2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

A Frequency Response Parametrization of All Stabilizing Controllers for Continuous Time Systems

L.H. Keel, B. Shafai, and S.P. Bhattacharyya

Abstract— **In this paper, we present an alternative and new interpretation of the Nyquist criterion in terms of Bode plots of the plant and the controller. This result gives a nonparametric characterization of the frequency response of arbitrary order stabilizing controllers. The result shows that the frequency response of any stabilizing controller must satisfy constraints on its magnitude, phase, and rate of change of phase at certain frequencies that are imposed by the frequency response of the inverse plant.**

I. INTRODUCTION

The Nyquist criterion [1] provides a powerful test for closed-loop stability in terms of open-loop measured data. When applied to a plant-controller pair however, it requires the testing of the combined transfer function. This is not convenient in some synthesis and design problems, where explicit conditions are required on the controller to be designed, in terms of given plant data. In this paper, we develop new criteria for controller design to precisely address and fix the above problems.

We are motivated by the following question: Given only the frequency response of the plant as available data, is it possible to derive, without constructing an identified transfer function model of the plant, conditions on the frequency response of an arbitrary order controller for it to qualify as a stabilizing controller? In this paper, we develop *nonparametric* conditions on an arbitrary order stabilizing controller in terms of its frequency response. Such a characterization is done by interpreting the Nyquist criterion via separate Bode plots of the plant and the controllers. This result shows that the frequency response of the inverse plant imposes constraints on the magnitude, phase and rate of change of phase of the controller at certain frequencies. We also show that such conditions can easily be extended to meet performance requirements such as gain and phase margin specifications.

These results reflect the resurgence of classical control ideas in control theory with a "modern twist". In this general philosophy, we should mention the related works of Hara [2], Ikeda [3], [4] and Jayasuriya [5]. In Hara [2] a frequency dependent version of the KYP Lemma is developed and used for synthesis. Ikeda [3], [4] advocates a model free approach to design. In Jayasuriya [5] a Quantitative Feedback Theory (QFT) approach to design is discussed, wherein robustness bounds are imposed at various frequencies that are relevant to loop shaping. It will be seen that the new methods developed here also involve various frequencies where specific conditions must hold.

This work was supported in part by the National Science foundation Grant No. HRD-05311490.

L.H. Keel is with Center of Excellence in Information Systems, Tennessee State University, Nashville, TN 37203, USA.

B. Shafai is with Department of Electrical & Computer Engineering, Northeastern University, Boston, MA 02115, USA.

S.P. Bhattacharyya is with Department of Electrical & Computer Engineering, Texas A&M University, College Station, TX 77843, USA.

II. NOTATION AND PRELIMINARIES

Let us begin by considering a finite dimensional rational proper plant G with p^+ open right half plane (RHP) poles, in a unity feedback configuration as shown in Fig. 1.

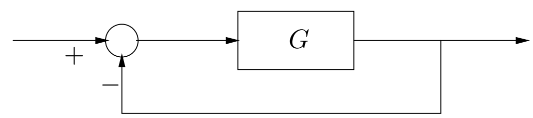

Fig. 1. A unity feedback system

Let $G(j\omega)$ be the frequency response of the plant and let ω_i, $i = 0, 1, 2, \cdots, k+1$ with $\omega_0 = 0$ and $\omega_{k+1} = \infty$ denote the frequencies where the Nyquist plot of $G(s)$ cuts the negative real axis of the complex plane. In other words, these frequencies are the solutions of the following equation:

$$\angle G(j\omega) = n\pi, \quad \text{for } n = \pm 1, \pm 3, \pm 5, \cdots. \tag{1}$$

Define the set

$$\Omega = \{\omega_0, \omega_1, \cdots, \omega_k, \omega_{k+1}\} \tag{2}$$

where

$$0 =: \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_k < \omega_{k+1} := \infty \tag{3}$$

and ω_0 and ω_{k+1} are included only if they satisfy the above angle condition. Introduce the corresponding sequence of integers

$$\{i_0, i_1, i_2, \cdots, i_k, i_{k+1}\} \tag{4}$$

where

$$i_t = 0, \quad \text{if } |G(j\omega_t)| < 1 \tag{5}$$

and otherwise,

$$i_t = \begin{cases} +1, & \text{if } \left.\dfrac{d}{d\omega}\angle G(j\omega)\right|_{\omega=\omega_t} > 0 \\ 0, & \text{if } \left.\dfrac{d}{d\omega}\angle G(j\omega)\right|_{\omega=\omega_t} = 0 \\ -1, & \text{if } \left.\dfrac{d}{d\omega}\angle G(j\omega)\right|_{\omega=\omega_t} < 0 \end{cases} \tag{6}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

for $t = 0, 1, 2, \cdots, k+1$.

