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Linearized Analysis versus Optimization-based Nonlinear Analysis for Nonlinear Systems

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Abstract— **For autonomous nonlinear systems stability and input-output properties in small enough (infinitesimally small) neighborhoods of (linearly) asymptotically stable equilibrium points can be inferred from the properties of the linearized dynamics. On the other hand, generalizations of the S-procedure and sum-of-squares programming promise a framework potentially capable of generating certificates valid over quantifiable, finite size neighborhoods of the equilibrium points. However, this procedure involves multiple relaxations (unidirectional implications). Therefore, it is not obvious if the sum-of-squares programming based nonlinear analysis can return a feasible answer whenever linearization based analysis does. Here, we prove that, for a restricted but practically useful class of systems, conditions in sum-of-squares programming based region-of-attraction, reachability, and input-output gain analyses are feasible whenever linearization based analysis is conclusive. Besides the theoretical interest, such results may lead to computationally less demanding, potentially more conservative nonlinear (compared to direct use of sum-of-squares formulations) analysis tools.**

I. Introduction

Internal stability, input-to-state, and input-to-output properties of dynamical systems are commonly analyzed by constructing Lyapunov/storage functions satisfying certain conditions (such as dissipation inequalities) [1], [2], [3], [4]. Generalizations of the S-procedure [5], [4] and sum-of-squares (SOS) relaxations for polynomial nonnegativity [6] provide a framework for the search of such Lyapunov/storage functions for systems with polynomial vector fields based on (linear or bilinear) semidefinite programming (SDP) problems [7], [8], [9], [10], [11], [12], [13], [14], [16], [17], [18].

On the other hand, it is well known that if there exist Lyapunov/storage functions for the linearized dynamics (around an asymptotically stable equilibrium point) then, by certain continuity assumptions, these functions (always) serve as Lyapunov/storage functions for the nonlinear system possibly only locally, i.e., corresponding Lyapunov or dissipation inequalities only hold in a "sufficiently small" neighborhood of the equilibrium point. The promise of SOS programming based nonlinear analysis is that it may be possible to construct Lyapunov/storage functions that satisfy the Lyapunov or dissipation inequalities not only in a "sufficiently small" neighborhood of the equilibrium point but also over quantifiable, non-trivial subsets of the state space. However, the transformation from system analysis questions to corresponding SDP problems (in nonlinear analysis) involves a series of sufficient (but not necessarily necessary) conditions. For example, except certain special or hypothetical cases, S-procedure is not lossless and not all nonnegative polynomials are SOS [9], [6], [19]. Therefore, it is not obvious if (SOS programming based) nonlinear analysis yields a certificate for the nonlinear system whenever the linear analysis does.

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In this paper, we propose conditions for the feasibility of SDP problems (equivalently SOS programming problems), proposed in [5], [20], [7] in the context of stability robustness, reachability, and input-output gain analyses of nonlinear systems around asymptotically stable equilibrium points, based on the properties of the corresponding linearized dynamics. We focus on systems with cubic polynomial vectors fields mainly due to practical reasons. Although SOS programming based analysis can theoretically be used for systems with polynomial vector fields of any finite degree, there are practical bounds on the degree imposed by the capabilities of current SDP solvers and computational resources (see [7], [14] for a more detailed discussion). Therefore, nonlinear analysis with cubic vectors fields is a pragmatic extension for linearization based analysis with tighter approximations for the actual dynamics and richer families of Lyapunov/storage functions.

The motivation is primarily theoretical, showing that the optimization-based (S-procedure/SOS) methods for nonlinear analysis (as proposed in [5], [20], [7]) always involve feasible bilinear SDP problems whenever the linearization is asymptotically stable. Furthermore, these results may also have some limited practical value in actually constructing (possibly conservative) quantitative results for the nonlinear system as outlined in section VI.

