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Upper Bound Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control and Integrated Design for Collocated Structural Systems

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***Abstract*— The present paper addresses the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm analysis and feedback control design problem for structural systems with collocated actuators and sensors. The mixed norm formulation provides a trade-off measure of a system performance and robustness in the presence of uncertainties in the system model. First, we develop an explicit upper bound expression for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of collocated structural systems and an explicit parametrization of output feedback control gains to guarantee such bounds. The results offer computationally efficient solutions for system analysis and multi-objective controller design that are especially suitable for large-scale collocated systems where traditional analysis and design methods fail. The second part of the paper uses the proposed bounds to address the simultaneous design of structural damping parameters and feedback control gains for optimized closed-loop mixed-norm performance. A linear matrix inequality (LMI) formulation is provided for the integrated damping and control gain optimization. Structural control design numerical examples are presented to demonstrate the advantages and computational efficiency of the proposed bounds and the integrated design approach.**

I. Introduction

$\mathcal{H}_\infty$ controllers are robust with respect to external disturbances, since they use no statistical information; however, they are conservative. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ approach is an attempt to mitigate this problem by taking advantage of the non-uniqueness of the $\mathcal{H}_\infty$ controllers to improve other aspects of the closed-loop system, such as the average performance. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ designs seek to minimize the $\mathcal{H}_2$ norm of a transfer matrix of the closed-loop system while simultaneously maintaining a constraint on the $\mathcal{H}_\infty$ norm. There has been some past work addressing the problem of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design for flexible structures [2], [3], [5], [6], [10], [12]. The corresponding methods follow computationally demanding approaches based on Lyapunov and Riccati equations.

Collocated direct velocity feedback (DVF) control is considered an attractive approach for active structural vibration isolation due to its simplicity of implementation and inherent robustness. Closed-loop stability and robustness to modeling uncertainty is a desirable outcome of the dissipative nature of collocated DVF [4]. Despite the popularity of DVF control, the selection of appropriate DVF control gains to satisfy closed-loop system performance specifications is a challenging problem. A main reason is that the corresponding control design formulation represents a static output feedback control design that is inherently an extremely difficult computational problem in its generality [11].

It is known that the overall system performance for closed-loop system can be significantly improved if the design process of the open-loop system parameters and the controller is integrated. The integrated design strategy corresponds to a simultaneous optimization of the design parameters of both the plant and the controller to satisfy desired design specifications and to optimize the performance of the closed-loop system. Past research work has verified that the integrated strategy provides closed-loop systems with improved performance compared to the sequential method of design. However, often the integrated plant/controller design optimization problem results in a complex nonlinear nonconvex optimization problem that does not guarantee convergence to the global optimum of the design variables. This makes the integrated design strategy computationally very challenging. The interested reader is referred to [7], [8] for a literature review and thorough discussion on the integrated design methods in structural systems.

In this paper, we first address the analysis and synthesis problems for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of collocated structural systems leading to a computationally efficient approach. First, we propose an upper bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of such systems, and subsequently we derive an explicit formula for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback control gain of such systems. Next, we use the results of the first part to present an effective and computationally tractable approach to integrate the structural and control design in collocated structural systems using a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm closed-loop performance criterion. The objective is to determine the optimal values of the damping parameters of the structure and to simultaneously design optimal output feedback gains such that an upper bound of the closed-loop system $\mathcal{H}_2$ norm is minimized subject to an upper bound on the $\mathcal{H}_\infty$ norm from the disturbance input signals to the desired outputs.

