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Dynamic Vehicle Routing with Moving Demands – Part II: High speed demands or low arrival rates

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***Abstract*— In the companion paper we introduced a vehicle routing problem in which service demands arrive stochastically on a line segment. Upon arrival, the demands translate perpendicular to the line with a fixed speed. A vehicle, with speed greater than that of the demands, seeks to provide service by reaching each mobile demand. In this paper we study a first-come-first-served (FCFS) policy in which the service vehicle serves demands in the order in which they arrive. When the demand arrival rate is very low, we show that the FCFS policy can be used to minimize the expected time, or the worst-case time, to service a demand. We determine necessary and sufficient conditions on the arrival rate of the demands (as a function of the problem parameters) for the stability of the FCFS policy. When the demands are much slower than the service vehicle, the necessary and sufficient conditions become equal. We also show that in the limiting regime when the demands move nearly as fast as the service vehicle; (i) the demand arrival rate must tend to zero; (ii) every stabilizing policy must service the demands in the order in which they arrive, and; (iii) the FCFS policy minimizes the expected time to service a demand.**

I. Introduction

In companion paper [1] we introduced a vehicle routing problem in which demands arrive via a temporal Poisson arrival process with rate λ at a uniformly distributed location on a line segment of length W, see Fig. 1. The demands move in a fixed direction perpendicular to the line with fixed speed $v < 1$. A service vehicle, modeled as a unit speed first-order integrator, seeks to serve these mobile demands by reaching each demands location. The goal is to determine conditions on the arrival rate λ, which ensure stability of the system (i.e., ensure a finite expected time spent by a demand in the environment). We refer the reader to [1] for related work and motivation. In [1] we showed that to ensure the existence of a stabilizing policy, we must have $\lambda \leq 4/vW$. We proposed a service policy which relied on the computation of the translational minimum Hamiltonian path (TMHP) through unserviced demands, and showed that for small v the policy ensures stability for all arrival rates up to a constant factor of the necessary condition.

In this paper we focus on the case when the arrival rate is low (if v is close to one we will see that this is a necessary condition for stability). For this case we propose a first-come-first-served (FCFS) policy; such policies are common

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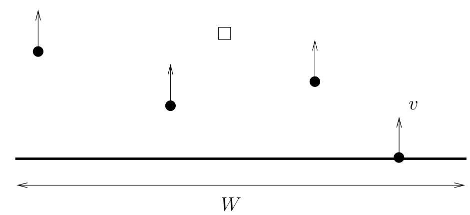

Fig. 1. The problem setup. The line segment is the generator of mobile demands. The dark circle denotes a demand and the square denotes a vehicle.

in classical queuing theory [2], [3]. In our proposed policy, the service vehicle also seeks to optimize its position to respond to the arrival of new demands. Determining the optimal position is a coverage problem, and related works include geometric location problems such as [4], and [5], and robotic sensor coverage and deployment problems [6].

The contributions of this paper can be summarized as follows. We study a first-come-first-served (FCFS) policy in which demands are served in the order in which they arrive, and when the environment contains no outstanding demands, the vehicle moves to a location which minimizes the expected (or worst-case) travel time to a demand. We show that for fixed v, as the demand arrival rate λ tends to zero, the FCFS policy is the optimal policy in terms of minimizing the expected (or worst-case) delay between a demands arrival and its service completion. We determine necessary and sufficient conditions on λ for the stability of the FCFS policy. As $v \to 0^+$, the necessary and sufficient conditions become equal. When v approaches one, we show that: (i) for existence of a stabilizing policy, λ must tend to zero as $1/\sqrt{-\log(1-v)}$, (ii) every stabilizing policy must service the demands in the order in which they arrive, and (iii) the FCFS policy minimizes the expected time to service a demand. When compared to the TMHP-based policy introduced in companion paper [1], the FCFS policy has a larger stability region when v is large, but a smaller stability region when v is small. This is summarized in Fig. 2.

This paper is organized as follows: the problem is formalized in Section II. The FCFS policy is introduced in Section II-B. In Section III we determine the optimal placement for minimizing the expected and worst-case travel time. In Section IV we determine a necessary condition for stability as v tends to one, and in Section V we determine a sufficient condition for the stability of the FCFS policy. In Section VI, we present simulation results. Due to space constraints, we limit the presentation of some proofs to a sketch. The complete proofs are presented in [7].

