2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Dynamic Vehicle Routing with Moving Demands – Part I: Low speed demands and high arrival rates

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Abstract— **We introduce a dynamic vehicle routing problem in which demands arrive uniformly on a segment and via a temporal Poisson process. Upon arrival, the demands translate perpendicular to the segment in a given direction and with a fixed speed. A service vehicle, with speed greater than that of the demands, seeks to serve these translating demands. For the existence of any stabilizing policy, we determine a necessary condition on the arrival rate of the demands in terms of the problem parameters: (i) the speed ratio between the demand and service vehicle, and (ii) the length of the segment on which demands arrive. Next, we propose a novel policy for the vehicle that involves servicing the outstanding demands as per a translational minimum Hamiltonian path (TMHP) through the moving demands. We derive a sufficient condition on the arrival rate of the demands for stability of the TMHP-based policy, in terms of the problem parameters. We show that in the limiting case in which the demands move much slower than the service vehicle, the necessary and the sufficient conditions on the arrival rate are within a constant factor.**

I. Introduction

Vehicle routing is concerned with planning optimal vehicle routes for providing service to a given set of customers. In contrast, Dynamic Vehicle Routing (DVR) considers scenarios in which not all customer information is known *a priori*, and thus routes must be re-planned as new customer information becomes available. An early DVR problem is the Dynamic Traveling Repairperson Problem (DTRP) [1], in which service requests, or demands arrive sequentially in a region and a service vehicle seeks to serve them by reaching each demand location. In this two-part paper, we introduce a DVR problem in which the demands move with a specified velocity upon arrival. This problem has applications in areas such as perimeter defense, wherein the demands are moving targets trying to cross a region under surveillance by a UAV. Another application is in the automation industry where the demands are objects arriving on a conveyor belt and a robotic arm seeks to perform a pick-and-place operation on them.

The goal in the DTRP [1] is to minimize the expected time spent by each demand before being served. In [1], the authors propose a policy that is optimal in the case of low arrival rate, and several policies within a constant factor of the optimal in the case of high arrival rate. In [2], they study multiple service vehicles, and vehicles with finite service capacity. In [3], a single policy is proposed which is optimal for the case of low arrival rate and performs within a constant factor of the best known policy for the case of high arrival rate. In [4], decentralized policies are developed for the multiple service vehicle versions of the DTRP.

The Euclidean Traveling Salesperson Problem (ETSP) consists of determining the minimum length tour through a given set of static points in a region [5]. Vehicle routing with objects moving on straight lines was introduced in [6], in which a fixed number of objects move in the negative y-direction with fixed speed, and the motion of the service vehicle is constrained to be parallel to either the x- or the y-axis. For a version of this problem wherein the vehicle has arbitrary motion, termed as the translational Traveling Salesperson Problem, a polynomial-time approximation scheme was presented in [7] to catch all objects in minimum time. [8] and [9] address other versions of ETSP with moving objects.

We introduce a dynamic vehicle routing problem in which demands arrive uniformly on a segment of length W, via a temporal Poisson process with rate λ. Upon arrival, the demands translate in a fixed direction perpendicular to the line and with a fixed speed $v < 1$. A service vehicle, modeled as a first-order integrator with unit speed, seeks to serve these mobile demands. Our contributions are as follows. First, we derive a necessary condition on the arrival rate for the existence of a stabilizing policy, i.e., a finite expected time spent by a demand in the environment. Second, we propose a novel policy which involves servicing all of the outstanding demands as per a translational minimum Hamiltonian path (TMHP) through them. We derive a sufficient condition for stability of this TMHP-based policy, and also obtain an upper bound on the steady-state expected time a demand spends in the environment. As the arrival rate $\lambda \to +\infty$, the necessary stability condition implies that the demands must have $v \to 0^+$. This regime of low demand speed is the focus of this paper. In this regime, we show that the sufficient stability condition for the TMHP-based policy is within a constant factor of the necessary condition for stability.

In companion paper [10], we analyze a first-come-first-served (FCFS) policy in which the demands are served in the order of their arrival. We show that in the regime of $\lambda \to 0^+$, the FCFS policy minimizes the expected time spent by a demand before being served; while in the regime of $v \to 1^-$, we show that every stable policy must serve demands in the FCFS order, and hence FCFS is optimal. Thus, for low demand speeds the TMHP-based policy can stabilize higher arrival rates, while for high demand speeds the FCFS can stabilize higher arrival rates. This is summarized in Fig. 1.