Remark 1 It is easy to see that Nyquist plot of $G(s)$ at $s = j\omega_t$ cuts the negative real axis when (1) holds and it cuts the negative real axis to the left of $-1+j0$ when $|G(j\omega_t)| > 1$. The conditions in (6) along with the condition $|G(j\omega_t)| > 1$ indicate that $i_t = +1$ when the Nyquist plot cuts the negative real axis to the left of $-1+j0$ downward, corresponding to a counterclockwise encirclement, and $i_t = -1$ when the plot cuts the negative real axis to the left of $-1+j0$ upward, corresponding to a clockwise encirclement of $-1+j0$.

III. A Bode Equivalent of the Nyquist Criterion

We first assume that the plant G has no imaginary axis poles. We assume as usual that the Nyquist contour shown in the Figure 2 is traversed in the clockwise direction, that is with ω increasing.

Lemma 1 *Under the assumption that the plant G has no imaginary axis poles, let N be the number of counterclockwise encirclements of $-1+j0$ by the Nyquist plot of $G(s)$. Then*

$$N = i_0 + \sum_{t=1}^{k} 2i_t + i_\infty =: i(G). \tag{7}$$

Proof: Consider the negative real axis cuts of the Nyquist plot of $G(s)$. The number of counterclockwise encirclements of $-1+j0$ is equal to the net count of downward cuts of the plot $G(j\omega)$ which occur on the negative real axis to the left of $-1+j0$. Furthermore, such a cut must satisfy the following two conditions:

$$|G(j\omega_k)| > 1 \tag{8}$$

and

$$\frac{d}{d\omega}\angle G(j\omega)\bigg|_{\omega=\omega_k} > 0 \tag{9}$$

For $\omega \in (-\infty, \infty)$, the $G(j\omega)$ plot passes through these points, twice whereas $G(j0)$ and $G(j\infty)$ can only induce one cut, positive or negative. Therefore, the expression of $i(G)$ in (10) is nothing but a number of net counterclock encirclements around the point $-1+j0$ by the Nyquist plot. ∎

Using this lemma, we can now state the condition for stability of the feedback system.

Theorem 1 *Under the assumption that the plant has no imaginary axis poles, the unity feedback system in Fig. 1 is stable iff*

$$i(G) = p^+ \tag{10}$$

where p^+ is the number of open RHP poles of the plant G.

Proof: From the Nyquist criterion, the feedback system is stable iff the complex plane plot of $G(j\omega)$ produces p^+ net counterclockwise encirclements around the point $-1+j0$ and therefore the theorem is evident from Lemma 1. ∎

We now consider the case when the plant G has one or more poles at the origin. This addresses the class of systems with one of more integrators that are required for a system to track a step or ramp.

A. Plants with Poles at the Origin

Let m_0 be the number of poles at the origin, and let i_0 denote the corresponding number of encirclements in the counterclockwise direction of the Nyquist plot of $G(s)$. Note that here

$$G(0^+) \neq G(j0^+). \tag{11}$$

As typically done in Nyquist theory, we use right indentation of the Nyquist Γ-contour when the contour approaches imaginary axis poles (see Fig. 2).

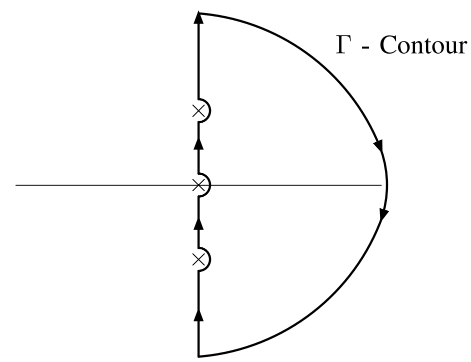

Fig. 2. Γ - Contour for Nyquist plot

Since the cases when the plant has odd and even numbers of poles at the origin are quite different, we separately state the results.