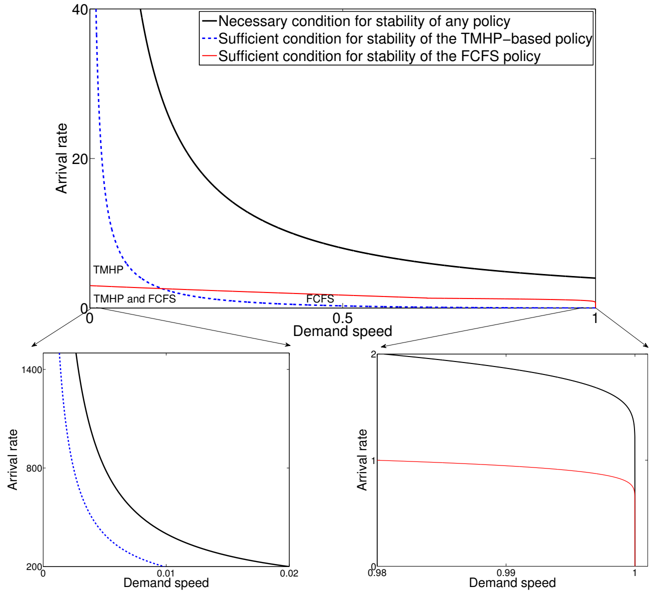

Fig. 2. A summary of stability regions for the TMHP-based policy and the FCFS policy. Stable service policies exist only for the region under the solid black curve. In the top and the bottom right figures, the red curve is due to Theorem V.1. The solid black curve and the dashed blue curve in the top figure are described in [1]. In the asymptotic regime shown in the bottom right, the solid black curve is due to Theorem IV.3, and is different from the solid black curve in the top figure. In the asymptotic regime shown in the bottom left, the dashed blue curve is described in [1], and is different than the one in the top figure.

II. Problem Formulation and Service Policy

We consider a single service vehicle that seeks to service mobile demands that arrive via a spatio-temporal process on a segment with length W along the x-axis, termed the *generator*. The vehicle is modeled as a first-order integrator with speed upper bounded by one. The demands arrive uniformly distributed on the generator via a temporal Poisson process with intensity $\lambda > 0$, and move with constant speed $v \in]0, 1[$ along the positive y-axis. We assume that once the vehicle reaches a demand, the demand is served instantaneously. The vehicle is assumed to have unlimited fuel and demand servicing capacity.

We define the environment as $\mathcal{E} := [0, W] \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$, and let $\mathbf{p}(t) = [X(t), Y(t)]^T \in \mathcal{E}$ denote the position of the service vehicle at time t. Let $\mathcal{Q}(t) \subset \mathcal{E}$ denote the set of all demand locations at time t, and $n(t)$ the cardinality of $\mathcal{Q}(t)$. Servicing of a demand $\mathbf{q}_i \in \mathcal{Q}$ and removing it from the set $\mathcal{Q}$ occurs when the service vehicle reaches the location of the demand. A static feedback control policy for the system is a map $\mathcal{P} : \mathcal{E} \times \text{FIN}(\mathcal{E}) \to \mathbb{R}^2$, where $\text{FIN}(\mathcal{E})$ is the set of finite subsets of $\mathcal{E}$, assigning a commanded velocity to the service vehicle as a function of the current state of the system: $\dot{\mathbf{p}}(t) = \mathcal{P}(\mathbf{p}(t), \mathcal{Q}(t))$. Let D_i denote the time that the ith demand spends within the set $\mathcal{Q}$, i.e., the delay between the generation of the ith demand and the time it is serviced. The policy $\mathcal{P}$ is *stable* if under its action, $\limsup_{i\to+\infty} \mathbb{E}[D_i] < +\infty$, i.e., the steady state expected delay is finite. Equivalently, the policy $\mathcal{P}$ is stable if under

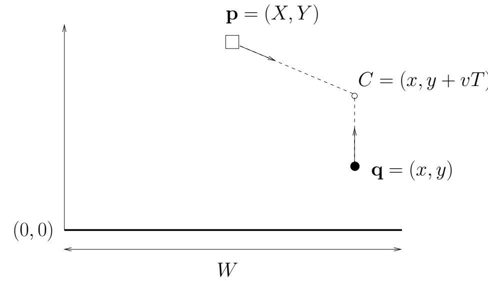

Fig. 3. Constant bearing control. The vehicle motion towards the point $C := (x, y + vT)$ minimizes the time taken to reach the demand $\mathbf{q}$.

its action,

$$\limsup_{t\to+\infty} \mathbb{E}[n(t)] < +\infty,$$

that is, if the vehicle is able to service demands at a rate that is—on average—at least as fast as the rate at which new demands arrive. In what follows, our goal is to *design stable control policies* for the system.