II. Plant Formulation and Preliminaries

Consider a structural system with collocated sensors and actuators with velocity measurement in a vector second-order form represented by

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= F(u(t) + w(t)) \\ z(t) &= F^T\dot{q}(t) \\ y(t) &= F^T\dot{q}(t) \end{aligned} \tag{1}$$

where $q(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^m$ are the vectors representing displacements, control input,

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external disturbance input, and performance output, respectively. Also, $y(t)$ represents the measured output vector. The matrices $F \in \mathbb{R}^{n\times m}$, $M > 0$, $K > 0$, and $D > 0$ represent the input distribution matrix with full column rank, mass matrix, stiffness matrix and damping matrix, respectively. If we assume that $x^T(t) = \begin{bmatrix} q^T(t) & \dot{q}^T(t)\end{bmatrix}$, then the state-space representation of the system (1) will be

$$\begin{aligned}\dot{x}(t) &= Ax + B(u(t) + w(t))\\ z(t) &= Cx(t)\end{aligned} \tag{2}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D\end{bmatrix}, B = \begin{bmatrix} 0 \\ M^{-1}F\end{bmatrix}, C = \begin{bmatrix} 0 \\ F\end{bmatrix}^T \tag{3}$$

It can be readily shown that the above system is an externally symmetric system, *i.e.*, assuming $u(t) = 0$, the transfer function from $W(s)$ to $Z(s)$ given by $G(s) = sF^T(Ms^2 + Ds + K)^{-1}F$ is symmetric.

In the following, we present two lemmas that represent LMI-based formulations for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norm of the system (2).

Lemma 1: (Bounded Real Lemma [1]) A stable system (2) with $u(t) = 0$ has an $\mathcal{H}_\infty$ norm from $w(t)$ to $z(t)$ less than or equal to γ if and only if there exists a symmetric matrix $P > 0$ such that the following LMI holds.

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & 0 \\ C & 0 & -\gamma I\end{bmatrix} \leq 0. \tag{4}$$

Lemma 2: ($\mathcal{H}_2$ performance specification [9]) Suppose that the system (2) with $u(t) = 0$ is asymptotically stable, and let $G(s) = C(sI - A)^{-1}B$ denote its transfer function. Then the following statements are equivalent:
i) $\|G\|_2 \leq \eta$
ii) There exists a symmetric matrix $P > 0$ such that

$$\begin{bmatrix} PA + A^TP & PB \\ B^TP & -I\end{bmatrix} \leq 0 \tag{5}$$

$$\begin{bmatrix} P & C^T \\ C & Z\end{bmatrix} \geq 0 \tag{6}$$

$$trace(Z) \leq \eta^2. \tag{7}$$

The next lemma will be useful in the proofs of the main results of the paper.

Lemma 3: [9] Consider matrices Γ and Λ such that Γ has full column rank, and Λ is symmetric positive definite. Then $\Lambda \geq \Gamma\Gamma^T$ if and only if $\lambda_{max}(\Gamma^T\Lambda^{-1}\Gamma) \leq 1$.

III. An Explicit Expression for the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Norm of the Collocated Structural System

As discussed earlier in this paper, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem addresses the minimization of an $\mathcal{H}_2$ norm criterion, subject to an $\mathcal{H}_\infty$ norm constraint. In this section we propose an explicit upper bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of the collocated structural systems represented by (1). The following theorem provides such an explicit bound avoiding the need of burdensome computations.

Theorem 1: Consider the unforced system in (1), i.e., $u(t) = 0$. For any given $\gamma \geq \gamma_{bound} = \lambda_{max}(F^TD^{-1}F)$, an upper bound η_{bound} on the $\mathcal{H}_2$ norm of the system, while the $\mathcal{H}_\infty$ norm satisfies the condition $\|G_{zw}\|_\infty \leq \gamma$, can be computed from the following expression

$$s\eta_{bound} = \frac{[\lambda_{max}(F^TD^{-1}F)]^{\frac{1}{2}}}{\sqrt{2}}[trace(F^TM^{-1}F)]^{\frac{1}{2}} \tag{8}$$

if $\gamma \geq 1$. Otherwise, for $\gamma < 1$, an upper bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of the system is determined from

$$\eta_{bound} = [\max(\delta, \sigma) \times trace(F^TM^{-1}F)]^{\frac{1}{2}} \tag{9}$$

where

$$\begin{aligned}\delta &= \frac{1}{2}\lambda_{max}(F^TD^{-1}F)\\ \sigma &= \frac{\lambda_{max}(F^TD^{-1}F)}{\gamma + [\gamma^2 - \lambda^2_{max}(F^TD^{-1}F)]^{1/2}}.\end{aligned} \tag{10}$$

