2, \ldots, n$, i.e., $\psi = \sum_{k=1}^n a_k z_k$, which means that it is possible to find a combination of the observation data (i.e., to determine the coefficients a_k) to extract dominant characteristics. Now we substitute the snapshots expansion $\psi = \sum_{k=1}^n a_k z_k$ into (9), then we can obtain

$$\sum_{j=1}^n \left[\sum_{k=1}^n \int_0^L z_j(x') z_k(x') dx' a_k \right] z_j(x) = \widetilde{\lambda} \sum_{j=1}^n a_j z_j(x). \quad (10)$$

By introducing the following matrix notation

$$C_{jk} = \int_0^L z_j(x') z_k(x') dx', \quad \mathbf{a} = [a_1, a_2, \ldots, a_n]^{\mathrm{T}} \quad (11)$$

then we can rewrite (10) as

$$\sum_{j=1}^n \left[\sum_{k=1}^n C_{jk} a_k - \widetilde{\lambda} a_j \right] z_j(\hat{\rho}) = 0, \quad \text{i.e.,} \quad \mathcal{C}\mathbf{a} = \lambda \mathbf{a}, \quad (12)$$

where $\mathcal{C} = [C_{jk}] \in \mathbf{R}^{n\times n}$. Since $\mathcal{C}$ is a nonnegative Hermitian matrix, i.e., $\mathcal{C} = \mathcal{C}^T$, it has a complete set of orthogonal eigenvectors $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and each POD basis function can be expressed as $\psi_i = [z_1, \ldots, z_n]\,\mathbf{a}_i$, $i = 1, 2, \ldots, l$.

III. POD/Galerkin Method

We let $\psi_j \in V_{POD}^*$ be the test function, where $V_{POD}^* = \mathrm{span}\{\psi_1, \ldots, \psi_l\}$ is the test function space spanned by the POD modes. Then, we multiply both sides of (1) by the test function $\psi_j(x) \in V_{POD}^*$, for $j = 1, \ldots, l$, and integrate by parts taking into account that $\psi_j(0) = \psi_j(L) = 0$, to obtain the following weak form

$$\begin{aligned} &\int_0^L \frac{\partial z}{\partial t}\psi_j(x)dx + (1+v)\int_0^L \zeta \frac{\partial z}{\partial x}\frac{\partial \psi_j}{\partial x}dx \\ &= \int_0^L \xi(x) u \psi_j(x) dx + \int_0^L \lambda(x) z(x,t)\psi_j(x)dx. \end{aligned} \quad (13)$$

We implement the Galerkin approximation $z(x,t) \approx y(x,t) = \sum_{k=1}^l \alpha_k(t)\psi_k(x)$ and substitute this expression for $z(x,t)$ into the weak form (13). Then, we can obtain the following finite dimensional system:

$$\frac{d\mathrm{y}}{dt} = (K+G)\mathrm{y} + K\mathrm{y}v(t) + Fu(t), \quad (14)$$

where

$$M_{jk} = \int_0^1 \psi_j(x)\psi_k(x)dx = \delta_{jk}, K_{jk} = -\int_0^1 \frac{\partial(\zeta\psi_j)}{\partial x}\frac{\partial \psi_k}{\partial x}dx, \quad (15)$$

$$F_j = \int_0^1 \xi(x)\psi_j(x)dx, G_{jk} = \int_0^1 \lambda(x)\psi_j(x)\psi_k(x)dx, \quad (16)$$

where $\mathrm{y}(t) = (\alpha_1(t), \ldots, \alpha_l(t))^T \in \mathbb{R}^l$, $G, K \in \mathbb{R}^{l\times l}$. The vector $\mathrm{y}(t)$ is the finite dimensional approximation, with respect to the obtained POD modes, of the variable $z(x,t)$ in (1). The initial values are given by $\alpha_j(0) = \langle z(\cdot,0), \psi_j \rangle$, $j = 1, 2, \cdots, l$.