A. Constant Bearing Control

The vehicle uses the following motion, referred to as *constant bearing control*, to reach a moving demand.

Proposition II.1 (Constant bearing control, [8]) *Given the locations $\mathbf{p} := (X, Y) \in \mathcal{E}$ and $\mathbf{q} := (x, y) \in \mathcal{E}$ at time t of the vehicle and a demand, respectively, then the motion of the vehicle towards the point $(x, y + vT)$, where*

$$T(\mathbf{p}, \mathbf{q}) := \frac{\sqrt{(1 - v^2)(X - x)^2 + (Y - y)^2}}{1 - v^2} - \frac{v(Y - y)}{1 - v^2},$$

minimizes the time taken by the vehicle to reach the demand.

Constant bearing control is illustrated in Fig. 3.

B. The First-Come-First-Served (FCFS) Policy

We are now ready introduce the FCFS policy, which will be the focus of this paper. In this policy the service vehicle uses constant bearing control and services the demands in the order in which they arrive. If the environment contains no demands, the vehicle moves to the location (X^*, Y^*) which minimizes the expected, or worst-case, time to catch the next demand to arrive. We can state this policy as follows.

The FCFS policy

Assumes: Given the optimal location $(X^*, Y^*) \in \mathcal{E}$.

if no unserviced demands in $\mathcal{E}$ **then**
 Move toward (X^*, Y^*) until the next demand arrives.
else
 Move using the constant bearing control to service the furthest demand from the generator.
Repeat.

Fig. 4 illustrates an instance of the FCFS policy. The first question is, how do we compute the optimal position (X^*, Y^*)? This will be answered in the following section.

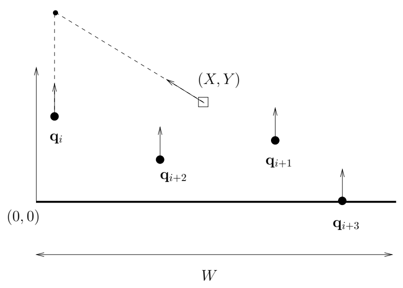

Fig. 4. The FCFS policy. The vehicle services the demands in the order of their arrival in the environment, using the constant bearing control.

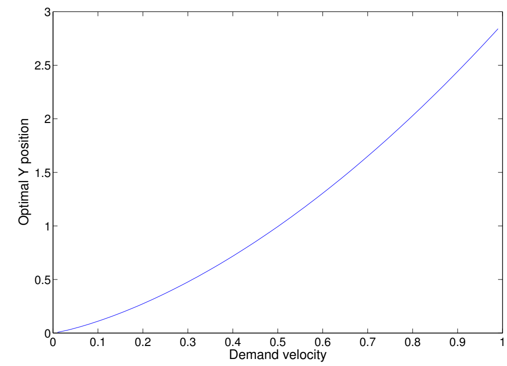

Fig. 5. The Y position of the service vehicle which minimizes the expected distance to a demand, as a function of v. In this plot the generator has length $W = 10$.

III. Optimal Vehicle Placement

In this section we study the FCFS policy when $v < 1$ is fixed and $\lambda \to 0^+$. In this regime stability is not an issue, as demands arrive very rarely, and the problem becomes one of optimally placing the service vehicle (ie., determining (X^*, Y^*) in the statement of the FCFS policy). We determine placements that minimize the expected time and the worst-case time.

A. Minimizing the Expected Time

We seek to place the vehicle at location that minimizes the expected time to service a demand once it appears on the generator. Demands appear at uniformly random positions on the generator and the vehicle uses the constant bearing control to reach the demand. Thus, the expected time $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$ to reach a demand generated at position $\mathbf{q} = (x, 0)$ from vehicle position $\mathbf{p} = (X, Y)$ is given by

$$\frac{1}{W(1-v^2)} \int_0^W \left(\sqrt{(1-v^2)(X-x)^2 + Y^2} - vY \right) dx.$$

The following lemma characterizes the way in which this expectation varies with the position $\mathbf{p}$.