Proof. Let us consider the following Lyapunov matrix

$$P = \alpha\begin{bmatrix} K & 0 \\ 0 & M\end{bmatrix} \tag{11}$$

where α is a positive scalar. To determine a bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of the system (2), the LMIs (4)-(7) should be satisfied simultaneously. To this end, we fix a prescribed level of $\mathcal{H}_\infty$ performance to be γ, and the $\mathcal{H}_2$ norm of the system is sought such that LMIs (4)-(7) hold. Substituting the state-space matrices (3) into the LMI (4) and using the Lyapunov matrix (11) results in

$$\begin{bmatrix} -2\alpha D & \alpha F & F \\ \alpha F^T & -\gamma I & 0 \\ F^T & 0 & -\gamma I\end{bmatrix} \leq 0. \tag{12}$$

Making use of the Schur complement formula yields

$$\frac{\alpha^2 + 1}{\gamma}FF^T - 2\alpha D \leq 0 \tag{13}$$

which can be rewritten as

$$\frac{\alpha^2 + 1}{2\alpha\gamma}D^{-\frac{1}{2}}FF^TD^{-\frac{1}{2}} \leq I_n$$

where I_n is the $n \times n$ identity matrix. The latter inequality implies that

$$0 \leq \frac{1}{\gamma}\lambda_{max}(D^{-\frac{1}{2}}FF^TD^{-\frac{1}{2}}) \leq \frac{2\alpha}{\alpha^2 + 1} \leq 1. \tag{14}$$

Taking into account the fact that $\lambda_i(AB) = \lambda_i(BA)$ for all nonzero eigenvalues and any pair of matrices A and B of compatible dimensions, we obtain that

$$\lambda_{max}(F^TD^{-1}F) \leq \gamma \tag{15}$$

which provides an upper bound on the $\mathcal{H}_\infty$ norm of the collocated structural systems [7]. In addition, assuming $\beta = \frac{1}{\gamma}\lambda_{max}(F^T D^{-1} F)$, inequality (14) results in

$$\frac{1}{\alpha} \geq \frac{\beta}{1+\sqrt{1-\beta^2}}. \tag{16}$$

It is noted that inequality (15) implies that $0 < \beta \leq 1$, and hence (16) always results in real values.

Now, substituting the state -space matrices (3) into the LMI (5) by taking (11) into account results in

$$\begin{bmatrix} -2\alpha D & \alpha F \\ \alpha F^T & -I \end{bmatrix} \leq 0. \tag{17}$$

Applying the Schur complement formula to (17) and using Lemma 3 yields

$$\frac{1}{\alpha} \geq \frac{1}{2}\lambda_{max}(F^T D^{-1} F). \tag{18}$$

In order to satisfy both inequalities (16) and (18), $\frac{1}{\alpha}$ must be chosen to be greater than the maximum of the right hand sides of the inequalities (16) and (18), which results in a bound on α. Finally, inequalities (6) and (7) together result in the bound (9). The upper bound (9) can be simplified further. Note that if (13) holds, then the following inequality holds as well.

$$FF^T \leq \frac{2\gamma}{\alpha} D.$$

Applying Lemma 3 to the latter inequality, it is easy to obtain the following inequality

$$\frac{1}{\alpha} \geq \frac{1}{2\gamma}\lambda_{max}(F^T D^{-1} F). \tag{19}$$

Since inequality (18) holds, condition (19) will hold if $\gamma \geq 1$. This is equivalent to the condition that $\delta \geq \sigma$ in (10). Hence, for $\gamma \geq 1$ the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm bound is simplified to the $\mathcal{H}_2$ norm bound as given in (8), and this completes the proof. □

IV. Explicit Characterization of the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controllers

Consider the controlled vector second-order system represented by (1). The collocated mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis problem is to design a symmetric static feedback gain G such that the output feedback control law

$$u(t) = -Gy(t) \tag{20}$$

renders the closed-loop system stable and

$$\|T_{zw}\|_\infty \leq \gamma_g \tag{21}$$

$$\|T_{zw}\|_2 \leq \eta_g \tag{22}$$

where γ_g and η_g are given scalars, and T_{zw} represents the closed-loop transfer function mapping $w(t)$ to $z(t)$.