Notation: For $\xi \in \mathcal{R}^n$, $\xi \succeq 0$ means that $\xi_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $\xi^T Q \xi \geq 0$ (> 0) for all $\xi \in \mathcal{R}^n$. For $x_1 \in \mathcal{R}^{n_1}$ and $x_2 \in \mathcal{R}^{n_2}$, $[x_1; x_2] \in \mathcal{R}^{n_1+n_2}$ denotes the concatenation of x_1 and x_2. $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients. The subset $\Sigma[\xi] := \{\pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_M^2 \ : \ \pi_1, \cdots, \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of SOS polynomials. For $\eta > 0$ and a function $g : \mathcal{R}^n \to \mathcal{R}$, define the η-sublevel set $\Omega_{g,\eta}$ of g as

$$\Omega_{g,\gamma} := \{x \in \mathcal{R}^n \ : \ g(x) \leq \eta\}.$$

For a piecewise continuous map $u : [0, \infty) \to \mathcal{R}^m$, define the $\mathcal{L}_2$ norm as

$$\|u\|_2 := \sqrt{\int_0^\infty u(t)^T u(t) dt}.$$

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In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. ◁

II. PRELIMINARIES

Following two lemmas are straightforward generalizations of the S-procedure [4]. See [21], [7] for the proofs.

Lemma II.1. *Given $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \cdots, s_m \in \Sigma[x]$ such that $g_0 - \sum_{i=1}^m s_i g_i \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$.*

Lemma II.2. *Given $g_0, g_1, g_2 \in \mathbb{R}[x]$ such that g_0 is positive definite and $g_0(0) = 0$, if there exist $s_1, s_2 \in \Sigma[x]$ such that $g_1 s_1 + g_2 s_2 - g_0 \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x) \leq 0\} \backslash \{0\} \subset \{x \in \mathcal{R}^n : g_2(x) > 0\}$.* ◁

The following fact will be used in the subsequent sections.

Lemma II.3. *Let $Q = Q^T \in \mathcal{R}^{n \times n}$ be positive definite, $f : \mathcal{R}^n \to \mathcal{R}$ be defined as $f(x) = x^T Q x$, $c_1, \ldots, c_m$ be positive real numbers, and $g : \mathcal{R}^m \to \mathcal{R}$ be defined as $g(y) = c_1 y_1^2 + c_2 y_2^2 + \ldots + c_m y_m^2$. Then, $f(x)g(y)$ can be written in the form*

$$f(x)g(y) = \mathbf{z}(x, y)^T H \mathbf{z}(x, y),$$

where $\mathbf{z}(x, y) = y \otimes x$ and $H \succ 0$. ◁

Proof:

$$\begin{aligned} f(x)g(y) &= x^T Q x (c_1 y_1^2 + \ldots + c_m y_m^2) \\ &= \textstyle\sum_{i=1}^m c_i (y_i x)^T Q (y_i x) \\ &= \mathbf{z}(x, y)^T H \mathbf{z}(x, y), \end{aligned}$$

where $H = H^T \in \mathcal{R}^{nm \times nm}$ is

$$H := \begin{bmatrix} c_1 Q & & \\ & \ddots & \\ & & c_n Q \end{bmatrix}.$$

Clearly, H is positive definite since Q is positive definite. □

Lemma II.4. *Let Q and f be as in Lemma II.3, $c_1, \ldots, c_n$ be positive real numbers, and $g : \mathcal{R}^n \to \mathcal{R}$ be defined as $g(x) = c_1 x_1^2 + \ldots + c_n x_n^2$. Then, $f(x)g(x)$ can be written in the form*

$$f(x)g(x) = \mathbf{z}(x)^T H \mathbf{z}(x),$$

where $\mathbf{z}(x)$ is a vector of all monomials of the form $x_i y_j$ for $i = 1, \ldots, n$ and $j \geq i$ with no repetition. ◁

Lemma II.5. *Let Q and f be as in Lemma II.3, $c_1, \ldots, c_{n+m}$ be positive real numbers, and $g : \mathcal{R}^{m+n} \to \mathcal{R}$ be defined as $g(x, y) = c_1 y_1^2 + c_2 y_2^2 + \ldots + c_m y_m^2 + c_{m+1} x_1^2 + \ldots + c_{m+n} x_n^2$. Then, $f(x)g(x, y)$ can be written in the form*

$$f(x)g(x, y) = \mathbf{z}(x, [x; y])^T H \mathbf{z}(x, [x; y]),$$

where $\mathbf{z}(x, [x; y])$ is a vector of all monomials of the form x_i^2 for $i = 1, \ldots, n$ and $x_i y_j$ for $i = 1, \ldots n$ and $j = 1, \ldots m$ with no repetition. ◁

Although $\mathbf{z}$ (as defined above)depends on x and/or y, this dependence will not be explicitly notated whenever it is convenient and does not cause confusion.