This material is based upon work supported in part by ARO-MURI Award W911NF-05-1-0219, ONR Award N00014-07-1-0721 and by the Institute for Collaborative Biotechnologies through the grant DAAD19-03-D-0004 from the U.S. Army Research Office.

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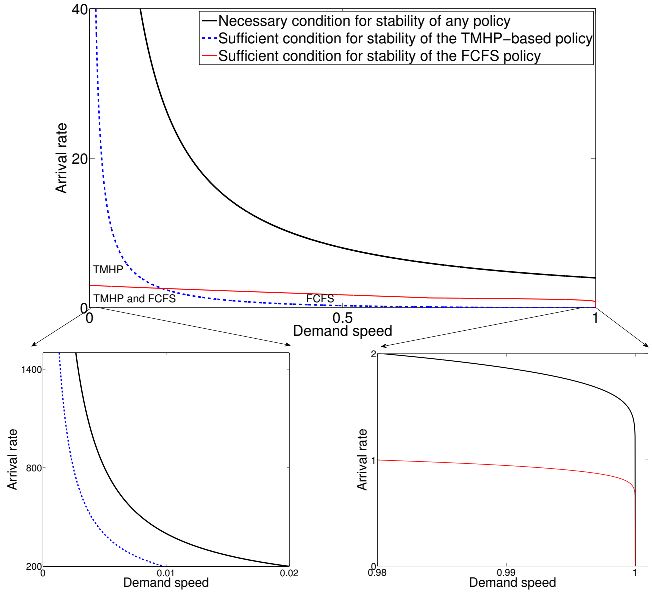

Fig. 1. A summary of stability regions for the TMHP-based policy and the FCFS policy. Stable service policies exist only for the region under the solid black curve. In the top figure, the solid black curve is due to Theorem IV.1, the dashed blue curve is due to part (i) of Theorem V.1, and the red curve is described in [10]. In the asymptotic regime shown in the bottom left, the dashed blue curve is due to Theorem V.2, and is different than the one in the top figure. In the asymptotic regime shown in the bottom right, the solid black curve is described in [10], and is different from the solid black curve in the top figure.

This paper is organized as follows: we begin with background results on the traveling salesperson problems in Section II. The problem formulation is presented in Section III. The necessary condition for stability is derived in Section IV. The TMHP-based policy and its stability results are presented in Section V. Simulation results are presented in Section VI. Due to space constraints, we include only a sketch of proofs for some intermediate results. The complete proofs are presented in [11].

II. Preliminary results

We use the following motion to reach a demand.

Proposition II.1 (Constant bearing control, [12]) *Given the locations $\mathbf{p} := (X, Y) \in \mathcal{E}$ and $\mathbf{q} := (x, y) \in \mathcal{E}$ at time t of the vehicle and a demand, respectively, then the motion of the vehicle towards the point $(x, y + vT)$, where*

$$T(\mathbf{p}, \mathbf{q}) := \frac{\sqrt{(1-v^2)(X-x)^2 + (Y-y)^2}}{1-v^2} - \frac{v(Y-y)}{1-v^2},$$

minimizes the time taken by the vehicle to reach the demand.

Constant bearing control is illustrated in Fig. 2.

We now review several results on determining shortest paths through sets of points.

A. The Euclidean Minimum Hamiltonian Path (EMHP)

Given a set of points in the plane, a Euclidean Hamiltonian path is a path that visits each point exactly once. A Euclidean

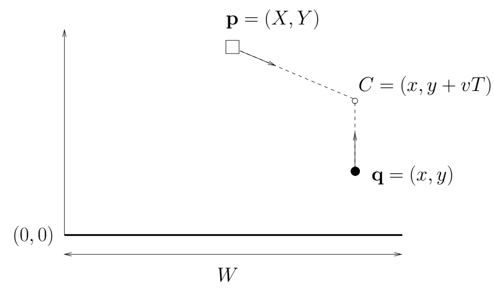

Fig. 2. Constant bearing control. The vehicle motion towards the point $C := (x, y + vT)$ minimizes the time taken to reach the demand $\mathbf{q}$.

minimum Hamiltonian path (EMHP) is a Euclidean Hamiltonian path that has minimum length. In this paper, we also consider the problem of determining a constrained EMHP which starts at a specified start point, visits a set of points and terminates at a specified end point.

More specifically, the EMHP problem is as follows.

> Given n static points placed in $\mathbb{R}^2$, determine the length of the shortest path which visits each point exactly once.