1) m_0 is odd: For the case of a plant with an odd number of poles at the origin, the Nyquist plot starts from the negative or positive imaginary axis as ω increases from zero, depending upon the values of m_0. Furthermore, the Nyquist plot turns 180^o clockwise for every pole at the origin. The first clockwise half circle is located in the LHP or RHP depending upon the sign of $G(0^+)$. For simplicity, let us consider the case when $m_0 = 1$. If $G(0^+) < 0$, the clockwise half circle is located in the LHP and it results in a negative real axis cut to the left of $-1+j0$. Since this cut is upward, we have $i_0 = -1$. On the other hand, the clockwise half circle is located in the RHP for $G(0^+) > 0$ and this results in no negative real axis cut, that is, $i_0 = 0$. From such considerations, we derive the following general formulas: For m_0 odd,

$$i_0 = \begin{cases} -\left(\dfrac{m_0-1}{2}\right) & \text{if } \ (-1)^{\frac{m_0-1}{2}} G(0^+) > 0 \\ \\ -\left(\dfrac{m_0+1}{2}\right) & \text{if } \ (-1)^{\frac{m_0+1}{2}} G(0^+) > 0 \end{cases} \tag{12}$$

2) m_0 is even: For the case of a plant with an even number of poles at the origin, the Nyquist plot begins from the negative or positive real axis as ω increases from zero, depending upon the value of m_0. With considerations similar to the previous case we derive the following general conditions: For m_0 even,

$$i_0 = \begin{cases} -\left(\frac{m_0}{2} - 1\right) & \text{if } \ (-1)^{\frac{m_0}{2}-1} G(0^+) > 0 \ \text{ and } \\ & \quad \left.\frac{d}{d\omega} \angle G(j\omega)\right|_{\omega=0^+} > 0 \\ -\left(\frac{m_0}{2}\right) & \text{otherwise} \end{cases} \tag{13}$$

B. Plants with Poles on the Imaginary Axis

Let the denominator of the plant transfer function be

$$(s^2 + u_1^2)^{m_1} (s^2 + u_2^2)^{m_2} \cdots (s^2 + u_k^2)^{m_k} \tag{14}$$

that is, the plant has m_i pairs of poles at $\pm j u_i$ for $i = 1, 2, \cdots, k$. We now define integer quantities j_i for $j = 1, 2, \cdots, k$ which denote the number of corresponding counterclockwise encirclements by the Nyquist plot of the point $-1 + j0$ as follows. The verification of these is left to the reader and is based on arguments outlined in the previous cases.

- $G(ju_i^-)$ is complex and m_i is odd,

$$j_i = \begin{cases} -(m_i - 1) & \text{if } \ \angle G(ju_i^-) \in (0, \pi) \\ -(m_i + 1) & \text{if } \ \angle G(ju_i^-) \in (\pi, 2\pi) \end{cases} \tag{15}$$

- $G(ju_i^-)$ is complex and m_i is even,

$$j_i = -m_i \tag{16}$$

- $G(ju_i^-)$ is real and m_i is odd,

$$j_i = \begin{cases} -(m_i - 1) & \text{if } \ G(ju_i^-) > 0 \text{ and } \\ & \quad \left.\frac{d}{d\omega} \angle G(j\omega)\right|_{\omega=u_i^+} > 0 \\ -(m_i + 1) & \text{otherwise} \end{cases} \tag{17}$$

- $G(ju_i^-)$ is real and m_i is even,

$$j_i = \begin{cases} -m_i & \text{if } \ G(ju_i^-) > 0 \text{ or } \\ & \left(\begin{array}{c} G(ju_i^-) < 0 \text{ and } \\ \left.\frac{d}{d\omega} \angle G(j\omega)\right|_{\omega=u_i^+} > 0 \end{array} \right) \\ -(m_i + 2) & \text{otherwise} \end{cases} \tag{18}$$

Theorem 1 can now be restated without restrictions on the location of poles of the plant.

Theorem 2 *The unity feedback system in Fig. 1 is stable iff*

$$i(G) := i_0 + \sum_{k=1}^{l} 2 i_k + i_\infty + \sum_{r=1}^{k} j_r = p^+ \tag{19}$$

where p^+ is the number of open RHP poles of the plant G.

IV. Arbitrary Order Controllers

We now consider a finite dimensional rational proper controller with frequency response $C(j\omega)$ and with c^+ RHP poles and ask when it can stabilize a finite dimensional rational, proper plant with frequency response $P(j\omega)$.