IV. Bilinear Quadratic Optimal Control

The finite horizon optimal control problem defined in (3) can now be rewritten as

$$\min_{u,v} J = \frac{1}{2}\mathrm{y}^T(t_f)S\mathrm{y}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathrm{y}^T(t)Q\mathrm{y}(t) + r_u^2 u^2 + r_v v^2\right] dt,$$

where $S_{ij} = \int_0^L \mathcal{S}(x)\psi_i(x)\psi_j(x)dx$ and $Q_{ij} = \int_0^L \mathcal{Q}(x)\psi_i(x)\psi_j(x)dx$, $i, j = 1, \ldots, l$.

Introducing the Lagrange multiplier $\mathrm{p} \in \mathbb{R}^l$, we can define the system Hamiltonian $\mathcal{H}(\mathrm{y}, u, v, \mathrm{p}) = \frac{1}{2}(\mathrm{y}^T Q \mathrm{y} + r_u u^2 + r_v v^2) + \mathrm{p}^T (A\mathrm{y} + Fu + K\mathrm{y}v)$, where $A = K + G$. The minimizing control law is given by

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow u^*(t) = -r_u^{-1}F^T\mathrm{p}, \\ \dfrac{\partial \mathcal{H}}{\partial v} = 0 \Rightarrow v^*(t) = -r_v^{-1}(K\mathrm{y})^T\mathrm{p}. \end{cases} \quad (17)$$

Thus, using the maximum principle, a canonical optimality condition can be obtained,

$$\begin{cases} \dot{\mathrm{y}} = \dfrac{\partial \mathcal{H}}{\partial \mathrm{p}} = A\mathrm{y} - Fr_u^{-1}F^T\mathrm{p} - K\mathrm{y}r_v^{-1}(K\mathrm{y})^T\mathrm{p}, \\ \dot{\mathrm{p}} = -\dfrac{\partial \mathcal{H}}{\partial \mathrm{y}} = -Q\mathrm{y} - A^T\mathrm{p} + r_v^{-1}(K\mathrm{y})^T\mathrm{p}K^T\mathrm{p}, \\ \mathrm{y}(t_0) = \mathrm{y}_0, \quad \mathrm{p}(t_f) = S\mathrm{y}(t_f). \end{cases} \quad (18)$$

which is a nonlinear two-point boundary value problem (TBVP) and usually impossible to be solved explicitly.

To compute the optimal control for the bilinear system (14), we propose the following successive scheme based on the Picard approximation,

$$\dot{\mathrm{y}}^{(k+1)} = A\mathrm{y}^{(k+1)} - W\mathrm{p}^{(k+1)} - G^{(k)}, \quad (19)$$

$$\dot{\mathrm{p}}^{(k+1)} = -Q\mathrm{y}^{(k+1)} - A^T\mathrm{p}^{(k+1)} + H^{(k)}, \quad (20)$$

$$\mathrm{y}^{(k+1)}(t_0) = \mathrm{y}_0, \quad \mathrm{p}^{(k+1)}(t_f) = S\mathrm{y}^{(k+1)}(t_f), \quad (21)$$

where the superscript (k) denotes the iteration number and $W = Fr_u^{-1}F^T = W^T$, $G^{(k)} = K\mathrm{y}^{(k)}r_v^{-1}\left[K\mathrm{y}^{(k)}\right]^T\mathrm{p}^{(k)}$, $H^{(k)} = r_v^{-1}\left[K\mathrm{y}^{(k)}\right]^T\mathrm{p}^{(k)}K^T\mathrm{p}^{(k)}$. To solve the linear two boundary value problem (19)-(21), it is standard to assume $\mathrm{p}^{(k+1)} = P\mathrm{y}^{(k+1)} + q^{(k+1)}$, $P^T = P$ and to obtain the equations