An upper bound on the length of such a path for points in a unit square was given by Few [13]. By mimicking the technique of Few, we can extend the bound to the case of points in a rectangular region, which is described in the following lemma (cf. [11] for proof).

Lemma II.2 (EMHP length) *Given n points in a $1 \times h$ rectangle in the plane, where $h \in \mathbb{R}_{>0}$, there exists a path that starts from a unit length edge of the rectangle, passes through each of the n points exactly once, and terminates on the opposite unit length edge, having length upper bounded by*

$$\sqrt{2hn} + h + 5/2.$$

We will also require the following result on the length of a path through a large number of points. Given a set $\mathcal{Q}$ of n points in $\mathbb{R}^2$, the Euclidean Traveling Salesperson Problem (ETSP) is to determine the shortest tour, i.e., a closed path that visits each point exactly once. Let $\mathrm{ETSP}(\mathcal{Q})$ denote the length of the ETSP tour through $\mathcal{Q}$. The following is the classic result by Beardwood, Halton, and Hammersly [14].

Theorem II.3 (Asymptotic ETSP length, [14]) *If a set $\mathcal{Q}$ of n points are distributed independently and uniformly in a compact region of area A, then there exists a constant β_{TSP} such that, almost surely,*

$$\lim_{n \to +\infty} \frac{\mathrm{ETSP}(\mathcal{Q})}{\sqrt{n}} = \beta_{\mathrm{TSP}} \sqrt{A}.$$

The constant β_{TSP} has been estimated numerically as $\beta_{\mathrm{TSP}} \approx 0.7120 \pm 0.0002$, [15].

B. The Translational Minimum Hamiltonian Path (TMHP)

Next, we describe the TMHP problem which was proposed and solved in [7]. This problem is posed as follows.

> Given initial coordinates; $\mathbf{s}$ of a start point, $\mathcal{Q} := \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ of a set of points, and $\mathbf{f}$ of a finish point, all translating with the same constant speed v and in the same direction, determine a path that starts at time zero from point $\mathbf{s}$, visits all points in the set $\mathcal{Q}$ exactly once and ends at the finish point, and the length $\mathcal{L}_{T,v}(\mathbf{s}, \mathcal{Q}, \mathbf{f})$ of which is minimum.

In what follows, we wish to determine the TMHP through points which translate in the positive y direction. We also assume the speed of the service vehicle to be normalized to unity, and hence consider the speed of the points $v \in]0, 1[$. A solution for the TMHP problem is: (i) for $v \in]0, 1[$, define the map $g_v : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g_v(x, y) = \Big(\frac{x}{\sqrt{1-v^2}}, \frac{y}{1-v^2}\Big).$$

(ii) Compute the EMHP that starts at $g_v(\mathbf{s})$, passes through the set of points given by $\{g_v(\mathbf{q}_1), \ldots, g_v(\mathbf{q}_n)\} =: g_v(\mathcal{Q})$ and ends at $g_v(\mathbf{f})$. (iii) Move between any two demands using constant bearing control. The following result is established.

Lemma II.4 (TMHP length, [7]) *Let the initial coordinates $\mathbf{s} = (x_\mathbf{s}, y_\mathbf{s})$ and $\mathbf{f} = (x_\mathbf{f}, y_\mathbf{f})$, and the speed of the points $v \in]0, 1[$. The length of the TMHP is*

$$\mathcal{L}_{T,v}(\mathbf{s}, \mathcal{Q}, \mathbf{f}) = \mathcal{L}_E(g_v(\mathbf{s}), g_v(\mathcal{Q}), g_v(\mathbf{f})) + \frac{v(y_\mathbf{f} - y_\mathbf{s})}{1-v^2},$$

where $\mathcal{L}_E(g_v(\mathbf{s}), g_v(\mathcal{Q}), g_v(\mathbf{f}))$ denotes the length of the EMHP with starting point $g_v(\mathbf{s})$, passing through the set of points $\{g_v(\mathbf{q}_1), \ldots, g_v(\mathbf{q}_n)\}$, and ending at $g_v(\mathbf{f})$.

III. Problem Formulation

We consider a single service vehicle that seeks to service mobile demands that arrive via a spatio-temporal process on a segment with length W along the x-axis, termed the *generator*. The vehicle is modeled as a first-order integrator with speed upper bounded by one. The demands arrive uniformly distributed on the generator via a temporal Poisson process with intensity $\lambda > 0$, and translate with constant speed $v \in]0, 1[$ along the positive y-axis, as shown in Figure 3. We assume that once the vehicle reaches a demand, the demand is served instantaneously. The vehicle is assumed to have unlimited fuel and demand servicing capacity.