Lemma III.1 (Properties of the expected time) *(i) The expected time $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$ is convex in $\mathbf{p}$, for all $\mathbf{p} \in [0, W] \times \mathbb{R}_{>0}$. (ii) There exists a unique point $\mathbf{p}^* := (W/2, Y^*) \in \mathbb{R}^2$ that minimizes $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$.*

Proof: [Sketch] Part (i) follows from the fact that the Hessian of $T((X, Y), (0, x))$ with respect to X and Y, is positive semi-definite. Further, $T(\mathbf{p}, \mathbf{q})$ is strictly convex for all $x \neq W/2$. But, letting $\mathbf{p} = (W/2, Y)$ and $\mathbf{q} = (0, x)$ we can write

$$\mathbb{E}[T(\mathbf{p}, \mathbf{q})] = \frac{1}{W(1-v^2)} \int_{x \in [0,W] \setminus \{W/2\}} T(\mathbf{p}, \mathbf{q}) dx.$$

The integrand is strictly convex for all $x \in [0, W] \setminus \{W/2\}$, implying $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$ is strictly convex on the line $X = W/2$, and the existence of a unique minimizer $(W/2, Y^*)$, and part (ii) is proven. ∎

Lemma III.1 tells us that there exists a unique point $\mathbf{p}^* := (X^*, Y^*)$ which minimizes the expected travel time. In addition, we know that $X^* = W/2$. Obtaining a closed form expression for Y^* does not appear to be possible. $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$ with $X = W/2$, yields

$$\mathbb{E}[T(\mathbf{p}, \mathbf{q})] = \frac{Y}{2a}\Big(b - \frac{2Y}{\sqrt{a}W} \log\Big(b - \sqrt{\frac{aW^2}{4Y^2}}\Big) - 2v\Big),$$

where $a = 1 - v^2$, and $b = \sqrt{1 + aW^2/4Y^2}$. For each value of v and W, this convex expression can be easily numerically minimized over Y, to obtain Y^*. A plot of Y^* as a function of v for $W = 10$ is shown in Fig. 5.

For the optimal position $\mathbf{p}^*$, the expected delay between a demand's arrival and its service completion is

$$D^* := \mathbb{E}[T(\mathbf{p}^*, (0, x))].$$

Thus, a lower bound on the steady-state expected delay of any policy is D^*. We now characterize the steady-state expected delay of the FCFS policy D_{FCFS}, as λ tends to zero.

Theorem III.2 (FCFS optimality for low arrival rates) *Fix any $v < 1$. Then as $\lambda \to 0^+$, $D_{\text{FCFS}} \to D^*$, and the FCFS policy minimizes the expected time to service a demand.*

Proof: We have shown how to compute the position $\mathbf{p}^* := (X^*, Y^*)$ which minimizes $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$. Thus, if the vehicle is located at $\mathbf{p}^*$, then the expected time to service the demand is minimized. But, as $\lambda \to 0^+$, the probability that demand $i + 1$ arrives before the vehicle completes service of demand i and returns to $\mathbf{p}^*$ tends to zero. Thus, the FCFS policy is optimal as $\lambda \to 0^+$. ∎

B. Minimizing the Worst-Case Time

The expected time to service a demand was the metric studied in the companion paper, and will be the metric of interest in Section V when we study the FCFS policy for $\lambda > 0$. However, another metric that can be used to determine (X^*, Y^*) is the worst-case time to service a demand.

Lemma III.3 (Optimal placement for worst-case) *The location (X^*, Y^*) that minimizes the worst-case time to service the demand is $(W/2, vW/2)$.*

Proof: [Sketch] The fact that $X^* = W/2$ follows from symmetry. We then substitute this value for X in Eq. (II.1), along with $y = 0$, to obtain the worst-case time expression as a function of Y. Performing a standard minimization of the worst-case time, we get $Y^* = vW/2$. ■

Using an argument identical to that in the proof of Theorem III.2 we have the following: For fixed $v < 1$, and as $\lambda \to 0^+$, the FCFS policy, with $(X^*, Y^*) = (W/2, vW/2)$, minimizes the worst-case time to service a demand.