$$\begin{aligned} &\dot{P} = -PA - A^TP + PWP - Q, \quad P(t_f) = S, \\ &\dot{q}^{(k+1)} = -(A - WP)^Tq^{(k+1)} + PG^{(k)} + H^{(k)}, \\ &q^{(k+1)}(t_f) = 0, \end{aligned} \quad (22)$$

where

$$G^{(k)} = r_v^{-1}K\mathrm{y}^{(k)}\left[K\mathrm{y}^{(k)}\right]^T\left[P\mathrm{y}^{(k)} + q^{(k)}\right],$$

$$H^{(k)} = r_v^{-1}\left[K\mathrm{y}^{(k)}\right]^T\left[P\mathrm{y}^{(k)} + q^{(k)}\right]K^T\left[P\mathrm{y}^{(k)} + q^{(k)}\right].$$

Then, at each iteration step, the quasi-closed-loop system becomes

$$\begin{aligned} &\dot{\mathrm{y}}^{(k+1)} = \left(A - W^TP\right)\mathrm{y}^{(k+1)} - Wq^{(k+1)} - G^{(k)}, \\ &\mathrm{y}^{(k+1)}(t_0) = \mathrm{y}_0. \end{aligned} \quad (23)$$

When the iteration index (k) is large enough, we can achieve the following feedback laws,

$$u=-r_u^{-1}F^T(P\mathrm{y}+q^*)\,,\; v=-r_v^{-1}(K\mathrm{y})^T(P\mathrm{y}+q^*)\,, \quad (24)$$

where $\lim_{k\to\infty} q^{(k)} = q^*$.

Remark 1: The solution of the Riccati matrix equation (P-equation) actually requires the solution of l^2 coupled ODEs, where l denotes the system dimension. The advantage of this new algorithm resides on the fact that it is not necessary to compute the Riccati equation in each iteration step. Only the vector equation for the feed-forward control term (q-equation) needs to be solved iteratively in each step. However, the solution of this equation requires the solution of only l coupled ODEs.

V. Convergence Study

In the rest of this section, it remains to prove the convergence of the iteration scheme in solving the optimal control problem. Namely, we will show the following limits in appropriate functional spaces

$$\lim_{k\to\infty} \mathrm{y}^{(k)} = \mathrm{y}^*,\; \lim_{k\to\infty} q^{(k)} = q^*. \quad (25)$$

The associated spaces are three Banach spaces (see, e.g., [15], [18]) $\mathfrak{B}_1 = C([t_0,t_f],\mathbb{R}^l)$, $\mathfrak{B}_2 = C([t_0,t_f],\mathbb{R}^{l\times l})$, $\mathfrak{B}_3 = C([t_0,t_f],\mathbb{R}^l)$ with norms $\|\mathrm{y}\|_{\mathfrak{B}_1} = \sup_{s\in[t_0,t_f]}\|\mathrm{y}(s)\|$, $\|P\|_{\mathfrak{B}_2} = \sup_{s\in[t_0,t_f]}\|P(s)\|$ and $\|q\|_{\mathfrak{B}_3} = \sup_{s\in[t_0,t_f]}\|q(s)\|$, where $\|\mathrm{y}\| = \sqrt{\sum_{i=1}^l \mathrm{y}_i^2}$, $\|P\| = \sqrt{\sum_{i,j=1}^l P_{ij}^2}$ and $\|q\| = \sqrt{\sum_{i=1}^l q_i^2}$. To show (25), we only need to show that both $\{\mathrm{y}^{(k)}\}$ and $\{P^{(k)}\}$ are Cauchy sequences. Thus, the convergence follows due to the completeness of the Banach spaces. The convergence proof is based on the contraction mapping theorem for Banach spaces [19].

Theorem 1: If the control weight factor r_v is large enough, then the iteration scheme is convergent, i.e., $\lim_{k\to\infty}\mathrm{y}^{(k)} = \mathrm{y}^*$, $\lim_{k\to\infty} q^{(k)} = q^*$.