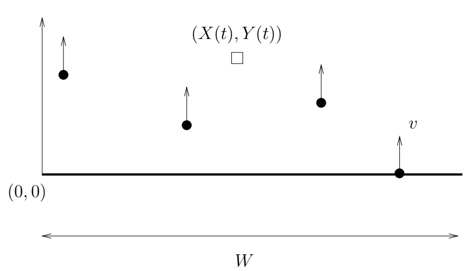

Fig. 3. The problem setup. The thick line segment is the generator of mobile demands. The dark circle denotes a demand and the square denotes the service vehicle.

We define the environment as $\mathcal{E} := [0, W] \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$, and let $\mathbf{p}(t) = [X(t), Y(t)]^T \in \mathcal{E}$ denote the position of the service vehicle at time t. Let $\mathcal{Q}(t) \subset \mathcal{E}$ denote the set of all demand locations at time t, and $n(t)$ the cardinality of $\mathcal{Q}(t)$. Servicing of a demand $\mathbf{q}_i \in \mathcal{Q}$ and removing it from the set $\mathcal{Q}$ occurs when the service vehicle reaches the location of the demand. A static feedback control policy for the system is a map $\mathcal{P} : \mathcal{E} \times \mathrm{FIN}(\mathcal{E}) \to \mathbb{R}^2$, where $\mathrm{FIN}(\mathcal{E})$ is the set of finite subsets of $\mathcal{E}$, assigning a commanded velocity to the service vehicle as a function of the current state of the system: $\dot{\mathbf{p}}(t) = \mathcal{P}(\mathbf{p}(t), \mathcal{Q}(t))$. Let D_i denote the time that the ith demand spends within the set $\mathcal{Q}$, i.e., the delay between the generation of the ith demand and the time it is serviced. The policy $\mathcal{P}$ is *stable* if under its action,

$$\limsup_{i \to +\infty} \mathbb{E}[D_i] < +\infty,$$

i.e., the steady state expected delay is finite. Equivalently, the policy $\mathcal{P}$ is stable if under its action, $\limsup_{t \to +\infty} \mathbb{E}[n(t)] < +\infty$, that is, if the vehicle is able to service demands at a rate that is—on average—at least as fast as the rate at which new demands arrive. In what follows, our goal is to *design stable policies* for the system.

IV. A Necessary condition for stability

In this section, we provide a necessary condition on the arrival rate for the existence of a stabilizing policy. We begin by stating the main result of the section, with the remainder of the section dedicated to its proof.

Theorem IV.1 (Necessary condition for stability) *A necessary condition for the existence of a stabilizing policy is that*

$$\lambda \leq \frac{4}{vW}.$$

To prove Theorem IV.1, we begin by looking at the distribution of demands in the service region.

Lemma IV.2 (Poisson point process) *Suppose the generation of demands commences at time 0 and no demands are serviced in the interval $[0, t]$. Let $\mathcal{Q}$ denote the set of all demands in $[0, W] \times [0, vt]$ at time t. Then, given a compact region $\mathcal{R}$ of area A contained in $[0, W] \times [0, vt]$,*

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = \frac{\mathrm{e}^{-\bar{\lambda}A}(\bar{\lambda}A)^n}{n!}, \quad \textit{where } \bar{\lambda} := \lambda/(vW).$$

Proof: [Sketch] This proof requires the calculation of the probability that at time t, $|\mathcal{R} \cap \mathcal{Q}| = n$, where $\mathcal{R} = [\ell, \ell + \Delta\ell] \times [h, h + \Delta h]$ (that is, the probability that the region $\mathcal{R}$ contains n points in $\mathcal{Q}$). Since the generation process is temporally Poisson and spatially uniform, this is equal to the probability that the region $[0, \Delta\ell] \times [0, \Delta h]$ contains n points in $\mathcal{Q}$. After some simplifications, we obtain that the result is true for the above considered rectangular region. Finally, since every measurable, compact region can be written as a countable union of rectangles, the result is established. ■

Remark IV.3 (Uniformly distributed demands) Lemma IV.2 shows us that the number of demands in an unserviced region is Poisson distributed with rate

$\lambda/(vW)$, and conditioned on this number, the demands are distributed uniformly. $\square$

We now establish a result on the expected time to travel from a demand to its nearest neighbor.