The equations of the closed-loop system of the plant (1) and the controller (20) read

$$\begin{aligned} M\ddot{q} + (D + FGF^T)\dot{q} + Kq &= Fw \\ z &= F^T\dot{q}. \end{aligned} \tag{23}$$

The following result provides an explicit expression for the output feedback controller gain that guarantees the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the closed-loop to be less than given bounds γ_g and η_g, respectively.

Theorem 2: Consider the vector second-order system (1). For any given positive scalars γ_g and η_g there exists a symmetric output feedback control law (20) that guarantees the closed-loop $\mathcal{H}_\infty$ norm to be less than γ_g and the $\mathcal{H}_2$ norm to be less than η_g. Assuming $\alpha \geq \alpha_g = \frac{1}{\eta_g^2} trace(F^T M^{-1} F)$, one of the following holds:

(*i*) If F is square and invertible, then G can be selected as

$$\begin{cases} G \geq \frac{\alpha_g^2+1}{2\alpha_g\gamma_g} I - F^{-1}DF^{-T} & \gamma_g \leq 1 + \frac{1}{\alpha_g^2} \\ G \geq \frac{\alpha_g}{2} I - F^{-1}DF^{-T} & \text{otherwise} \end{cases} \tag{24}$$

(*ii*) If FF^T is singular, then G can be selected as

$$\begin{cases} G \geq \frac{\alpha_g^2+1}{2\alpha_g\gamma_g} I - F^{\dagger}DF^{\dagger^T} + \Omega & \text{if } \gamma_g \leq 1 + \frac{1}{\alpha_g^2} \\ G \geq \frac{\alpha_g}{2} I - F^{\dagger}DF^{\dagger^T} + \Omega & \text{otherwise} \end{cases} \tag{25}$$

where $\Omega = (F^{\dagger}DF^{\perp^T})(F^{\perp}DF^{\perp^T})^{-1}(F^{\perp}DF^{\dagger^T})$.

Proof. Considering the Lyapunov matrix (11), inequalities (6) and (7) together yield

$$\alpha \geq \alpha_g = \frac{1}{\eta_g^2} trace(F^T M^{-1} F).$$

Then, substituting the closed-loop system matrices determined from (1) in BRL (4), we obtain

$$-2\alpha_g(D + FGF^T) + \frac{\alpha_g^2+1}{\gamma_g} FF^T \leq 0. \tag{26}$$

If F is invertible, the inequality (26) implies that

$$G \geq \frac{\alpha_g^2+1}{2\alpha_g\gamma_g} I - F^{-1}DF^{-T}. \tag{27}$$

The control gain given in (27) is the one that guarantees the $\mathcal{H}_\infty$ norm of the closed-loop system to be less than γ_g. Substituting the closed-loop system matrices into LMI (5) results in

$$\begin{bmatrix} -2\alpha_g(D + FGF^T) & \alpha_g F \\ \alpha_g F^T & -I \end{bmatrix} \leq 0. \tag{28}$$

Applying the Schur complement formula to (28) leads to

$$-2\alpha_g(D + FGF^T) + \alpha_g^2 FF^T \leq 0. \tag{29}$$

If F^{-1} exists, then we have

$$G \geq \frac{\alpha_g}{2} I - F^{-1}DF^{-1^T} \tag{30}$$

which is the feedback control gain that guarantees the $\mathcal{H}_2$ norm of the closed-loop system to be less than η_g. To design a static feedback controller (20) such that the closed-loop system satisfies both (21) and (22), the controller gain G must satisfy both inequalities obtained in (27) and (30). It is noted that if $\frac{\alpha_g^2+1}{2\alpha_g\gamma_g} > \frac{\alpha_g}{2}$, then the control gain associated with mixed $\mathcal{H}_2/\mathcal{H}_\infty$ specification becomes the one associated with the $\mathcal{H}_\infty$ specification. Otherwise, the control gain associated with the $\mathcal{H}_2$ specification is chosen as the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control gain, and this results in (24).