Proof: By direct computations, we can obtain

$$\begin{aligned}&\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)} \\ &=-\int_{t_0}^{t} e^{(A-WP)(t-\tau)}\Big\{W\Big[q^{(k+1)}-q^{(k)}\Big]+\Big[G^{(k)}-G^{(k-1)}\Big]\Big\}\,d\tau,\end{aligned} \quad (26)$$

and

$$\begin{aligned}q^{(k+1)}-q^{(k)}=&-\int_t^{t_f} e^{(A-WP)^T(t-\tau)}P\left[G^{(k)}-G^{(k-1)}\right]d\tau \\ &-\int_t^{t_f} e^{(A-WP)^T(t-\tau)}\left[H^{(k)}-H^{(k-1)}\right]d\tau.\end{aligned} \quad (27)$$

Then, we compute the norms,

$$\left\|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right\|\le\int_{t_0}^{t_f}\!\Big(\gamma_1\left\|q^{(k+1)}-q^{(k)}\right\|+\gamma_2\left\|G^{(k)}-G^{(k-1)}\right\|\Big)d\tau,$$

$$\left\|q^{(k+1)}-q^{(k)}\right\|\le\int_{t_0}^{t_f}\!\Big(\gamma_3\left\|G^{(k)}-G^{(k-1)}\right\|+\gamma_4\left\|H^{(k)}-H^{(k-1)}\right\|\Big)d\tau,$$

where $\gamma_1(t) = \left\|e^{(A-WP)(t-\tau)}W\right\|$, $\gamma_2(t) = \left\|e^{(A-WP)(t-\tau)}\right\|$, $\gamma_3(t) = \left\|e^{(A-WP)^T(t-\tau)}P\right\|$ and $\gamma_4(t) = \left\|e^{(A-WP)^T(t-\tau)}\right\| = \gamma_2(t)$.

We rewrite $G^{(k)}=r_v^{-1}K\mathrm{Y}^{(k)}K^T\left[P\mathrm{y}^{(k)}+q^{(k)}\right]$ and $H^{(k)}=r_v^{-1}\delta_{\mathrm{yy}}^{(k)}K^T\left[P\mathrm{y}^{(k)}+q^{(k)}\right]+r_v^{-1}\delta_{\mathrm{y}q}^{(k)}K^T\left[P\mathrm{y}^{(k)}+q^{(k)}\right]$, where $\mathrm{Y}^{(k)} = \mathrm{y}^{(k)}\left[\mathrm{y}^{(k)}\right]^T$, $\delta_{\mathrm{yy}}^{(k)} = \left[\mathrm{y}^{(k)}\right]^T K^TP\mathrm{y}^{(k)}$ and $\delta_{\mathrm{y}q}^{(k)} = \left[\mathrm{y}^{(k)}\right]^T K^Tq^{(k)}$. Now we evaluate $G^{(k)}-G^{(k-1)}$ and $H^{(k)}-H^{(k-1)}$ in terms of $q^{(k)}-q^{(k-1)}$ and $\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}$,

$$\begin{aligned}&\frac{\left\|G^{(k)}-G^{(k-1)}\right\|}{r_v^{-1}} \\ &\le\left\|K\left[\mathrm{Y}^{(k)}-\mathrm{Y}^{(k-1)}\right]K^T\left[P\mathrm{y}^{(k)}+q^{(k)}\right]\right\| \\ &+\left\|K\mathrm{Y}^{(k-1)}K^TP\left[\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right]\right\| \\ &+\left\|K\mathrm{Y}^{(k-1)}K^T\left[q^{(k)}-q^{(k-1)}\right]\right\|,\end{aligned} \quad (28)$$