Define the set of distinct nonnegative frequencies

$$\Omega^+(\phi) := \{\omega_0, \ \omega_1, \ \cdots, \ \omega_l, \ \omega_{l+1}\} \tag{20}$$

with

$$0 =: \omega_0 < \omega_1 < \cdots < \omega_l < \omega_{l+1} := \infty \tag{21}$$

satisfying the phase condition

$$\angle C(j\omega) = \angle P^{-1}(j\omega) \pm n\pi, \ n = \pm 1, \pm 3, \pm 5, \pm 7, \cdots. \tag{22}$$

and the magnitude condition

$$|C(j\omega_k)| > |P^{-1}(j\omega_k)|. \tag{23}$$

Note that these are the frequencies where the Nyquist plot of $P(s)C(s)$ intersects the negative real axis to the left of $-1 + j0$.

For the case when $P(s)C(s)$ has no imaginary axis poles, we introduce the integers i_k for $k = 0, 1, \cdots, l+1$:

$$i_k = \begin{cases} +1, & \text{if } \left.\frac{d}{d\omega}\angle C(j\omega)\right|_{\omega=\omega_k} > \left.\frac{d}{d\omega}\angle P^{-1}(j\omega)\right|_{\omega=\omega_k} \\ 0, & \text{if } \left.\frac{d}{d\omega}\angle C(j\omega)\right|_{\omega=\omega_k} = \left.\frac{d}{d\omega}\angle P^{-1}(j\omega)\right|_{\omega=\omega_k} \\ -1, & \text{if } \left.\frac{d}{d\omega}\angle C(j\omega)\right|_{\omega=\omega_k} < \left.\frac{d}{d\omega}\angle P^{-1}(j\omega)\right|_{\omega=\omega_k} \end{cases}$$

This is obtained by replacing $G(j\omega)$ by $P(j\omega)C(j\omega)$ in (6). Consider the first condition

$$\left.\frac{d}{d\omega}\angle P(j\omega)C(j\omega)\right|_{\omega=\omega_k} > 0 \tag{24}$$

which is equivalent to

$$\left.\frac{d}{d\omega}\angle P(j\omega)\right|_{\omega=\omega_k} + \left.\frac{d}{d\omega}\angle C(j\omega)\right|_{\omega=\omega_k} > 0 \tag{25}$$

or

$$\begin{aligned} \left.\frac{d}{d\omega}\angle C(j\omega)\right|_{\omega=\omega_k} &> -\left.\frac{d}{d\omega}\angle P(j\omega)\right|_{\omega=\omega_k} \\ &= \left.\frac{d}{d\omega}\angle P^{-1}(j\omega)\right|_{\omega=\omega_k} \end{aligned} \tag{26}$$

Similarly, we can also restate (12) - (18) in terms of the controller.

A. Plants with Poles at the Origin

Let m_0 be the number of poles at the origin, and i_0 the corresponding number of encirclements in the counterclockwise direction of $-1 + j0$ by the Nyquist plot of $P(s)C(s)$. Note that

$$P(0^+)C(0^+) \neq P(j0^+)C(j0^+). \tag{27}$$

Similar to the previous section, we can derive the following:

1) m_0 is odd:

$$i_0 = \begin{cases} -\left(\dfrac{m_0-1}{2}\right) \\ \quad \text{if } (-1)^{\frac{m_0-1}{2}}\mathrm{sgn}[C(0^+)] = \mathrm{sgn}[P(0^+)] \\ -\left(\dfrac{m_0+1}{2}\right) \\ \quad \text{if } (-1)^{\frac{m_0+1}{2}}\mathrm{sgn}[C(0^+)] = -\mathrm{sgn}[P(0^+)] \end{cases} \tag{28}$$

2) m_0 is even:

$$i_0 = \begin{cases} -\left(\dfrac{m_0}{2}-1\right) \\ \quad \text{if } (-1)^{\frac{m_0}{2}-1}\mathrm{sgn}[P(0^+)] = \mathrm{sgn}[C(0^+)] \\ \quad \text{and } \frac{d}{d\omega}\angle C(j\omega)\big|_{\omega=0^+} > \frac{d}{d\omega}\angle P^{-1}(j\omega)\big|_{\omega=0^+} \\ -\left(\dfrac{m_0}{2}\right) \qquad \text{otherwise} \end{cases} \tag{29}$$

B. Plants with Poles on the Imaginary Axis

Let the denominator of the plant transfer function be

$$(s^2+u_1^2)^{m_1}(s^2+u_2^2)^{m_2}\cdots(s^2+u_k^2)^{m_k}. \tag{30}$$

Such an expression is equivalent that the plant has m_i pairs of poles at $\pm ju_i$ for $i=1,2,\cdots,k$. Define integer quantities j_i for $j=1,2,\cdots,k$ as follows.