For the case that FF^T is not invertible, consider the following transformation matrix

$$N = \begin{bmatrix} F^\dagger \\ F^\perp \end{bmatrix}. \tag{31}$$

Pre- and post- multiplying the left hand side of (28) by N and N^T, respectively, we obtain

$$\begin{bmatrix} H + F^\dagger D F^{\dagger^T} - \frac{\alpha_g}{2} I & F^\dagger D F^{\perp^T} \\ F^\perp D F^{\dagger^T} & F^\perp D F^{\perp^T} \end{bmatrix} \geq 0. \tag{32}$$

Applying Schur complement formula to (32) results in the following inequality

$$H \geq \frac{\alpha_g}{2} I - F^\dagger D F^{\dagger^T} + (F^\dagger D F^{\perp^T})(F^\perp D F^{\perp^T})^{-1}(F^\perp D F^{\dagger^T})$$

which is the output feedback control gain associated with the $\mathcal{H}_2$ specification. Using the transformation (31), the inequality (26) leads to

$$\begin{bmatrix} H + F^\dagger D F^{\dagger^T} - \frac{\alpha_g^2+1}{2\alpha_g\gamma_g} I & F^\dagger D F^{\perp^T} \\ F^\perp D F^{\dagger^T} & F^\perp D F^{\perp^T} \end{bmatrix} \geq 0. \tag{33}$$

Applying Schur complement formula to (33) results in the following inequality

$$H \geq \frac{\alpha_g^2 + 1}{2\alpha_g\gamma_g} I - \Omega$$

which is the output feedback control gain associated with the $\mathcal{H}_\infty$ specification. Above discussion along with the similar lines as explained in the proof of (24) results in the explicit feedback control gain (25). More details on the intermediate steps of the proof can be found in [7]. □

V. Integrated Damping Parameter and Control Design for Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Specification

Consider the following vector second-order representation of a structural system with collocated actuators and sensors

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= Fu(t) + Ew(t) \\ y(t) &= F^T\dot{q}(t) \\ z(t) &= E^T\dot{q}(t) \end{aligned} \tag{34}$$

where $q(t), u(t), w(t), y(t), z(t)$ are as defined before. We consider the integrated design problem of simultaneously designing the damping parameters and the output feedback control gain of the collocated structural system obtained from the closed-loop interconnection of the plant (34) and the control law (20) that satisfies a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm closed-loop specification. For lumped parameter systems, the damping matrix D can be expressed in terms of the elemental damping coefficients as follows

$$D = \sum_{i=1}^{l} c_i \mathfrak{T}_i \tag{35}$$

where c_i denotes the viscous damping constant of the ith damper and $\mathfrak{T}_i$ represents the distribution matrix of the corresponding damper in the structural system. The distribution matrices $\mathfrak{T}_i$ are known symmetric matrices with elements 0, 1 and -1 that define the structural connectivity of the damping elements in the structure. Our objective is to formulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ integrated damping parameter and control gain design problems as LMI optimization problems.

Practical structural system design specifications impose upper bound constraints on the values of the damping coefficients, that is

$$0 \leq c_i \leq c_{i\,\max} \quad , \quad i = 1, \ldots, l \tag{36}$$

Also, often an upper bound on the total available damping resources is enforced, that is

$$\sum_{i=1}^{l} c_i \leq c_{\text{cap}}. \tag{37}$$

Another constraint that is needed in the proposed output feedback control design is a bound on the norm of the feedback gain matrix. This restriction is placed to constrain the amount of control effort required by the controller. For this purpose, we include the following constraint in the integrated design problem.

$$\|G\| \leq g_{bound} \tag{38}$$

We assume that $c_{i\,\max}$, c_{cap} and g_{bound} are given scalar bounds determined by the physical constraints of the design problem.