where $\mathrm{Y}^{(k)} - \mathrm{Y}^{(k-1)} = \left[\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right]\left[\mathrm{y}^{(k)}\right]^T + \mathrm{y}^{(k-1)}\left[\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right]^T$. Then, we have

$$\frac{\left\|G^{(k)}-G^{(k-1)}\right\|}{r_v^{-1}}\le\gamma_5^{(k)}\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|+\gamma_6^{(k)}\left\|q^{(k)}-q^{(k-1)}\right\|,$$

where $\gamma_5^{(k)}(t) = \left[\|\mathrm{y}^{(k)}\|+\|\mathrm{y}^{(k-1)}\|\right]\|K\|^2\left\|P\mathrm{y}^{(k)}+q^{(k)}\right\|+\|K\|^2\|P\|\left\|\mathrm{Y}^{(k-1)}\right\|$, $\gamma_6^{(k)}(t) = \|K\|^2\left\|\mathrm{Y}^{(k-1)}\right\|$.

Similarly, we have

$$\begin{aligned}&\frac{\left\|H^{(k)}-H^{(k-1)}\right\|}{r_v^{-1}} \\ &\le\left|\delta_{\mathrm{yy}}^{(k)}-\delta_{\mathrm{yy}}^{(k-1)}+\delta_{\mathrm{y}q}^{(k)}-\delta_{\mathrm{y}q}^{(k-1)}\right|\|K^T\|\left\|P\mathrm{y}^{(k)}+q^{(k)}\right\| \\ &+\left[\left|\delta_{\mathrm{yy}}^{(k-1)}\right|+\left|\delta_{\mathrm{y}q}^{(k-1)}\right|\right]\|K^TP\|\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\| \\ &+\left[\left|\delta_{\mathrm{yy}}^{(k-1)}\right|+\left|\delta_{\mathrm{y}q}^{(k-1)}\right|\right]\|K^T\|\left\|q^{(k)}-q^{(k-1)}\right\|.\end{aligned} \quad (29)$$

Noting $\delta_{\mathrm{yy}}^{(k)} - \delta_{\mathrm{yy}}^{(k-1)} = \left[\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right]^T K^TP\mathrm{y}^{(k)} + \left[\mathrm{y}^{(k-1)}\right]^T K^TP\left[\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right]$ and $\delta_{\mathrm{y}q}^{(k)} - \delta_{\mathrm{y}q}^{(k-1)} = \left[\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right]^T K^Tq^{(k)} + \left[\mathrm{y}^{(k-1)}\right]^T K^T\left[q^{(k)}-q^{(k-1)}\right]$, then we can obtain

$$\frac{\left\|H^{(k)}-H^{(k-1)}\right\|}{r_v^{-1}}\le\gamma_7^{(k)}\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|+\gamma_8^{(k)}\left\|q^{(k)}-q^{(k-1)}\right\|,$$

where $\gamma_7^{(k)}(t)$, $\gamma_8^{(k)}(t)$ can be obtained by direct computations

$$\begin{aligned}&\gamma_7^{(k)}(t) \\ &=\|K^T\|\left\|P\mathrm{y}^{(k)}+q^{(k)}\right\|\|K^TP\|\left(\|\mathrm{y}^{(k)}\|+\|\mathrm{y}^{(k-1)}\|\right) \\ &+\left[\left|\delta_{\mathrm{yy}}^{(k)}\right|+\left|\delta_{\mathrm{y}q}^{(k-1)}\right|\right]\|K^TP\|+\|K\|\|P\mathrm{y}^{(k)}+q^{(k)}\|\|K^Tq^{(k)}\|,\end{aligned} \quad (30)$$

$$\begin{aligned}&\gamma_8^{(k)}(t) \\ &=\|K^T\|\left\|P\mathrm{y}^{(k)}+q^{(k)}\right\|\|K^TP\|\|\mathrm{y}^{(k-1)}\| \\ &+\left[\left|\delta_{\mathrm{yy}}^{(k)}\right|+\left|\delta_{\mathrm{y}q}^{(k-1)}\right|\right]\|K^T\|.\end{aligned} \quad (31)$$

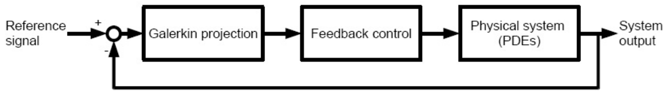

Fig. 1. Closed-loop control system.