Lemma IV.4 (Travel time bound) *Consider the set $\mathcal{Q}$ of demands in $\mathcal{E}$ at time t. Let T_d be a random variable giving the minimum amount of time required to travel to a demand in $\mathcal{Q}$ from a vehicle position (X, Y), selected* a priori. *Then*

$$\mathbb{E}\left[T_d\right] \geq \frac{1}{2}\sqrt{\frac{vW}{\lambda}}.$$

Proof: Using Proposition II.1, we can write the travel time T from an *a priori* vehicle position $\mathbf{p} := (X, Y)$ to a demand location $\mathbf{q} := (x, y)$ implicitly as

$$T(\mathbf{p}, \mathbf{q})^2 = (X - x)^2 + ((Y - y) - vT(\mathbf{p}, \mathbf{q}))^2. \quad (1)$$

Thus, any demand in S_T, where

$$S_T := \{(x, y) \in \mathcal{E} : (X - x)^2 + ((Y - vT) - y)^2 \leq T^2\},$$

can be reached from (X, Y) in T time units. In general, the area of S_T satisfies $|S_T| \leq \pi T^2$. By Lemma IV.2 the demands in an unserviced region are uniformly randomly distributed with density $\bar{\lambda} = \lambda/(vW)$. Thus, for every vehicle position $\mathbf{p}$ chosen before the generation of demands,

$$\mathbb{P}[T_d > T] = \mathbb{P}[|S_T \cap \mathcal{Q}| = 0] \geq \mathrm{e}^{-\bar{\lambda}|S_T|} \geq \mathrm{e}^{-\lambda\pi T^2/(vW)}.$$

Therefore,

$$\mathbb{E}\left[T_d\right] \geq \int_0^{+\infty} \mathbb{P}[T_d > T] dT \geq \int_0^{+\infty} \mathrm{e}^{-\lambda\pi T^2/(vW)} dT$$
$$= \frac{\sqrt{\pi}}{2\sqrt{\lambda\pi/(vW)}} = \frac{1}{2}\sqrt{\frac{vW}{\lambda}}.$$

$\blacksquare$

We can now prove Theorem IV.1.

Proof: [Proof of Theorem IV.1] A necessary condition for the stability of any policy (see, for example [1]) is that

$$\lambda \mathbb{E}\left[T\right] \leq 1,$$

where $\mathbb{E}\left[T\right]$ is the steady-state expected travel time between demands i and $i+1$. For every policy $\mathbb{E}\left[T\right] \geq \mathbb{E}\left[T_d\right] \geq \frac{1}{2}\sqrt{\frac{vW}{\lambda}}$. Thus a necessary condition for stability is that

$$\lambda \frac{1}{2}\sqrt{\frac{vW}{\lambda}} \leq 1 \quad \Leftrightarrow \quad \lambda \leq \frac{4}{vW}.$$

$\blacksquare$

V. The TMHP-based policy and its stability

In this section, we present the TMHP-based policy for the vehicle along with the sufficient condition for its stability. The TMHP-based policy is summarized in Algorithm 1, and an instance of the policy is illustrated in Fig. 4.

The TMHP-based policy gives the following result.

Algorithm 1: **TMHP-based policy**

Assumes: Service vehicle has initial position (X, Y), and all demands have lower y-coordinates.

if no outstanding demands in the environment **then**
- Move towards the generating line for a time interval of $Y/(1+v)$.

else
- Let V be a "virtual" demand located at $(X, 0)$ translating with speed v in the positive y-direction.
- Service all the outstanding demands by following a TMHP starting from (X, Y), and terminating at virtual demand V. Use the constant bearing control to move between demands.

Repeat.

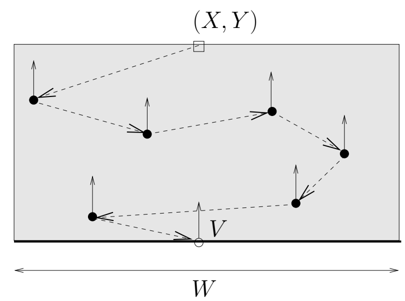

Fig. 4. The TMHP-based policy. The vehicle serves all outstanding demands inside the shaded rectangular region $\mathcal{R}(X, Y)$ as per the TMHP that begins at (X, Y) and terminates at the virtual demand V.