Using the above formulation, the solution of the integrated design of the damping parameters and the output feedback controller to satisfy closed-loop mixed $\mathcal{H}_2/\mathcal{H}_\infty$ specifications is obtained from the following result.

Theorem 3: Consider the collocated structural system (34) with the damping distribution (35). For a given positive scalar γ, the optimum damping coefficients c_i and the controller gain G that minimize the $\mathcal{H}_2$ norm bound of the closed-loop system of the collocated structural system (34) and the output feedback controller (20) and result in an upper bound γ on the closed-loop $\mathcal{H}_\infty$ norm, i.e., $\|G_{zw}\|_\infty \leq \gamma$, can be obtained by solving the following LMI optimization problem with respect to α, G, Z, and c_i:

$$\begin{cases} \min_{c_i,\alpha,Z,G} & \mu^2 \\ \text{subject to} & (40a) - (40g) \end{cases} \tag{39}$$

where the LMI constraints are as follows.

$$-2\Big(\sum_{i=1}^{l} c_i \mathfrak{T}_i + FGF^T\Big) + \alpha EE^T \leq 0 \tag{40a}$$

$$\begin{bmatrix} \alpha M & E \\ E^T & Z \end{bmatrix} \geq 0 \tag{40b}$$

$$\begin{bmatrix} \sum_{i=1}^{l} c_i \mathfrak{T}_i + FGF^T - \frac{\alpha}{2\gamma} EE^T & E \\ E^T & 2\alpha\gamma \end{bmatrix} \geq 0 \tag{40c}$$

$$trace(Z) \leq \mu^2 \tag{40d}$$

$$0 \le c_i \le c_{i\,\max} \quad , \quad i = 1, \ldots, l \tag{40e}$$

$$\sum_{i=1}^{l} c_i \le c_{cap} \tag{40f}$$

$$\|G\| \le g_{bound} \tag{40g}$$

Proof. Using the Lyapunov matrix P as in (11) and substituting it into the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ inequality conditions along with the appropriate application of the Schur complement formula results in the LMIs of Theorem 3. □

Remark 1: The total number of unknown parameters in the LMI optimization problem of Theorem 3 is $m(m+1)+l+1$ that correspond to the independent elements of the symmetric feedback gain matrix G, the damping constants c_i, the parameter α, and the positive definite matrix Z. It is also noted that the total number of LMI constraints is $l+6$.

Remark 2: The control gain matrix norm bound condition in (40g) can be written in an LMI form as follows

$$\begin{bmatrix} g_{bound}^2 I & G^T \\ G & I \end{bmatrix} \ge 0 \tag{41}$$

Remark 3: Following similar lines as above, the results of Theorem 3 can be used to minimize the available damping resources c_{cap} or the control gain norm g_{bound} subject to a given bound on the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ norm of the closed-loop system. For example, minimization of the control gain norm g_{bound} subject to a given bound γ of the $\mathcal{H}_\infty$ norm and μ of the $\mathcal{H}_2$ norm of the closed-loop system can be achieved by solving the following LMI optimization problem.

$$\begin{cases} \min_{c_i, G} & g_{bound} \\ \text{subject to} & (40a)-(40g) \end{cases} \tag{42}$$

VI. Numerical Examples

In this section, we validate and compare our proposed bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of open-loop collocated structural systems and the corresponding static output feedback control gain computation by providing an illustrative example. Next, we validate the proposed integrated damping parameter and control design methodology in an $\mathcal{H}_2/\mathcal{H}_\infty$ setting using a computational example. The MATLAB Robust Control Toolbox is used to solve the corresponding LMI optimization problems.