Therefore, by taking $\mathfrak{B}$-norms both sides, we obtain

$$\begin{bmatrix} \|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\|_{\mathfrak{B}_1} \\ \|q^{(k+1)}-q^{(k)}\|_{\mathfrak{B}_3} \end{bmatrix} \leq \frac{T}{r_v} \begin{bmatrix} \|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\|_{\mathfrak{B}_1} \\ \|q^{(k)}-q^{(k-1)}\|_{\mathfrak{B}_3} \end{bmatrix} \quad (32)$$

where the elements of the transform matrix T are given by

$$T_{11} = \max_{\tau\in[t_0,t_f]} \left[\gamma_2(\tau)\gamma_5^{(k)}(\tau)\right], \quad (33)$$

$$T_{12} = \max_{\tau\in[t_0,t_f]} \left[\gamma_1(\tau)+\gamma_2(\tau)\gamma_6^{(k)}(\tau)\right], \quad (34)$$

$$T_{21} = \max_{\tau\in[t_0,t_f]} \left[\gamma_3(\tau)\gamma_5^{(k)}(\tau)+\gamma_4(\tau)\gamma_7^{(k)}(\tau)\right], \quad (35)$$

$$T_{22} = \max_{\tau\in[t_0,t_f]} \left[\gamma_3(\tau)\gamma_6^{(k)}(\tau)+\gamma_4(\tau)\gamma_8^{(k)}(\tau)\right]. \quad (36)$$

Therefore, if all of the eigenvalues of T, $\sigma(T)$ satisfy $r_v^{-1}\max|\sigma(T)| < 1$, then we can conclude that the sequences $\{\mathrm{y}^{(k)}\}$ and $\{q^{(k)}\}$ are convergent. ∎

Remark 2: In the proof of Theorem 1, we note that the transformation matrix T calculated in (33)-(36) depends on the iteration index (k) and also includes the evolutions of y and q. Although it is difficult to compute the eigenvalues of T explicitly in each iteration step, to ensure convergence of the iteration scheme we can just make the control weighting factor r_v large enough. It is possible to prove that a large enough r_v also guarantees boundness for the matrix T. Increasing the value of r_v is also a way to ensure $|v| < 1$.

VI. Simulation Study

Closing the control loop with the iteration-based feedback laws is not as direct as in the finite dimensional case (see, Fig. 1). After the N-th iteration, we can obtain the feedback controllers

$$u^{(N)}=-r_u^{-1}F^T\left[P\mathrm{y}+q^{(N)}\right], v^{(N)}=-r_v^{-1}(K\mathrm{y})^T\left[P\mathrm{y}+q^{(N)}\right],$$

based on (24), where $y(t)$ is the finite-dimensional approximation, with respect to the l POD modes, of $z(x,t)$. Before being able to substitute the feedback laws into the original system (1), with the physical domain defined over $(0 \leq x \leq L = 1)$, we need to rewrite the control laws in terms of $z(x,t)$, or at least in terms of a higher-order approximation $\mathbf{Y}(t)$ of $z(x,t)$.