III. $\mathcal{L}_2 \to \mathcal{L}_2$ INPUT-OUTPUT GAIN ANALYSIS

Consider the dynamical system governed by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^{n_w}$, and f is a n-vector with elements in $\mathbb{R}[(x, w)]$ such that $f(0, 0) = 0$ and h is an n_y-vector with elements in $\mathbb{R}[x]$ such that $h(0) = 0$. Let $\phi(t; \mathbf{x}_0, w)$ denote the solution to (1) at time t with the initial condition $x(0) = \mathbf{x}_0$ driven by the input/disturbance w.

Lemma III.1. *[22] If there exist real scalars $\gamma > 0$ and $R \geq 0$ and a continuously differentiable function V such that*

$$V(0) = 0 \text{ and } V(x) > 0 \quad \text{for all nonzero } x \in \mathcal{R}^n, \tag{2}$$

$$\Omega_{V,R^2} \text{ is bounded}, \tag{3}$$

$$\nabla V f(x, w) \leq w^T w - \gamma^{-2} y^T y \quad \forall x \in \Omega_{V,R^2},\ w \in \mathcal{R}^{n_w}, \tag{4}$$

then for the system in (1)

$$x(0) = 0 \text{ and } \|w\|_2 \leq R \Rightarrow \|y\|_2 \leq \gamma \|w\|_2. \tag{5}$$

◁

In other words, γ is an upper bound for the "local" input-output gain for the system in (1). For given $\gamma > 0$, we restrict V to be a polynomial of some fixed degree and use Proposition III.1 to compute lower bounds on the maximum value of R such that (2)-(4) hold.

Proposition III.1. *[21] For given $\gamma > 0$ and positive definite polynomial l_1 satisfying $l_1(0) = 0$, let $R_{\mathcal{L}_2}$ be defined through*

$$R_{\mathcal{L}_2}^2 := \max_{V \in \mathcal{V}_{poly}, R \geq 0, s \in \mathcal{S}} R^2 \quad \text{subject to} \tag{6}$$

$$V(0) = 0, \quad s_1 \in \Sigma[(x, w)], \tag{7}$$

$$V - l_1 \in \Sigma[x], \tag{8}$$

$$-\left[(R^2 - V)s_1 + \nabla V f(x, w) - w^T w + \gamma^{-2} y^T y\right] \in \Sigma[(x, w)], \tag{9}$$

where $\mathcal{V}_{poly} \subseteq \mathcal{V}$ and $\mathcal{S}$ are prescribed finite-dimensional subsets of $\mathbb{R}[x]$. Then,

$$x(0) = 0 \text{ and } \|w\|_2 \leq R_{\mathcal{L}_2} \Rightarrow \|y\|_2 \leq \gamma \|w\|_2.$$

◁

Now, consider the case where f and h are of the form

$$\begin{aligned} f(x, w) &= Ax + Bw + f_2(x) + f_3(x) \\ &\quad + (g_1(x) + g_2(x))w, \\ h(x) &= Cx + h_2(x), \end{aligned} \tag{10}$$

where f_2, f_3, g_1, g_2, and h_2 are matrices (of appropriate dimension) of (purely) quadratic, cubic, linear, quadratic, and quadratic polynomials in their arguments respectively and A, B, and C are matrices (of reals) of appropriate dimension. Then, the following proposition gives conditions on the feasibility of the constraints in (6)-(9) based on the analysis of the corresponding linearized dynamics.

Proposition III.2. *For given $\gamma > 0$, $l_1(x) = x^T L_1 x$ with $L_1 \succ 0$, f and h in the form in (10), if there exist a symmetric matrix Q and $\epsilon > 0$ such that $Q \succeq L_1$ and*

$$D_0 := \begin{bmatrix} A^TQ + QA + \gamma^{-2}C^TC & QB \\ B^TQ & -I \end{bmatrix} \preceq -\epsilon I,$$

then the constraints in (6)-(9) are feasible. ◁

Proof: Define $V(x) := x^TQx$. Let $\mathbf{z} = \mathbf{z}(x, [x; w])$ be as defined in section II. Then, there exist $H \succ 0$, M_1, and M_2 such that

$$\begin{aligned} \mathbf{z}^T H \mathbf{z} &= (w^Tw + x^Tx)(x^TQx) \\ x^T M_1 \mathbf{z} &= x^TQ(f_2(x) + g_1(x)w) + x^TC^Th_2(x) \\ \mathbf{z}^T M_2 \mathbf{z} &= 2x^TQ(f_3(x) + g_2(x)w) + h_2(x)^Th_2(x). \end{aligned}$$