Example 1: We consider a simple 3-DOF structural system of masses, dampers and springs as depicted in Figure 1. The distribution matrix is given by $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$, and the system parameters are assumed to be $m_i = 1$, $k_i = 1$, and $d_i = \beta k_i$ for $i = 1, 2, 3$. We assume the desired $\mathcal{H}_\infty$ norm bound on the open-loop system to be 0.9. The objective is to minimize the $\mathcal{H}_2$ norm of the system, while the condition $\|G_{zw}\|_\infty \le 0.9$ holds. Varying the value of the parameter β, we plot the upper bounds on the achievable $\mathcal{H}_2$ norm of the open-loop system determined from Theorem 1 compared with the actual $\mathcal{H}_2$ norm of the system computed using MATLAB, versus parameter β. For a known level of $\mathcal{H}_\infty$ norm γ, the $\mathcal{H}_2$ norm of the system is computed by solving the optimization problem

$$\begin{aligned} &min \quad \eta \\ &s.t. \quad (4)-(7) \end{aligned}$$

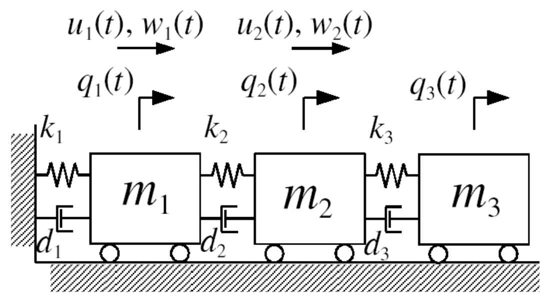

Fig. 1. Mass, spring and damper system

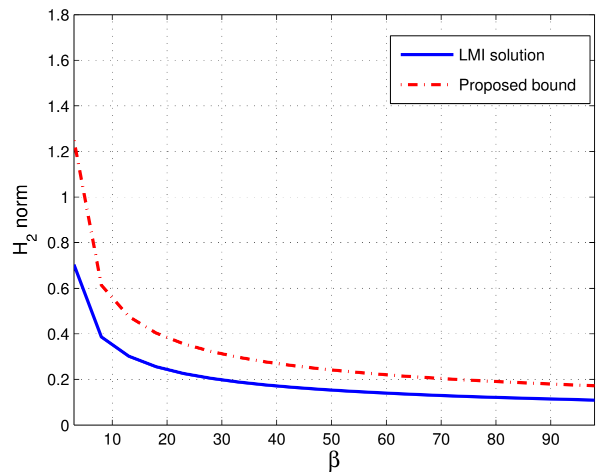

Fig. 2. Comparison of the actual $\mathcal{H}_2$ norm and the $\mathcal{H}_2$ bound vs. the parameter β.

This comparison has been illustrated in Figure 2 where it is observed that the proposed analytical bound of this paper provides a good approximation for the $\mathcal{H}_2$ norm of the system subject to an $\mathcal{H}_\infty$ norm bound.

For synthesis purposes, we fix $\beta = 0.015$ and design a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller that guarantees the $\mathcal{H}_2$ norm of the system to be less than or equal to 1.15, and the $\mathcal{H}_\infty$ norm to be less than or equal to 2.32. Note that the design becomes suboptimal and the feedback control gain is determined, using the result of Theorem 2, to be

$$G \ge G_0 = \begin{bmatrix} 0.4175 & 0.0150 & 0 \\ 0.0150 & 0.4175 & 0.0150 \\ 0 & 0.0150 & 0.4325 \end{bmatrix}.$$

The actual $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the closed-loop system with the control law $u(t) = -G_0 y(t)$ given above would be 1.0570 and 2.2346, respectively.