We use the pseudo-spectral method to simulate the nonlinear PDE system. Assuming that the evolution can be expanded by a series of harmonic functions, $z(x,t) \approx \sum_{j=1}^{e}\beta_j(t)\phi_j(x)$, where $\phi_j(x) = \sin(j\pi x)$, then we can derive a higher-order finite dimensional system using the Galerkin projection method

$$\mathbf{M}\frac{d\mathbf{Y}}{dt} = \mathbf{AY} + \mathbf{KY}v^{(N)} + \mathbf{F}u^{(N)}, \quad (37)$$

where the system state vector is defined by $\mathbf{Y} = [\beta_1,\ldots,\beta_e]^T$, with $e > l$. The system matrices can be obtained by following the same lines of (15)-(16) by replacing the POD modes with harmonics basis functions. By noting that

$$\alpha_i = \int_0^1 z(x,t)\psi_i(x)dx = \sum_{j=1}^{e}\beta_j\int_0^1\phi_j(x)\psi_i(x)dx$$

and introducing $\mathbf{C} \in \mathbf{R}^{l\times e}$, $[\mathbf{C}]_{ij} = \int_0^1\phi_j(x)\psi_i(x)dx$, then we have $\mathrm{y} = \mathbf{CY}$. Thus, we can formulate the feedback laws in terms of the new state vector $\mathbf{Y}$,

$$u=-r_u^{-1}F^T\left(P\mathbf{CY}+q^*\right), v=-r_v^{-1}(K\mathbf{CY})^T\left(P\mathbf{CY}+q^*\right).$$

Therefore, the closed loop system becomes

$$\begin{aligned}\mathbf{M}\frac{d\mathbf{Y}}{dt} &= \mathbf{AY} - r_u^{-1}\mathbf{F}F^T\left(P\mathbf{CY}+q^*\right) \\ &\quad - r_v^{-1}\mathbf{KY}(K\mathbf{CY})^T\left(P\mathbf{CY}+q^*\right). \end{aligned} \quad (38)$$

We first simulate the system (1) over $t_0 = 0 \leq t \leq t_f = 50$ with $\zeta(x) = 10^{-3}$, $\xi(x) = \sin(\pi x)$, $\lambda(x) \equiv 0$, $\varphi(x) = \sum_{k=1}^{5}\sin(k\pi x)$ and $u(t) = v(t) = 0$ to obtain the POD modes. The system evolution and the dominant POD modes are shown in Fig. 2 and Fig. 3, respectively. By using the first four POD modes ($l = 4$) we can construct a bilinear system and the approximation error is shown in Fig. 4. In validating the iteration algorithm, we choose $r_u = 1$, $r_v = 15$, $S = 0.5\mathbf{I}$ and $Q = 0.01\mathbf{I}$. The iteration scheme converges and the obtained feedback laws can enhance the dissipation of the system evolution. The simulation of the evolution of the closed-loop PDE system is shown in Fig. 5 using 12 sine wave basis functions in the pseudo-spectral approximation. A comparison of the spatial profiles of the controlled and uncontrolled cases at the final time t_f is shown in Fig. 6.

VII. Conclusions

In this paper we study a controlled parabolic system with two types of actuation: diffusivity and interior controls. By using the POD technique, we derive a low dimensional dynamical system which governs the dominant dynamics of the original parabolic system. The reduced order system is of a bilinear form. We propose a convergent successive scheme based on the Picard approximation to compute the solution of a finite-time sub-optimal control defined for the reduced-order bilinear system. This new algorithm avoids repeated numerical computation of the Riccati equation at each iteration step by introducing an iteration scheme for the feed-forward control component. In terms of the number of ODEs required to solve the Riccati matrix equation (P-equation) and the feed-forward vector equation (q-equation), this new method can decrease the number of ODEs to be computed at each iteration step from l^2 to l. Simulation studies show the effectiveness of the model reduction technique and the successive sub-optimal control laws.