Theorem V.1 (Stability of TMHP-based policy) *(i) The TMHP-based policy is stable if*

$$\lambda < \frac{(1 - v^2)^{3/2}}{2vW(1 + v)^2}, \quad \textit{and,}$$

(ii) assuming that the TMHP-based policy is stable, the steady state expected time spent by a demand in the environment is upper bounded by

$$\frac{5W}{2\sqrt{1 - v^2}} \left(\frac{1}{1/(1 + v) - \sqrt{2Wv\lambda/(1 - v^2)^{3/2}}} \right).$$

Proof: Let $\mathcal{R}(X, Y)$ denote the region $[0, W] \times [0, Y]$ defined by the position (X, Y) of the service vehicle, as shown in Fig. 4. Observe that at the end of every iteration of this policy, all outstanding demands have their y-coordinates less than or equal to that of the vehicle, and hence would be contained in $\mathcal{R}(X, Y)$. Let the vehicle be located at $\mathbf{p}(t_i) = (X(t_i), Y(t_i))$ at time instant t_i. If there are no outstanding demands in $\mathcal{R}(X(t_i), Y(t_i))$, then $\frac{Y(t_i)}{1+v}$ is the distance that the vehicle moves towards the generator. Thus, we have

$$Y(t_{i+1}) = Y(t_i) - \frac{Y(t_i)}{1 + v} = \frac{vY(t_i)}{1 + v},$$

if there are no unserviced demands in $\mathcal{R}(X(t_i), Y(t_i))$ at time t_i. Otherwise, for any unserviced demands $\{\mathbf{q}_1, \dots, \mathbf{q}_{n_i}\}$ where $n_i \geq 1$, in $\mathcal{R}(X(t_i), Y(t_i))$,

$$Y(t_{i+1}) = v\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1, \dots, \mathbf{q}_{n_i}\}, V(t_i)),$$

where $\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1, \dots, \mathbf{q}_{n_i}\}, V(t_i))$ is the time taken for the vehicle as per the TMHP that begins at $\mathbf{p}(t_i)$, serves all n_i demands and ends at the virtual demand $V(t_i)$. Since the distribution of the demands inside $\mathcal{R}(X(t_i), Y(t_i))$ is spatially Poisson (cf. Lemma IV.2 from Section IV),

$$\begin{aligned} Y(t_{i+1}) &= \frac{vY(t_i)}{1+v}, \quad \text{w.p. } \mathrm{e}^{-\bar{\lambda}A}, \\ &= v\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1\}, V(t_i)), \quad \text{w.p. } (\bar{\lambda}A)\mathrm{e}^{-\bar{\lambda}A}, \\ &= v\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1, \mathbf{q}_2\}, V(t_i)), \text{ w.p. } \frac{(\bar{\lambda}A)^2}{2!}\mathrm{e}^{-\bar{\lambda}A}, \end{aligned}$$

and so on, where $A = WY(t_i)$ is the area of $\mathcal{R}(X(t_i), Y(t_i))$. We now seek an upper bound for the length $\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1, \dots, \mathbf{q}_{n_i}\}, V(t_i))$ of the TMHP for which we use the method from Section II-B. For $n_i = k \geq 1$, invoking Lemma II.4 and writing $Y_i := Y(t_i)$ and $\mathcal{Q}_k := \{\mathbf{q}_1, \dots, \mathbf{q}_k\}$,

$$\begin{aligned} \mathcal{L}_{T,v}(\mathbf{p}, \mathcal{Q}_k, V) &= \mathcal{L}_E(g_v(\mathbf{p}), g_v(\mathcal{Q}_k), g_v(V)) - \frac{vY_i}{1-v^2} \\ &\leq \sqrt{\frac{2WY_ik}{(1-v^2)^{3/2}}} + \frac{Y_i}{1+v} + \frac{5W}{2\sqrt{1-v^2}}, \end{aligned}$$

where the second equality is due to $y_{V(t_i)} = 0$, and the inequality is obtained using Lemma II.2. Thus, we have

$$\begin{aligned} &\mathbb{E}\left[Y_{i+1}|Y_i\right] \leq v\frac{Y_i}{1+v}\mathrm{e}^{-\bar{\lambda}A} + \\ &v\sum_{k=1}^{\infty}\Big(\sqrt{\frac{2WY_ik}{(1-v^2)^{3/2}}} + \frac{Y_i}{1+v} + \frac{5W}{2\sqrt{1-v^2}}\Big)\frac{(\bar{\lambda}A)^k}{k!}\mathrm{e}^{-\bar{\lambda}A}, \end{aligned}$$