IV. Necessary Conditions for Stability

In this section, we consider $\lambda > 0$, and determine necessary conditions on λ to ensure that the FCFS policy remains stable. To establish these conditions we utilize a standard result in queueing theory (cf. [2]) which states that a necessary condition for the existence of a stabilizing policy is that $\lambda \mathbb{E}[T] \le 1$, where $\mathbb{E}[T]$ is the expected time to service a demand (i.e., the travel time between demands). We also determine a necessary condition on λ for the stability of any policy as $v \to 1^-$, and establish the optimality of the FCFS policy. We begin with the following.

Proposition IV.1 (Special case of equal speeds) *For $v = 1$ there does not exist a stabilizing policy.*

Proof: [Sketch] If $v = 1$, then a demand can be reached only if the service vehicle is above the demand. Note that the only policy that ensures that a demand's y-coordinate never exceeds that of the service vehicle is the FCFS policy. The travel time between demand i and $i+1$ is given by

$$T(\mathbf{q}_i, \mathbf{q}_{i+1}) = \frac{\Delta x^2 + \Delta y^2}{2\Delta y} = \frac{1}{2}\left(\frac{\Delta x^2}{\Delta y} + \Delta y\right),$$

where Δx and Δy are the differences between their x- and y-coordinates respectively. Taking expectation, and making use of the independence of Δx and Δy, we obtain $\mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] = +\infty$. This implies for every $\lambda > 0$, $\lambda\mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] = +\infty$. This means that the necessary condition for stability, i.e., $\lambda\mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] \le 1$, is violated. Thus, there does not exist a stabilizing policy. ■

Next we look at the FCFS policy and give a necessary condition for its stability.

Theorem IV.2 (Necessary stability condition for FCFS) *A necessary condition for the stability of the FCFS policy is*

$$\lambda \le \begin{cases} \dfrac{3}{W}, & \text{for } v \le v^*_{\text{nec}}, \\ \dfrac{3\sqrt{2v}}{W\sqrt{(1+v)\left(C_{\text{nec}} - \log\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, & \text{otherwise,} \end{cases}$$

where $C_{\text{nec}} = 0.5 + \log(2) - \gamma$, where γ is the Euler constant; and v^*_{nec} is the solution to the equation $2v - (1+v)(C_{\text{nec}} - 0.5 \cdot \log(1-v^2) + \log v) = 0$, and is approximately equal to $4/5$.

Proof: [Sketch] The travel time between consecutive demands is given by

$$T = \frac{1}{1-v^2}\left(\sqrt{(1-v^2)\Delta x^2 + \Delta y^2} - v\Delta y\right),$$

where Δx and Δy are the differences in the x- and y-coordinates of the demands respectively. Since T is convex in Δx and Δy, we apply Jensen's inequality, followed by substitution of the expressions for the expected values of Δx and Δy, and obtain

$$\mathbb{E}[T] \ge \frac{1}{1-v^2}\Big(\sqrt{(1-v^2)\frac{W^2}{9} + \frac{v^2}{\lambda^2}} - \frac{v^2}{\lambda}\Big).$$

Using the necessary condition for stability,

$$\lambda \le \frac{3}{W}. \tag{1}$$

This provides a good necessary condition for low v, but we will be able to obtain a much better necessary condition for large v.

Since T is convex in Δx, applying Jensen's inequality,

$$\mathbb{E}[T|\Delta y] \ge \frac{1}{1-v^2}\left(\sqrt{(1-v^2)W^2/9 + \Delta y^2} - v\Delta y\right), \tag{2}$$

where $\mathbb{E}[\Delta x] = W/3$. Now, the random variable Δy is distributed exponentially with parameter λ/v, un-conditioning Eq. (2) on Δy we obtain that $\mathbb{E}[T]$ is lower bounded by

$$\frac{v}{\lambda(1-v^2)}\int_0^{+\infty}\left(\sqrt{\frac{(1-v^2)W^2}{9} + y^2} - vy\right)\mathrm{e}^{-\lambda y/v}dy.$$