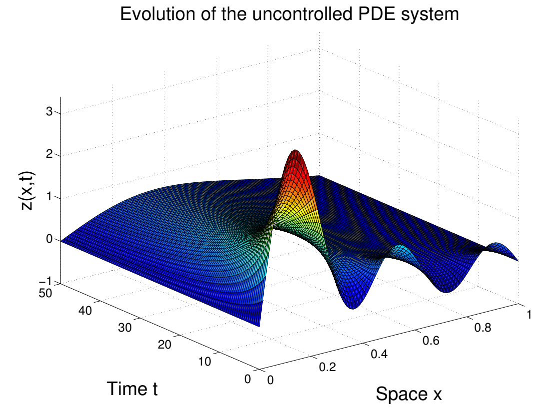

Fig. 2. Uncontrolled dynamics of $z(x,t)$ in system (1).

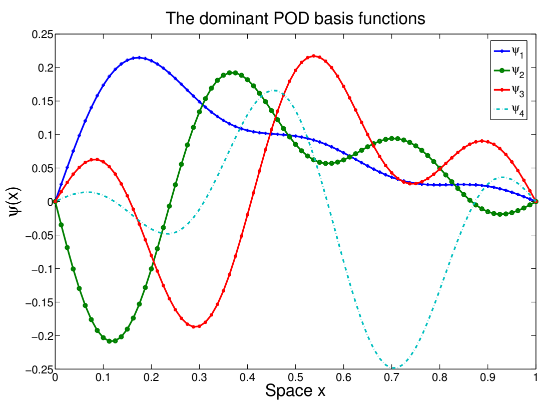

Fig. 3. The first four ($l = 4$) dominant POD modes.

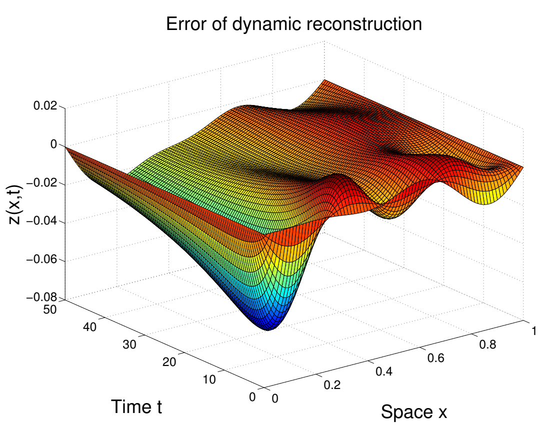

Fig. 4. Error between PDE and reduced-order ODE.

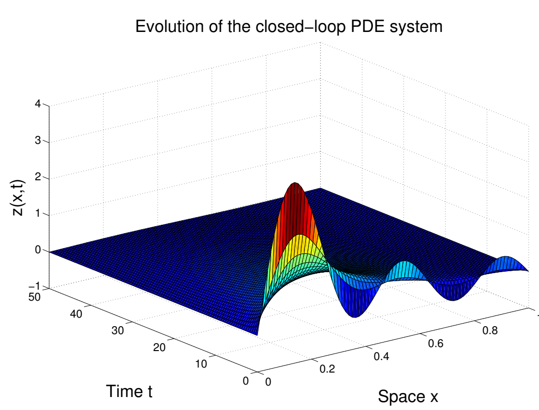

Fig. 5. Closed-loop dynamics of $z(x,t)$.

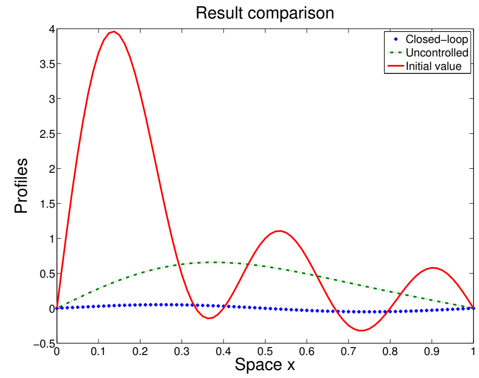

Fig. 6. Comparison of final spatial profiles $z(x,t_f)$ ($t_f = 50$). The initial spatial profile $z(x,t_0)$ ($t_0 = 0$) is also shown.

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