Let $\alpha > 0$ be such that

$$D_1 := \begin{bmatrix} -D_0 & -M_1 \\ -M_1^T & \alpha H - M_2 \end{bmatrix} \succeq \epsilon I$$

and $R := \sqrt{\epsilon/(2\alpha)}$. Define

$$s_1(x, w) := \alpha(x^Tx + w^Tw).$$

Then, $V - l_1$ and s_1 are SOS. Consider

$$\begin{aligned} b(x, w) := &- \left[\nabla V f(x, w) - w^Tw + h^T(x)h(x)\right] \\ &- \alpha(x^Tx + w^Tw)\left(R^2 - V\right), \end{aligned}$$

which can be decomposed as

$$b(x, w) = [x; w; \mathbf{z}]^T D_2 [x; w; \mathbf{z}],$$

where

$$D_2 := D_1 - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon I - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \frac{\epsilon}{2} I.$$

Hence, b is SOS. □

IV. Reachability analysis

For $R \geq 0$ and $\|w\|_2 \leq R$, the set $\mathcal{G}_{R^2}$ of points reachable from the origin under the flow of (1) is defined as

$$\mathcal{G}_{R^2} := \{\phi(T; 0, w) \in \mathcal{R}^n \;:\; T \geq 0,\; \|w\|_2 \leq R\}.$$

Lemma IV.1 adapted from a Lyapunov-like argument in [4, §6.1.1] provides a characterization of sets containing $\mathcal{G}_{R^2}$ [5], [22].

Lemma IV.1. *If there exists a continuously differentiable function V such that*

$$V(x) > 0 \text{ for all } x \in \mathcal{R}^n \backslash \{0\} \text{ with } V(0) = 0, \quad (11)$$

$$\Omega_{V,R^2} \text{ is bounded}, \quad (12)$$

$$\nabla V f(x, w) \leq w^Tw \;\; \forall x \in \Omega_{V,R^2},\; w \in \mathcal{R}^{n_w}, \quad (13)$$

then $\mathcal{G}_{R^2} \subseteq \Omega_{V,R^2}$. ◁

For given $\beta > 0$ and positive definite, convex polynomial p, the following proposition provides a lower bound for the maximum value of R such that $\mathcal{G}_{R^2} \subseteq \Omega_{p,\beta}$.

Proposition IV.1. *[22] Let $\beta > 0$, l_1 be a positive definite polynomial satisfying $l_1(0) = 0$, R_{reach} be defined through*

$$R_{reach}^2 := \max_{V \in \mathcal{V}_{poly}, R \geq 0, s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2} R^2 \text{ subject to} \quad (14)$$

$$V(0) = 0,\; s_1 \in \Sigma[x], \text{ and } s_2 \in \Sigma[(x, w)], \quad (15)$$

$$V - l_1 \in \Sigma[x], \quad (16)$$

$$(\beta - p) - (R^2 - V)s_1 \in \Sigma[x], \quad (17)$$

$$-\left[(R^2 - V)s_2 + \nabla V f(x, w) - w^Tw\right] \in \Sigma[(x, w)], \quad (18)$$

where $\mathcal{V}_{poly} \subset \mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subsets of $\mathbb{R}[x]$. Then,

$$\mathcal{G}_{R_{reach}^2} \subseteq \Omega_{V, R_{reach}^2} \subseteq \Omega_{p,\beta}.$$

◁

Proposition IV.2. *For $p(x) = x^TPx$ with $P \succ 0$, $l_1(x) = x^TL_1x$ with $L_1 \succ 0$, and f of the form in (10), if there exist $\epsilon > 0$ and $Q \succeq L_1$ such that*

$$D_0 := \begin{bmatrix} A^TQ + QA & QB \\ B^TQ & -I \end{bmatrix} \preceq -\epsilon I,$$

then the constraints in (14)-(18) are feasible. ◁

Proof: Define $V(x) := x^TQx$. Let $\mathbf{z} = \mathbf{z}(x, [x; w])$ be as defined in section II. Then, there exist $H \succ 0$ (by Lemma II.3), M_1, and M_2 such that

$$\begin{aligned} \mathbf{z}^T H \mathbf{z} &= (w^Tw + x^Tx)(x^TQx) \\ x^T M_1 \mathbf{z} &= x^TQ(f_2(x) + g_1(x)w) \\ \mathbf{z}^T M_2 \mathbf{z} &= 2x^TQ(f_3(x) + g_2(x)w). \end{aligned}$$