- $P(ju_i^-)C(ju_i^-)$ is complex and m_i is odd,
  - if $\angle P^{-1}(ju_i^-) < \angle C(ju_i^-) < \angle P^{-1}(ju_i^-)+\pi$
  $$j_i = -(m_i-1), \tag{31}$$
  - if $\angle P^{-1}(ju_i^-)+\pi < \angle C(ju_i^-) < \angle P^{-1}(ju_i^-)+2\pi$
  $$j_i = -(m_i+1). \tag{32}$$
- $P(ju_i^-)C(ju_i^-)$ is complex and m_i is even,
  $$j_i = -m_i \tag{33}$$
- $G(ju_i^-)$ is real and m_i is odd,
  - if $\left(\begin{array}{c} \mathrm{sgn}[P(ju_i^-)] = \mathrm{sgn}[C(ju_i^-)] \text{ and} \\ \dfrac{d}{d\omega}\angle C(j\omega)\Big|_{\omega=u_i^+} < \dfrac{d}{d\omega}\angle P^{-1}(j\omega)\Big|_{\omega=u_i^+} \end{array}\right)$
  $$j_i = -(m_i-1), \tag{34}$$
  - otherwise
  $$j_i = -(m_i+1). \tag{35}$$
- $G(ju_i^-)$ is real and m_i is even,
  - if $\mathrm{sgn}[P(ju_i^-)] = \mathrm{sgn}[C(ju_i^-)]$ or
  $$\left(\begin{array}{c} \mathrm{sgn}[P(ju_i^-)] = -\mathrm{sgn}[C(ju_i^-)] \text{ and} \\ \dfrac{d}{d\omega}\angle C(j\omega)\Big|_{\omega=u_i^+} > \dfrac{d}{d\omega}\angle P^{-1}(j\omega)\Big|_{\omega=u_i^+} \end{array}\right)$$
  $$j_i = -m_i, \tag{36}$$
  - otherwise
  $$j_i = -(m_i+2). \tag{37}$$

Define

$$i(C) := i_0 + \sum_{k=1}^{l} 2i_k + i_\infty + \sum_{r=1}^{k} j_r. \tag{38}$$

Theorem 3 *The controller C stabilizes the plant P if and only if*

$$i(C) = p^+ + c^+. \tag{39}$$

V. PERFORMANCE MEASURES

The performance of a controller is often determined by the closed-loop stability margins it provides. The gain margin is such a performance measure. To compute it, define the distinct frequencies:

$$\begin{aligned} &\left\{u \ : \ \angle C(ju) = \angle P^{-1}(ju) \pm n\pi, \ n=1,3,5,\cdots\right\} \\ &= \{u_1, \ u_2, \ \cdots, \ u_m\} \end{aligned} \tag{40}$$

and

$$\Omega(\phi) := \{u_0, \ u_1, \ \cdots, \ u_m, \ u_{m+1}\} \tag{41}$$

where $u_0 = 0$ and $u_{m+1} = \infty$. Let us denote magnitudes measured in decibels as:

$$C(\omega)_{\mathrm{db}} \ := \ 20\log_{10}|C(j\omega)| \tag{42}$$
$$P^{-1}(\omega)_{\mathrm{db}} \ := \ 20\log_{10}|P^{-1}(j\omega)|. \tag{43}$$

Define

$$\Omega^+(\phi) := \left\{\omega_k \in \Omega(\phi) \ : \ C(\omega_k)_{\mathrm{db}} > P^{-1}(\omega_k)_{\mathrm{db}}\right\} \tag{44}$$
$$\Omega^-(\phi) := \left\{\omega_k \in \Omega(\phi) \ : \ C(\omega_k)_{\mathrm{db}} < P^{-1}(\omega_k)_{\mathrm{db}}\right\}. \tag{45}$$

The upper (lower) gain margin is the smallest increase (decrease) in gain measured in decibels that destabilizes the closed-loop.