Using the software Maple®, this simplifies to

$$\frac{\pi W}{2\cdot 3\sqrt{1-v^2}}\left[\mathbf{H}_1\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right) - \mathbf{Y}_1\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right)\right] - \frac{v^2}{\lambda(1-v^2)},$$

where $\mathbf{H}_1(\cdot)$ is the 1st order Struve function and $\mathbf{Y}_1(\cdot)$ is 1st order Bessel function of the 2nd kind. A Taylor series expansion of $\mathbf{H}_1(z) - \mathbf{Y}_1(z)$ about $z = 0$ yields

$$\mathbf{H}_1(z) - \mathbf{Y}_1(z) \ge \frac{1}{\pi}\left(\frac{2}{z} + C_{\text{nec}}z - z\log(z)\right),$$

where $C_{\text{nec}} = 1/2 + \log(2) - \gamma \approx 0.62$. Thus,

$$\mathbb{E}[T] \ge \frac{v}{\lambda(1+v)} + \frac{\lambda W}{18v}\left(C_{\text{nec}} - \log\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right)\right),$$

Using the necessary condition for stability, and simplifying using the fact that $\lambda W/3 < 1$, we have for stability,

$$\lambda \le \frac{3\sqrt{2v}}{W\sqrt{(1+v)\left(C_{\text{nec}} - \log\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, \tag{3}$$

when $C_{\text{nec}} > \log(\sqrt{1-v^2}/v)$, i.e., when $v > v^*_{\text{nec}}$, where v^*_{nec} is obtained when we set the RHS of Eq. (1) equal to the RHS of Eq. (3), and is approximately equal to $4/5$. Thus, the necessary condition for stability is given by Eq. (1) when $v \le v^*_{\text{nec}}$, and by Eq. (3) when $v > v^*_{\text{nec}}$. ■

The necessary condition states that in the limit as $v \to 1^-$, λ goes to zero as $1/\sqrt{-\log(1-v)}$, which is slower than any polynomial in $(1-v)$. The following result shows that as $v \to 1^-$, that the condition in Theorem IV.2 is necessary for every policy.

Theorem IV.3 (Policy independent necessary condition) *For the limiting regime as $v \to 1^-$, every stabilizing policy must serve the demands in the order in which they arrive and hence,*

$$\lambda \le \frac{3\sqrt{2}}{W\sqrt{-\log(1-v)}}.$$

Proof: [Sketch] Suppose there is a policy P that is not does not serve demands FCFS, but stabilizes the system with

$$\lambda = B(1-v)^p,$$

for some $p > 0$, and $B > 0$. As per P, suppose the vehicle serves demand $i+j$ before demand $i+1$. The time to travel to demand $i+1$ from any demand $i+j$, where $j > 1$ is

$$T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) \ge \frac{\Delta y}{1-v^2} + \frac{v\Delta y}{1-v^2} = \frac{\Delta y}{1-v},$$

where Δx and Δy are now the minimum of the x- and y-distances from $\mathbf{q}_{i+j}$ to the $\mathbf{q}_{i+1}$. The random variable Δy is Erlang distributed with shape $j-1 \ge 1$ and rate λ, implying

$$\mathbb{P}[\Delta y \le c] \le 1 - \mathrm{e}^{-\lambda c/v}, \quad \text{for each } c > 0.$$

Now, since $\lambda = B(1-v)^p$ as $v \to 1^-$, almost surely $\Delta y > (1-v)^{1/2-p}$. Thus

$$\lambda T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) \ge B(1-v)^p(1-v)^{-(p+1/2)} \to +\infty,$$

as $v \to 1^-$, making P unstable. Thus, a necessary condition for a policy to stabilize with $\lambda = B(1-v)^p$, is that as $v \to 1^-$, the policy must serve demands in the order in which they arrive. This holds for every p, and by letting p go to infinity, $B(1-v)^p$ converges to zero for all $v \in (0,1]$. Thus, a non-FCFS policy cannot stabilize the system no matter how quickly $\lambda \to 0^+$ as $v \to 1^-$. The bound on λ follows from Theorem IV.2. ■

The following result is a consequence of Theorem IV.3 and the fact that the FCFS policy uses constant bearing control.

Corollary IV.4 (FCFS Optimality for high speed) *In the limiting regime as $v \to 1^-$, the FCFS policy minimizes the expected time to service a demand.*

V. A Sufficient Condition for FCFS stability

In this section, we derive a sufficient condition on the arrival rate that ensures stability for the FCFS policy. To establish this condition, we utilize a standard result in queueing theory (cf. [2]) which states that a sufficient condition for the existence of a stabilizing policy is that $\lambda\mathbb{E}[T] < 1$, where $\mathbb{E}[T]$ is the expected time to service a demand.