Let $\alpha > 0$ be such that

$$D_1 := \begin{bmatrix} -D_0 & -M_1 \\ -M_1^T & \alpha H - M_2 \end{bmatrix} \succeq \epsilon I,$$

and $R := \sqrt{\epsilon/(2\alpha)}$. Define

$$\begin{aligned} s_1(x) &:= \lambda_{max}(P)/\lambda_{min}(Q) \\ s_2(x, w) &:= \alpha(x^Tx + w^Tw). \end{aligned}$$

Then, $V - l_1$, s_1, s_2, and $(\beta - p) - (R^2 - V)s_1$ are SOS. Consider

$$b(x, w) := -\nabla V f(x, w) + w^Tw - \alpha(x^Tx + w^Tw)\left(R^2 - V\right),$$

which can be decomposed as

$$b(x, w) = [x; w; \mathbf{z}]^T D_2 [x; w; \mathbf{z}],$$

where

$$D_2 := D_1 - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon I - \begin{bmatrix} \alpha R^2 I & 0 \\ 0 & 0 \end{bmatrix} \succeq \frac{\epsilon}{2} I.$$

Hence, b is SOS. □

A. Extensions of the reachability analysis for systems with degenerate linearization

Consider the system

$$\begin{array}{rcl} \dot{x}(t) & = & A_m x(t) + B\Lambda \hat{K}_x^T(t)x(t) + Ew(t) \\ \dot{\hat{K}}_x & = & -\Gamma_x x(t)x^T(t)PB, \end{array} \tag{19}$$

where $x(t) \in \mathcal{R}^n$, $B \in \mathcal{R}^{n\times m}$, $w(t) \in \mathcal{R}^{n\times n_w}$, and P, E, Λ, A_m, and Γ_x are matrices of appropriate dimension with Hurwitz A_m. The dynamics in (19) can be considered as the closed loop dynamics for the system $\dot{x}(t) = A_m x(t) + B\Lambda u(t)$ regulated to the origin by a model reference adaptive controller of the form[23]

$$u(t) = \hat{K}_x(t)x(t)$$

in the presence of the disturbance w. Note that the results in Proposition IV.2 is not applicable to the system in (19) because its linearization at the origin is not asymptotically stable. Nevertheless, the nonlinear reachability analysis as outlined in Proposition IV.1 is still applicable.

Proposition IV.3. *Let $x_1 \in \mathcal{R}^{n_1}$, $x_2 \in \mathcal{R}^{n_2}$, and $w \in \mathcal{R}^{n_w}$ and consider*

$$\begin{array}{rcl} \dot{x}_1(t) & = & Ax_1(t) + b(x_1(t), x_2(t)) + Ew(t) \\ \dot{x}_2(t) & = & q(x_1(t)) \end{array} \tag{20}$$

where $b : \mathcal{R}^{n_1+n_2} \to \mathcal{R}^{n_1}$ whose entries are bilinear polynomials in x_1 and x_2, $q : \mathcal{R}^{n_1} \to \mathcal{R}^{n_2}$ whose entries are quadratic polynomials in x_1, and E and A are real matrices of appropriate dimension such that there exist $Q_1 = Q_1^T \succ 0$ and $\epsilon > 0$ with

$$\begin{bmatrix} A^TQ_1 + Q_1A & Q_1E \\ E^TQ_1 & -I \end{bmatrix} \preceq -\epsilon I.$$

. Then, there exist positive definite $V \in \mathbb{R}[(x_1, x_2)]$, $s \in \Sigma[x_1]$, and $R > 0$ such that $b_m \in \Sigma[(x_1, x_2, w)]$ where

$$b_m(x_1, x_2, w) := -\left[\nabla V f(x_1, x_2, w) - w^Tw + (R^2 - V)s\right].$$