Theorem 4 *If C is a stabilizing controller, the upper gain margin denoted K_{db}^+ is:*

$$K_{\mathrm{db}}^+ = \min_{\omega_k \in \Omega^+(\phi)} \left\{C(\omega_k)_{\mathrm{db}} - P^{-1}(\omega_k)_{\mathrm{db}}\right\}. \tag{46}$$

The lower gain margin denoted K_{db}^- is:

$$K_{\mathrm{db}}^- = \min_{u_k \in \Omega^-(\phi)} \left\{P^{-1}(u_k)_{\mathrm{db}} - C(u_k)_{\mathrm{db}}\right\}. \tag{47}$$

The phase margin is also an important performance measure. To compute it, introduce the distinct frequencies v_i:

$$\begin{aligned} \Omega(g) \ &= \ \left\{v \ : \ |C(jv)| = |P^{-1}(jv)|\right\} \\ &= \ \{v_1, \ v_2, \ \cdots, \ v_m\}. \end{aligned} \tag{48}$$

Similarly, we also define

$$\Omega^+(g) := \left\{v_k \in \Omega(g) \ : \ \angle C(jv_k) > \angle P^{-1}(jv_k) + n\pi\right\}$$
$$\Omega^-(g) := \left\{v_k \in \Omega(g) \ : \ \angle C(jv_k) < \angle P^{-1}(jv_k) + n\pi\right\}.$$

The positive (negative) phase margin Φ^+ (Φ^-) is the minimum phase decrease (increase) that destabilizes the loop.

Theorem 5 *If C is a stabilizing controller, the positive phase margin is*

$$\Phi^+ = \min_{n \text{ odd}} \min_{v_k \in \Omega^+(g)} \left\{\angle C(jv_k) - \angle P^{-1}(jv_k) - n\pi\right\}. \tag{49}$$

The negative phase margin is

$$\Phi^- = \min_{n \text{ odd}} \min_{v_k \in \Omega^-(g)} \left\{ \angle P^{-1}(ju_k) - \angle C(ju_k) + n\pi \right\}. \tag{50}$$

The proofs of Theorems 4 and 5 follow from interpreting the Nyquist criterion in terms of the Bode magnitude and phase conditions.

In the next section, we illustrate the usefulness of Theorems 3-5 for controller design.

VI. EXAMPLES

Example 1 Consider a plant with 4 RHP poles and known frequency response $P(j\omega)$ as shown in Fig. 3.

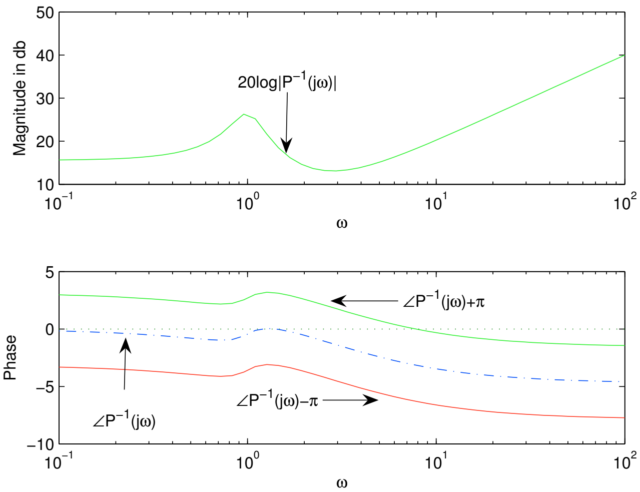

Fig. 3. Frequency response of the plant considered (Example 1)

Let us examine the conditions for a stable controller to stabilize the given plant. From the frequency response data given, we have

$$\angle P^{-1}(j\omega)\big|_{\omega=0} = 0 \quad \text{and} \quad \angle P^{-1}(j\omega)\big|_{\omega=\infty} = \frac{\pi}{2}.$$

Consider the following conditions:

(1) $\angle C(j0) = \pm n\pi$ for any integer n and $|C(j0)| > |P^{-1}(j0)|$,

(2) $\angle C(j\infty) = (1 \pm 2n)\pi/2$ and $|C(j\infty)| > |P^{-1}(j\infty)|$,

(3)

$$|C(j\omega_k)| > |P^{-1}(j\omega_k)| \tag{51}$$

and

$$\frac{d}{d\omega}\angle C(j\omega)\bigg|_{\omega=\omega_k} > \frac{d}{d\omega}\angle P^{-1}(j\omega)\bigg|_{\omega=\omega_k}. \tag{52}$$