where $\bar{\lambda} = \lambda/vW$ from Lemma IV.2. Collecting the terms with $vY_i/(1+v)$ together, we obtain

$$\begin{aligned} \mathbb{E}\left[Y_{i+1}|Y_i\right] &\leq \frac{vY_i}{1+v}\sum_{k=0}^{\infty}\frac{(\bar{\lambda}A)^k}{k!}\mathrm{e}^{-\bar{\lambda}A} + \\ &\qquad \sum_{k=1}^{\infty}\Big(\sqrt{\frac{2v^2WY_ik}{(1-v^2)^{3/2}}} + \frac{5vW}{2\sqrt{1-v^2}}\Big)\frac{(\bar{\lambda}A)^k}{k!}\mathrm{e}^{-\bar{\lambda}A} \\ &\leq \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\sqrt{Y_i}\,\mathbb{E}\left[\sqrt{n_i}|Y_i\right] + \frac{5vW}{2\sqrt{1-v^2}} \\ &\leq \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\sqrt{Y_i}\sqrt{\mathbb{E}\left[n_i|Y_i\right]} + \frac{5vW}{2\sqrt{1-v^2}} \\ &= \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\sqrt{Y_i}\sqrt{\frac{\lambda WY_i}{vW}} + \frac{5vW}{2\sqrt{1-v^2}} \\ &= \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\sqrt{\frac{\lambda}{v}}Y_i + \frac{5vW}{2\sqrt{1-v^2}}, \end{aligned}$$

where the inequality in the third step follows by applying Jensen's inequality to the conditional expectation and the equality in the fourth step is due to the arrival process is Poisson with rate λ and for a time interval Y_i/v. Using the law of iterated expectation, we have

$$\begin{aligned} &\mathbb{E}\left[Y_{i+1}\right] = \mathbb{E}\left[\mathbb{E}\left[Y_{i+1}|Y_i\right]\right] \\ &\leq \frac{v}{1+v}\mathbb{E}\left[Y_i\right] + \sqrt{\frac{2v\lambda W}{(1-v^2)^{3/2}}}\mathbb{E}\left[Y_i\right] + v\frac{5W}{2\sqrt{1-v^2}}, \end{aligned} \tag{2}$$

which is a linear recurrence in $\mathbb{E}\left[Y_i\right]$. Thus, $\lim_{i\to+\infty}\mathbb{E}\left[Y_i\right]$ is finite if

$$\frac{v}{1+v} + \sqrt{\frac{2Wv\lambda}{(1-v^2)^{3/2}}} < 1 \Leftrightarrow \lambda < \frac{(1-v^2)^{3/2}}{2Wv(1+v)^2}.$$

Thus, if λ satisfies the condition above, then expected number of demands in the environment is finite and the TMHP-based policy is stable.

The upper bound in part (ii) follows since the recurrence Eq. (2) is linear. ■

A. Limiting Case of Low Speed Demands

In this section, we focus on the case when $\lambda \to +\infty$ and, by the necessary stability condition in Theorem IV.1, $v \to 0^+$. Recall that for this case, the sufficient stability condition for the TMHP-based policy is that $\lambda < 1/(2vW)$. This differs by a factor of 8 from the policy independent necessary stability condition of $\lambda < 4/(vW)$. By utilizing the tight asymptotic expression for the length of the TSP path, given in Theorem II.3, in place of the bound in Lemma II.2, we can reduce this factor to approximately 2.

To begin, consider an iteration i of the TMHP-based policy, and let $Y_i > 0$ be the position of the service vehicle. In the limit as $v \to 0^+$, the length of the TMHP at the ith iteration equals the length of the corresponding static EMHP as described in Lemma II.4. Further, in the limit as $\lambda \to +\infty$, the number of outstanding demands in that iteration $n_i \to +\infty$, and hence the length of the EMHP tends to the length of the ETSP through all outstanding demands at the end of the iteration. Thus, applying Theorem II.3, the position of the vehicle at the end of the iteration is given by

$$Y_{i+1} = v\beta_{\mathrm{TSP}}\sqrt{n_iA} = v\beta_{\mathrm{TSP}}\sqrt{n_iY_iW},$$

where $A := Y_iW$ is the area of the region below the vehicle at the ith iteration. Conditioned on Y_i,

$$\mathbb{E}\left[Y_{i+1}\right] = v\beta_{\mathrm{TSP}}\mathbb{E}\left[\sqrt{Wn_iY_i}\right] \leq v\beta_{\mathrm{TSP}}\sqrt{WY_i\mathbb{E}\left[n_i\right]},$$

where we have applied Jensen's inequality. Using Lemma IV.2, $\mathbb{E}\left[n_i\right] = WY_i\lambda/(vW)$ and thus

$$\mathbb{E}\left[Y_{i+1}|Y_i\right] \leq v\beta_{\mathrm{TSP}}\sqrt{W^2Y_i^2\frac{\lambda}{vW}} = \beta_{\mathrm{TSP}}\sqrt{\lambda vW}Y_i.$$

Thus, we arrive at the following result.