Theorem V.1 (Sufficient stability condition for FCFS) *The FCFS policy is stable if*

$$\lambda < \begin{cases} \dfrac{3}{W}\sqrt{\dfrac{1-v}{1+v}}, & \text{for } v \le v^*_{\text{suf}}, \\ \dfrac{\sqrt{12v}}{W\sqrt{(1+v)\left(C_{\text{suf}} - \log\left(\frac{1-v}{v}\right)\right)}}, & \text{otherwise,} \end{cases}$$

where $C_{\text{suf}} = \pi/2 - \log(0.5\cdot\sqrt{3}/\sqrt{2})$, and v^*_{suf} is the solution to $\sqrt{12v^*} - 3\sqrt{(1-v^*)(C_{\text{suf}} - \log(1-v^*) + \log v^*)} = 0$, and is approximately equal to $2/3$.

Proof: We first upper bound the time taken T by the vehicle from position (X, Y), coinciding with a demand, to reach the next demand at (x, y), using the inequality $\sqrt{a^2+b^2} \le |a| + |b|$. Thus,

$$T \le \frac{|X-x|}{\sqrt{1-v^2}} + \frac{(Y-y)}{1-v^2}, \tag{4}$$

Taking expectation, and using the sufficient condition for stability,

$$\lambda\mathbb{E}[T] < 1 \Leftrightarrow \lambda < \frac{3}{W}\sqrt{\frac{1-v}{1+v}}. \tag{5}$$

Eq. (4) gives a very conservative upper bound except for the case when v is very small. Alternatively, taking expected value of T conditioned on Δy, and applying Jensen's inequality to the square-root part, and on following steps similar to those in the proof of Theorem IV.2, we obtain

$$\mathbb{E}[T] \le \frac{\pi W}{2\cdot\sqrt{6}\sqrt{1-v^2}}\left[\mathbf{H}_1\left(\frac{\lambda W\sqrt{1-v^2}}{\sqrt{6}v}\right) - \mathbf{Y}_1\left(\frac{\lambda W\sqrt{1-v^2}}{\sqrt{6}v}\right)\right] - \frac{v^2}{\lambda(1-v^2)}. \tag{6}$$

We use upper bounds on the Struve and Bessel functions from [9] when v is sufficiently large, i.e., when the argument of $\mathbf{H}_1(\cdot)$ and $\mathbf{Y}_1(\cdot)$ is small. It can be shown that

$$\begin{aligned} \mathbf{H}_1(z) &\le \frac{z}{2}, \\ \mathbf{Y}_1(z) &\ge \frac{2}{\pi}\left(\frac{z}{2}\log\frac{z}{2} - \frac{1}{z}\right), \quad \text{for } 0 \le z \le 2, \end{aligned} \tag{7}$$

where $z := \lambda W\sqrt{1-v^2}/(\sqrt{6}v)$. Substituting into Eq. (6), and upon simplification we obtain

$$\mathbb{E}[T] \le \frac{\lambda W^2}{12v}\left(\frac{\pi}{2} - \log\frac{\lambda W}{3} - \log\frac{\sqrt{3}\sqrt{1-v^2}}{2\sqrt{2}v}\right) - \frac{1}{\lambda(1+v)}. \tag{8}$$

Now, let λ^* be the least upper bound on λ for which the FCFS policy is unstable, i.e., for every $\lambda < \lambda^*$, the FCFS policy is stable. To obtain λ^*, we need to solve $\lambda^*\mathbb{E}[T] = 1$. Using Eq. (8) along with Eq. (5), we can obtain a lower bound on λ^*. Since $\lambda < \lambda^*$ implies stability, a sufficient condition for stability is

$$\lambda < \frac{\sqrt{12v}}{W\sqrt{(1+v)\left(C - \log\left(\frac{1-v}{v}\right)\right)}}, \tag{9}$$

where $C_{\mathrm{suf}} := \pi/2 - \log(0.5 \cdot \sqrt{3}/\sqrt{2}) \approx 2.06$. To determine the value of the speed v^* beyond which this is a less conservative condition than Eq. (5), we solve

$$\frac{\sqrt{12v^*}}{W\sqrt{(1+v^*)\left(C - \log\left(\frac{1-v^*}{v^*}\right)\right)}} = \frac{3}{W}\sqrt{\frac{1-v^*}{1+v^*}},$$

which gives $v^*_{\mathrm{suf}} \approx 2/3$. It can be verified that for $v > v^*_{\mathrm{suf}}$, the argument of the Struve and Bessel functions is less than 2, and hence the bounds in Eq. (7) are valid.Thus, a sufficient condition for FCFS stability is given by Eq. (5) for $v \le v^*_{\mathrm{suf}}$ and by Eq. (9) for $v > v^*_{\mathrm{suf}}$. ■