In order to achieve $i(C) = 4$, the frequency response of a stabilizing controller must satisfy the following:

A. conditions (1) and (2) are satisfied and condition (3) is satisfied at one frequency or

B. conditions (1) and (2) are violated and (3) is satisfied at two frequencies.

Since the magnitude of $P^{-1}(j\omega)$ is unbounded as ω tends to ∞ and the controller is proper it follows that $i_\infty = 0$. Thus i_0 must also be zero in order to generate an even number of encirclements. It is also easy to see that a constant gain cannot stabilize the plant. This is because the phase of a constant gain (zero angle) can only intersect the phase of $P^{-1}(j\omega)$ only once at a nonzero frequency.

We can apply a similar consideration for a first order controller. If a controller is stable and minimum phase, then the maximum number of positive phase crossover frequencies is 3. Of these, only one contributes counterclockwise encirclement and therefore, the required $i(C) = 4$ cannot be attained.

We now consider verifying stabilizability by a given controller using the above arguments.

Suppose the controller to be tested is

$$C(s) = \frac{15.7091s^4 + 6.8889s^3 + 83.8916s^2 + 171.2964s + 62.5847}{s^4 + 4.2909s^3 - 2.1069s^2 - 7.1366s - 0.4308}$$

which has one RHP pole. Since

$$\frac{d}{d\omega}\left[\angle C(j\omega)\right]_{\omega=0} < \frac{d}{d\omega}\left[\angle P^{-1}(j\omega)\right]_{\omega=0},$$

we have $i_0 = -1$. Similarly, we have $i_\infty = 0$. From Fig. 4,

$$i(C) = -1 + 2(1) + 2(0) + 2(1) + 2(0) + 2(1) + 0 = 5$$

and

$$p^+ + c^+ = 4 + 1 = 5.$$

Thus closed-loop system is stable.

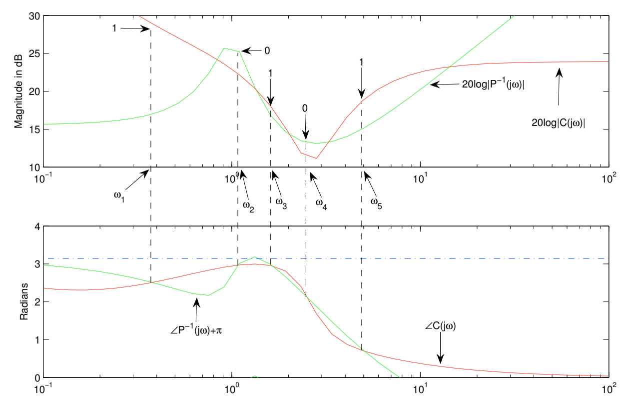

Fig. 4. No poles on the imaginary axis (Example 1)

Example 2 Consider a plant with 3 poles at the origin and the test PID controller is

$$C(s) = \frac{39.88s^2 + 50s + 49.71}{s}.$$

The poles of the open-loop plant are $\{0, 0, 0, -5, -2 \pm j4\}$. It is easy to see that $m_0 = 3 + 1 = 4$ (even) and $P(0^+)C(0^+) > 0$. Thus, from the condition in (29)

$$i_0 = -\left(\frac{m_0}{2}\right) = -\frac{4}{2} = -2.$$

Now let us observe Fig. 5. Since $|C(\infty)| < |P^{-1}(\infty)|$, $i_\infty = 0$. Therefore,

$$i(C) = i_0 + 2i_1 + i_\infty = -2 + 2(1) + 0 = 0$$

and

$$p^+ + c^+ = 0 + 0 = 0.$$

We conclude that the closed-loop system is stable.

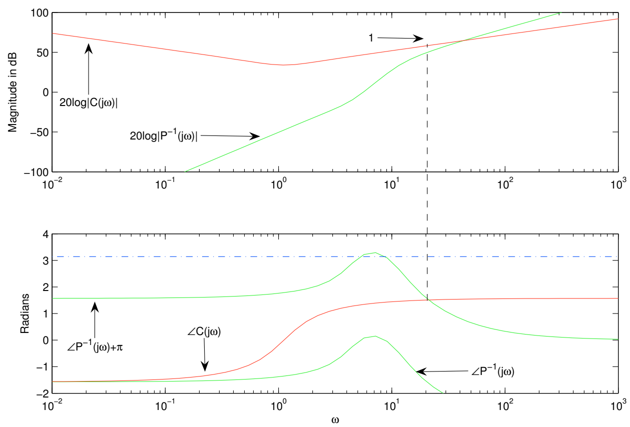

Fig. 5. Poles at the origin (Example 2)

The following example illustrates the gain and phase margin computations.