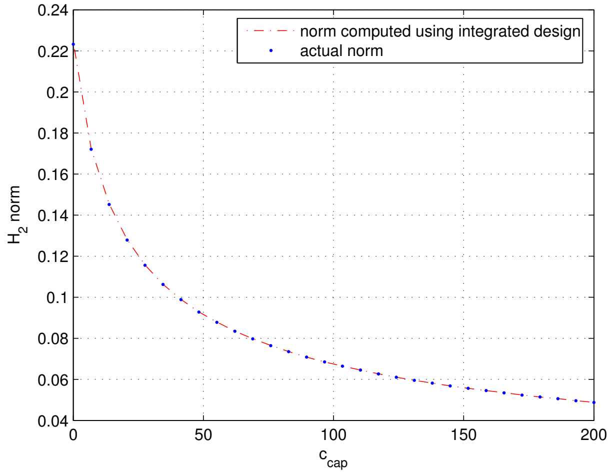

Fig. 3. Profiles of the $\mathcal{H}_2$ norm upper bound and the actual $\mathcal{H}_2$ norm for the optimized structure vs. the total damping capacity.

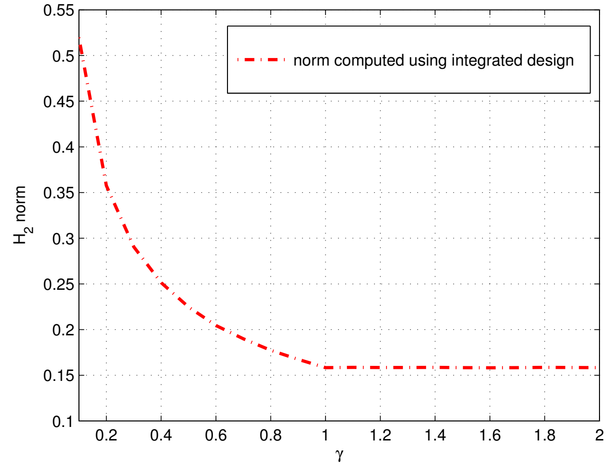

Fig. 4. optimum $\mathcal{H}_2$ norm vs. the bound on $\mathcal{H}_\infty$ norm.

Example 2: For the mass-spring-damper system discussed in Example 1, with the same M, K, and F matrices, and $E = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$, we use the integrated design procedure presented in Theorem 3 to determine the optimized damping coefficients and static output feedback controller gain so that the $\mathcal{H}_2$ norm of the closed-loop system is minimized subject to the $\mathcal{H}_\infty$ norm of the closed-loop system to be less than 1. As the first design scenario, we vary the total available damping resource c_{cap} and plot the $\mathcal{H}_2$ norm bound μ obtained from the solution to the LMIs of Theorem 3, as well as, the exact $\mathcal{H}_2$ norm of the closed-loop system corresponding to each design versus the total damping in Figure 3.

Next, we fix the parameters $c_{cap} = 10$ and $g_{bound} = 10$. The objective here is to minimize $\mathcal{H}_2$ norm of the closed-loop system for different levels of $\mathcal{H}_\infty$ norm performance. The simulation result is shown in Figure 4, where it verifies the trade-off between robustness resulted from employing an $\mathcal{H}_\infty$ specification and average energy performance resulted from an $\mathcal{H}_2$ specification. As observed, when the level of robustness increases, the optimum level of achievable $\mathcal{H}_2$ performance will decrease. Interestingly, beyond $\gamma = 1$, the $\mathcal{H}_2$ norm of the closed-loop system remains unchanged. This indeed confirms the explicit bound results we presented in Theorem 1, where we proved that for $\gamma > 1$ the $\mathcal{H}_2$ norm is independent of γ.

VII. Conclusion

In this paper, we examined the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ analysis and static output feedback control synthesis problems for structural systems with collocated sensors and actuators. Using a particular solution to the BRL and the equivalent LMI representation of the $\mathcal{H}_2$ norm for an open-loop collocated structural system we obtained an explicit expression that is useful to compute an explicit upper bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of such systems. Next, we obtained an explicit parametrization for the output feedback control gains that achieve a desired bound on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm. The paper also presented an efficient LMI-based computational methodology for the simultaneous design of the damping parameters and the control gain of a collocated structural system with velocity feedback to satisfy closed-loop $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance specifications. The computational examples illustrated that the proposed norm bounds provide a close approximation of the actual gains of the system and were effective for structural parameter and control design.

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