Remark V.2 (Limiting regimes) As $v \to 0^+$, the sufficient condition for FCFS stability becomes $\lambda < 3/W$, which is exactly equal to the necessary condition given by part (ii) of Theorem IV.2. Thus, the condition for stability is asymptotically tight in this limiting regime.

As $v \to 1^-$, the sufficient condition for FCFS stability becomes

$$\lambda < \frac{\sqrt{6}}{W\sqrt{-\log(1-v)}},$$

In comparison the necessary condition scales as

$$\lambda \le \frac{3\sqrt{2}}{W\sqrt{-\log(1-v)}}.$$

Thus, the necessary and sufficient conditions for the stability of the FCFS policy (and by Theorem IV.3, for any policy) differ by a factor $\sqrt{3}$. It should be noted that λ can converge to zero extremely slowly as $v \to 1^-$, and still satisfy the sufficient stability condition in Theorem V.1. For example, with $v = 1 - 10^{-6}$, the FCFS policy stabilizes the system for an arrival rate of $3/(5W)$. □

VI. Simulations

In this section, we present a numerical study to determine stability of the FCFS policy. We numerically determine the region of stability of the FCFS policy, and compare it with the theoretical results from the previous sections. For a given value of (v, λ), we begin with 1000 demands in the environment and determine the vehicle's average y-coordinate at the end of the iteration. If it exceeds the y-coordinate at the beginning of the iteration, then that particular data point of (v, λ) is classified as being unstable; otherwise, it is stable.

The results of this numerical experiment are presented in Figure 6. For the purpose of comparison, we overlay the plots for the necessary and the sufficient conditions for FCFS stability, which were established in Theorems IV.2 and V.1 respectively. We observe that the numerically obtained stability boundary for the FCFS policy falls between the two theoretically established curves.

VII. Conclusions and Future Directions

This two part paper has introduced a dynamic vehicle routing problem with moving demands. In this paper we studied the cases where the demands have high speed and where the arrival rate of demands is low. We introduced a first-come-first-served policy and gave necessary and sufficient conditions on the arrival rate for its stability. We also determined the optimal placement of the vehicle so as to minimize the worst-case, and the expected delay in servicing a demand. We showed that for fixed v, as the arrival rate tends to zero, the FCFS policy minimizes the worst-case service delay, and the expected service delay. Finally we showed that as v tends to one, FCFS minimizes the expected delay and that every stabilizing policy must service demands in the order in which arrive.

We have recently considered the case in which demands are approaching a deadline and the service vehicle seeks to stop them [10]. Future directions include studying the case when demands are generated according to a nonuniform distribution on the generator, and the case of multiple vehicles.

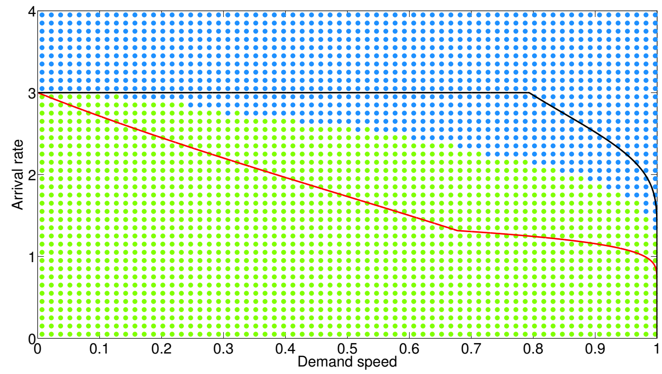

Fig. 6. Numerically determined region of stability for the FCFS policy. A lightly shaded (green-coloured) dot represents stability while a darkly shaded (blue-coloured) dot represents instability. The lower (red) curve is the sufficient stability condition in Theorem V.1. The upper (black) curve is the necessary stability condition in Theorem IV.2. The environment width is $W = 1$.

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