Example 3 Consider a plant with 2 RHP poles and let the frequency response of the system $P(j\omega)$ be known. Consider the test controller

$$C(s) = \frac{36s - 12}{s + 9}.$$

Since $C(0) < P^{-1}(0)$ and $C(\infty) < P^{-1}(\infty)$, we have $i_0 = i_\infty = 0$. From Fig. 6, we have

$$i(C) = 0 + 2(1) + 0 = 2$$

and

$$p^+ + c^+ = 2 + 0 = 2$$

and the closed-loop system is stable. The gain and phase margins are shown in Fig. 6 and by the combined Bode plots in Fig. 7.

VII. CONCLUDING REMARKS

In this paper we have presented an alternative to the traditional Nyquist criterion which may be useful for both synthesis and analysis of controllers. The result is an analytic version of the Nyquist criterion which provides useful insight and design information based on the plant frequency response data, which is shown to impose constraints on the magnitude, phase and rate of change of phase of any proposed stabilizing controller. We believe that these preliminary results are promising tools to develop analytical design methods based on measured data and these are presently under study.

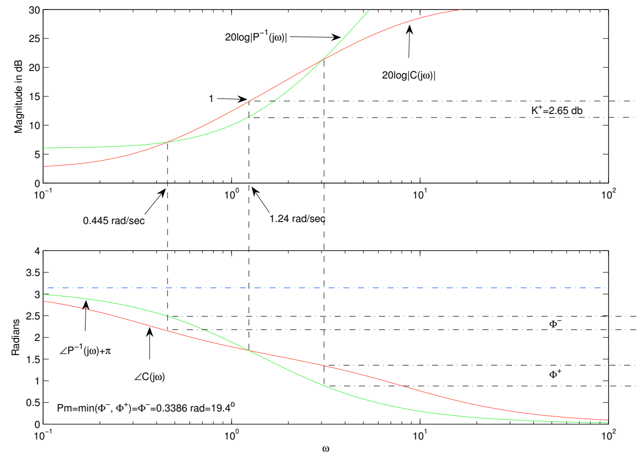

Fig. 6. Illustrating gain and phase margins (Example 3)

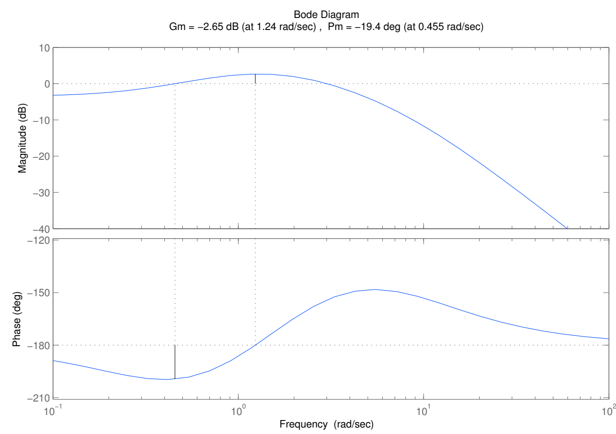

Fig. 7. Bode plot of $P(s)C(s)$ (Example 3)

REFERENCES

[1] H. Nyquist, "Regulator theory," *Bell System Technical Journal*, vol. 11, pp. 126–147, 1932.

[2] S. Hara, T. Iwasaki, and D. Shiokata, "Robust PID control using generalized KYP synthesis: Direct open-loop shaping in multiple frequency ranges," *IEEE Control Systems Magazine*, vol. 26, pp. 80–91, February, 2006.

[3] U.S. Park and M. Ikeda, "Data-based stability analysis for linear discrete-time system," *Proceedings of the 43rd IEEE Conference on Decision and Control*, Atlantis, Bahamas, December 14-17, 2004.

[4] F. Yasumasa, Y. Duanm and M. Ikeda, "System representation and optimal tracking in data space," *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 3-8, 2005.

[5] W. Wu and S. Jayasuriya, "A QFT design methodology for feedback systems under input saturation," *Journal of Dynamic Systems, Measurement, and Control*, vol. 123, pp. 225–232, June 2001.