◁

Proof: Let $V(x) := x_1^TQ_1x_1 + x_2^TQ_2x_2$, where $Q_2 = Q_2^T \succ 0$. Then, there exist B_1, B_2, $H_1 \succ 0$, and $H_2 \succ 0$ such that

$$\begin{array}{ll} x_1^TQ_1b(x_1,x_2) & = x_1^TB_1\mathbf{z}(x_1,x_2) \\ x_2^TQ_2q(x_1) & = x_1^TB_2\mathbf{z}(x_1,x_2) \\ x_1^TQ_1x_1x_1^Tx_1 & = \mathbf{z}(x_1,x_1)^TM_1\mathbf{z}(x_1,x_1) \\ x_2^TQ_2x_2x_1^Tx_1 & = \mathbf{z}(x_1,x_2)^TM_2\mathbf{z}(x_1,x_2) \end{array}$$

and $-b_m$ can be decomposed as

$$b_m = \begin{bmatrix} x_1 \\ w \\ \mathbf{z}(x_1,x_1) \\ \mathbf{z}(x_1,x_2) \end{bmatrix}^T D \begin{bmatrix} x_1 \\ w \\ \mathbf{z}(x_1,x_1) \\ \mathbf{z}(x_1,x_2) \end{bmatrix},$$

where D is

$$\begin{bmatrix} A^TQ_1 + Q_1A + \alpha R^2 I & Q_1E & 0 & B_1 + B_2 \\ E^TQ_1 & -I & 0 & 0 \\ 0 & 0 & -\alpha M_1 & 0 \\ B_1^T + B_2^T & 0 & 0 & -\alpha M_2 \end{bmatrix}$$

and D negative semidefinite by proper choice of α (sufficiently large) and R (sufficiently small). Consequently, $b_m \in \Sigma[(x_1, x_2, w)]$. □

V. REGION-OF-ATTRACTION ANALYSIS

The material of this section is adapted from [24] where similar results were proven in the context of robust region-of-attraction analysis for systems with parametric uncertainty. For simplicity, we focus on the case without uncertainty. Consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{21}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and f is an n-vector with entries in $\mathbb{R}[x]$ satisfying $f(0) = 0$, i.e., the origin is an equilibrium point of (21). Let $\phi(t; \mathbf{x}_0)$ denote the solution to (21) at time t with the initial condition $x(0) = \mathbf{x}_0$. The region-of-attraction of the origin for the system (21) is

$$\left\{\mathbf{x}_0 \in \mathcal{R}^n : \lim_{t\to\infty} \phi(t; \mathbf{x}_0) = 0\right\}.$$

A modification of a similar result in [2] provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriately chosen Lyapunov functions.

Lemma V.1. *Let $\gamma \in \mathcal{R}$ be positive. If there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ such that*

$$\Omega_{V,\gamma} \text{ is bounded, and} \tag{22}$$

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \mathcal{R}^n \tag{23}$$

$$\Omega_{V,\gamma}\backslash\{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\}, \tag{24}$$

then $\Omega_{V,\gamma}$ is an invariant subset of the ROA. ◁

In order to enlarge the computed invariant subset of the ROA, we define a variable sized region $\Omega_{p,\beta} = \{x \in \mathcal{R}^n : p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed positive definite convex polynomial, and maximize β while imposing the constraint $\Omega_{p,\beta} \subseteq \Omega_{V,\gamma}$ along with the constraints (22)-(24).

SOS programming and simple generalizations of the S-procedure (namely Lemmas II.1 and II.2) provide algebraic sufficient conditions for the constraints in Lemma V.1. Specifically, let l_1 and l_2 be a positive definite polynomials. Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \tag{25}$$

and $V(0) = 0$ are sufficient conditions for (22) and (23). By Lemma II.1, if $s_1 \in \Sigma[x]$, then

$$-[(\beta - p)s_1 + (V - \gamma)] \in \Sigma[x] \tag{26}$$

implies the set containment $\Omega_{p,\beta} \subseteq \Omega_{V,\gamma}$, and by Lemma

II.2, if $s_2, s_3 \in \Sigma[x]$, then

$$-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \in \Sigma[x] \qquad (27)$$

is a sufficient condition for (24). Consequently, $\Omega_{p,\beta^*_{ROA}}$ is a subset of the ROA and Ω_{V^*,γ^*} is an invariant subset of the ROA, where

$$\begin{array}{rl} \beta^*_{ROA} := & \max\limits_{V\in\mathcal{V},\beta,s_i\in\mathcal{S}_i} \beta \text{ subject to } (25)-(27), \\ & V(0)=0, s_i \in \Sigma[x], \beta > 0 \end{array} \qquad (28)$$

and V^* and γ^* are optimal values of V and γ in (28). Here, the sets $\mathcal{V}$ and $\mathcal{S}_i$ are prescribed finite-dimensional subspaces of polynomials.