Theorem V.2 (TMHP stability for low speeds) *In the limit as $v \to 0^+$, a sufficient condition for stability of the TMHP-based policy is*

$$\lambda < \frac{1}{\beta_{\mathrm{TSP}}^2vW} \approx \frac{1.9726}{vW}.$$

VI. SIMULATIONS

In this section, we present a numerical study to determine stability of the TMHP-based policy. We numerically determine the region of stability of the TMHP-based policy, and compare it with the theoretical results from the previous sections. The `linkern`[1] solver was used to generate approximations to the TMHP at every iteration of the policy. The linkern solver takes as an input an instance of the Euclidean Traveling salesperson problem. To transform the constrained EMHP problem into an ETSP, we replaced the distance between the start and end points with a large negative number, ensuring that this edge was included in the linkern output. For a given value of (v, λ), we begin with 1000 demands in the environment and determine the vehicle's average y coordinate at the end of the iteration. If it exceeds the y coordinate at the beginning of the iteration, then that particular data point of (v, λ) is classified as being unstable; otherwise, it is stable.

The results of this numerical experiment are presented in Figure 5. For the purpose of comparison, we overlay the plots for the theoretical curves, which were presented in Figure 1. We observe that the numerically obtained stability boundary for the TMHP-based policy falls between the necessary and the sufficient conditions for stability, and is well approximated by the curve established by Theorem V.2 for almost the entire interval of $]0, 1[$.

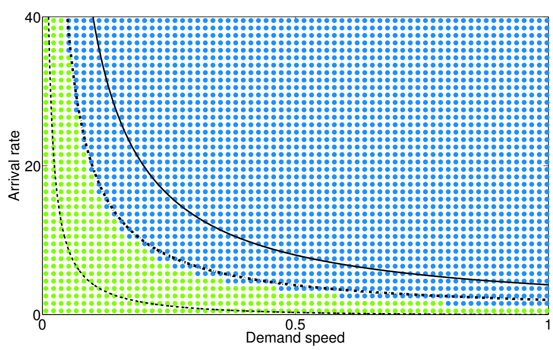

Fig. 5. Numerically determined region of stability for the TMHP-based policy. A lightly shaded (green-coloured) dot represents stability while a darkly shaded (blue-coloured) dot represents instability. The uppermost (thick solid) curve is the necessary condition for stability for any policy as derived in Theorem IV.1. The lowest (dashed) curve is the sufficient condition for stability of the TMHP-based policy as established by part (i) of Theorem V.1. The broken curve between the two curves is the sufficient stability condition of the TMHP-based policy in the low speed regime as established in Theorem V.2. The environment width is $W = 1$.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

We introduced a vehicle routing problem in which a service vehicle seeks to serve demands that arrive via a Poisson process on a line segment and that translate with a fixed speed in a direction perpendicular to the line. For the existence of a stabilizing policy, we first derived a necessary condition on the arrival rate of the demands as a function of the speed ratio between the demands and the vehicle, and the length of the line segment. Then, we proposed a novel service policy for the vehicle which involves servicing all the outstanding demands as per a TMHP through the translating demands. We derived a sufficient condition on the arrival rate of the demands for stability of the TMHP-based policy. In the limiting case of the relative speed tending to zero, we showed that the necessary and the sufficient conditions for stability are within a constant factor. In the companion paper [10], we analyze the first-come-first-served (FCFS) policy and show that in the regimes of high demand speeds, the policy is stable for higher arrival rates than the TMHP-based policy. Further, we show that in the high demand speed regime, the FCFS is an optimal policy.

In future, we plan to address versions of the present problem involving multiple service vehicles, and also with non-uniform spatial arrival of the demands. This extension has been completed for the placement problem.

[1]The TSP solver `linkern` is freely available for academic research use at `http://www.tsp.gatech.edu/concorde.html`.

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