We now focus on systems governed by ordinary differential equations of the form

$$\dot{x} = f(x) = Ax + f_2(x) + f_3(x), \qquad (29)$$

where f_2 and f_3 are vectors of (purely) quadratic and cubic polynomials, respectively, and $A \in \mathcal{R}^{n\times n}$, and prove that asymptotic stability of the linearized dynamics is also sufficient for the feasibility of the constraints in (28) (for sufficiently small $\gamma > 0$).

Proposition V.1. *Let f be an n-vector of cubic polynomials in x satisfying $f(0) = 0$, and let $P \succ 0$, $R_1 \succ 0$, $R_2 \succ 0$,*

$$p(x) := x^T P x, \quad l_1(x) := x^T R_1 x, \quad l_2(x) := x^T R_2 x.$$

If there exists $Q \succ 0$ such that

$$A^T Q + QA \prec 0,$$

then the constraints in (28) are feasible for some $R > 0$. $\triangleleft$

Proof: The proof is constructive. Let $\mathbf{z} = \mathbf{z}(x)$ be as defined in Lemma II.4, $\tilde{Q} \succ 0$ satisfy $A^T\tilde{Q} + \tilde{Q}A \preceq -2R_2$ and $\tilde{Q} \succeq R_1$ (such $\tilde{Q}$ can be obtained by properly scaling Q). Let

$$\epsilon := \lambda_{min}(R_2), \quad V(x) := x^T \tilde{Q} x,$$

and $H \succ 0$ be such that $(x^T x)V(x) = \mathbf{z}^T H \mathbf{z}$ (which exists by Lemma II.4). Let $M_2 \in \mathcal{R}^{n\times n_z}$ and symmetric $M_3 \in \mathcal{R}^{n_z\times n_z}$ satisfy

$$\begin{array}{ll} \nabla V f_2(x) & = x^T M_2 \mathbf{z} \\ \nabla V f_3(x) & = \mathbf{z}^T M_3 \mathbf{z}. \end{array}$$

Define

$$\begin{array}{ll} s_1(x) & := \frac{\lambda_{max}(\tilde{Q})}{\lambda_{min}(P)} \\ c_2 & := \frac{\lambda_{max}\left(M_3^+ + \frac{1}{2\epsilon}M_2^T M_2\right)}{\lambda_{min}(H)} \\ s_2(x) & := c_2 x^T x \\ \gamma & := \frac{\epsilon}{2c_2} \\ \beta & := \frac{\gamma}{2s_1} \\ s_3(x) & := 1, \end{array}$$

where for a symmetric matrix M, M^+ denotes its projection into the positive semidefinite cone. Clearly, $s_1 \in \Sigma[x]$, $s_2 \in \Sigma[x]$, and $s_3 \in \Sigma[x]$. Note that

$$V(x) - l_1(x) = x^T(\tilde{Q} - R_1)x \in \Sigma[x],$$

since $\tilde{Q} - R_1 \succeq 0$.

$$\begin{aligned} b_1(x) &:= -[(\gamma - V)s_2 + \nabla V f s_3 + l_2] \\ &= \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}^T B_1 \begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}, \end{aligned}$$

where

$$B_1 := \begin{bmatrix} -\gamma c_2 I - R_2 - (A^T\tilde{Q} + \tilde{Q}A) & -M_2/2 \\ -M_2^T/2 & c_2 H - M_3 \end{bmatrix}$$

and

$$B_1 \succeq \begin{bmatrix} \frac{\epsilon}{2} I & -M_2/2 \\ -M_2^T/2 & c_2 H - M_3 \end{bmatrix} \succeq 0$$

by the Schur's complement formula. Consequently, $b_1(x) \in \Sigma[x]$. Finally,

$$\begin{aligned} & -[(\beta - p)s_1 + (V - \gamma)] \\ & \quad = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \underbrace{\begin{bmatrix} -\beta s_1 + \gamma & 0 \\ 0 & s_1 P - \tilde{Q} \end{bmatrix}}_{B_2} \begin{bmatrix} 1 \\ x \end{bmatrix}, \end{aligned} \qquad (30)$$

where $B_2 \succeq 0$ and consequently $b_2 \in \Sigma[x]$. $\square$

VI. Interpretation and demonstration of results

It is worth re-stating that the motivation here is theoretical rather than practical. The conclusions can be summarized as that the nonlinear local analysis (based on S-procedure and SOS programming relaxations as proposed in [5], [20], [7]) is always capable of returning a feasible result (i.e., corresponding optimization problems are feasible) whenever corresponding conditions for the linearized dynamics are feasible. Alternatively, these results may also have some limited practical value in constructing (possibly conservative) quantitative results for the nonlinear system. For example, Propositions V.1, III.2, and IV.2 can be directly used to construct feasible solutions for the problems in Eq. (28) and Propositions III.1 and IV.1, respectively. Proofs of Propositions V.1, III.2, and IV.2 also suggest a recipe for constructing less conservative feasible solutions for these problems by searching for an "optimal" quadratic Lyapunov function (instead of fixing V to a Lyapunov function for the linearization). A construction in the case of region-of-attraction analysis can be summarized as follows: Choose the multipliers s_1, s_2, and s_3 in the form given in the proof of Proposition V.1 with the free parameter c_2. Affinely parameterize H, M_2, and M_3 in terms of Q (note that there may be multiple possible parameterizations for M_2 and M_3 and the choice may change the quantitative results - here we arbitrarily choose one parametrization). Then, Ω_{p,β^*} is a subset of the ROA where

$$\begin{gathered} \beta^* := \max_{\gamma,c_2,\beta,Q=Q^T\succeq R_1} \beta \quad \text{subject to} \\ \begin{bmatrix} -\gamma c_2 I - R_2 - A^T Q - QA & -M_2(Q)/2 \\ -M_2(Q)^T/2 & c_2 H(Q) - M_3(Q) \end{bmatrix} \succeq 0 \\ \begin{bmatrix} -\beta + \gamma & 0 \\ 0 & P - Q \end{bmatrix} \succeq 0. \end{gathered} \qquad (31)$$

Note that the above problem can be solved through a series of convex SDP problems by a line search on c_2. Construction of feasible solutions for the problems in Propositions III.1 and IV.1 can be developed in a similar manner.

The value of such "suboptimal" construction of feasible solutions for the problems in the context of nonlinear system analysis may be better appreciated by recalling the fact that one of the main difficulties in SOS programming based nonlinear system analysis is the computational complexity of the SOS programming. The procedure outlined above provides an ad hoc way of generating (possibly high quality) solutions for the corresponding optimization problems or initial seeds for further optimization. The following example demonstrates this construction for ROA analysis and compares the results with "optimal"solutions from (28).

Example VI.1. *Consider the Van der Pol dynamics*

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2, \end{aligned}$$

which have a stable equilibrium point at the origin and an unstable limit cycle around the origin which is the boundary of the ROA of the equilibrium point. In this example, we will construct invariant subsets of the ROA using the problem in Eq. (28) and Proposition V.1. Let $x = [x_1\ x_2]^T$, $p(x) = x^T x$, $l_1(x) = l_2(x) = 10^{-6} x^T x$. *The solution of the problem in Eq. (28) with a quadratic Lyapunov function candidate, (purely) quadratic* s_2, *and scalar* s_1 *and* s_3 *certifies* $\Omega_{p,1.57}$ *to be a subset of the ROA. The feasible solution provided in Proposition V.1 certifies* $\Omega_{p,0.20}$ *to be a subset of the ROA. Alternatively, by the procedure outlined above certifies that* $\Omega_{p,0.65}$ *is in the ROA.* ◁

VII. CONCLUSIONS

Sum-of-squares programming based analysis of nonlinear systems with polynomial vector fields may be regarded superior to analysis based on linearized dynamics in the sense that the former is capable of generating quantitative certificates as opposed to conclusions from the latter valid only over infinitesimally small neighborhoods of the equilibrium points. However, sum-of-squares based approach involves multiple relaxations. Therefore, it is not obvious if the sum-of-squares programming based nonlinear analysis can return feasible answers whenever linearization based analysis does. In this paper, we proved that, for a restricted but practically useful class of systems, conditions in sum-of-squares programming based region-of-attraction, reachability, and input-output gain analyses are feasible whenever linearization based analysis is conclusive. Besides the theoretical interest, such results may lead to computationally less demanding, potentially more conservative nonlinear (compared to direct use of sum-of-squares programming formulations) analysis tools.

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