

# A Class of Neutral-Type Delay Differential Equations that are Effectively Retarded

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**Abstract**— We demonstrate that some delay-differential equations of neutral type are, up to basis transformation, equivalent to retarded delay differential equations. In particular, for two classes of neutral delay differential equation models, we use state transformations to show that delayed derivatives can in some cases be expressed in terms of the model's state. Hence, we obtain conditions when neutral delay differential equations can be transformed into retarded delay differential equations.

## I. INTRODUCTION

The class of differential equations that involve delayed derivatives is classically referred to in the mathematics and control communities as neutral delay differential equations [3]. As opposed to retarded delay differential equations (ones that do not involve delayed derivatives), those of neutral type may exhibit such peculiarities as spectra with an infinite numbers of roots in certain right half planes with imaginary parts tending to infinity, which unfortunately brings stability and robustness to parameter variations of such systems into question. In this paper, we point out that some delay differential equations that are traditionally classified as neutral (i.e. having delayed derivatives in the equation) are essentially retarded. Specifically, we study two neutral delay differential equation models; the first is motivated in the study of output feedback control, while the second (and very classical) model arises in numerous feedback control as well as modeling applications. For both models, through using smart state transformations including the widely-used special coordinate basis (SCB) transformation [9] and more tailored transformations, we give conditions under which the delayed derivatives can be expressed in terms of the models' states, and hence show that such equations actually have retarded type dynamics. This study significantly helps clarify the definitions of neutral and retarded delay differential equations.

The remainder of the article is organized as follows. In Section 2, we motivate and describe the first neutral delay-differential equation model, namely one that arises when multiple output derivatives of an LTI system are used in feedback upon delay. Then we give a condition under which such a differential equation is equivalent to ones

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of retarded-type, using the SCB. In Section 3, we study a delay differential equation model that is classical in the study of neutral systems, namely one in which multiply-delayed first derivatives are present. We give the necessary and sufficient condition that such a differential equation can be made equivalent to one of retarded type through a state transformation.

## II. EQUIVALENT RETARDED REPRESENTATIONS FOR A MULTIPLE-DERIVATIVE-FEEDBACK MODEL

Time-derivatives of system outputs (up to a certain order) are well-understood to codify state information [9]. Thus, state estimation, which is needed for feedback controller design, requires the designer in one way or other to obtain derivatives of system outputs. For systems that are subject to time delays in observation, as well as ones where model-based observer design is impracticable and instead signal-based methods are needed (e.g., adaptive or decentralized systems), direct computation/approximation of output derivatives for feedback control may be a promising strategy [4], [5], [10]. One natural means for using output derivatives in feedback is through delayed measurement or delayed computation<sup>1</sup>. Moreover, various natural and engineered systems from such diverse domains as computational biology and electric power system management are modeled using differential equations with delayed-derivative terms (e.g., [1]). Motivated by these complementary control and modeling applications, we study the dynamics of a class of linear delay systems (linear delay differential equations) with delayed-derivative feedback. Our key result here is that these delayed-derivative dynamics emulate the drastically different characteristics of neutral-type and retarded-type dynamics, depending on the order of the derivative used in feedback.

The delayed-derivative model that we consider here comprises an LTI plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^m, \quad \mathbf{y} \in \mathbb{R}^p,$$

where the input  $\mathbf{u}$  is a linear combination of delayed output derivatives of multiple orders. In particular, the input is

$$\mathbf{u}(t) = \sum_{i=0}^{M-1} K_i \mathbf{y}^{(i)}(t-h), \quad t \geq 0,$$

where the delay  $h$  is strictly positive, the gains  $K_i \in \mathbb{R}^{m \times p}$  may be arbitrary,  $M$  is a positive integer, and the initial condition of the system is the signal  $\mathbf{x}(t)$  over the time-interval

<sup>1</sup>We stress that direct computations, just like any state estimation method, may be susceptible to sensor noise; we refer the reader to [7] for intelligent implementations.

$[-h, 0]$ . This class of feedback models is representative of systems where observations of outputs and their derivatives (e.g., velocity or position-derivative measurements) are subject to delay (e.g., due to the need for communication through a data channel). Substituting for the input in terms of the output and then the state, we automatically see the closed-loop dynamics are described by the following delay-differential equation:

$$\dot{\mathbf{x}} = A\mathbf{x} + B \sum_{i=0}^{M-1} K_i C \mathbf{x}^{(i)}(t-h). \quad (1)$$

This delay differential equation is of neutral type for  $M = 2$  and of advanced type for  $M > 2$ , since it involved first derivatives (respectively, higher derivatives) of the delayed state vector  $\mathbf{x}(t-h)$  for  $M = 2$  (respectively,  $M > 2$ ). We refer to this model as **multiple-derivative-feedback model**.

The multiple-derivative-feedback model, which is nominally described by neutral delay differential equations can in certain case be equivalently represented by retarded delay differential equations. That is, the delay differential equation can sometimes be rewritten without any delayed *derivative* terms. The concept underlying this reformulation is simple: derivatives of linear-system outputs (or their linear combinations) up to a certain order generally can be written as linear functions of the state variables, and hence in our case the delayed-derivative terms (up to a certain order) can be re-written in terms of the the state. The order of the derivatives of particular linear combinations of the output that can be written in this way follows immediately from a structural decomposition of linear systems known as the *special coordinate basis (SCB)* [8], [9]. This equivalence of output (linear combination) derivatives with states is well-established for finite-dimensional LTI plants. What our efforts here clarify is that such an equivalence is in force for delayed-derivative models, and in fact permits us to represent seemingly neutral/advanced-type systems as retarded ones.

To make the presentation clear to both control theorists and modelers, we focus our analysis on the control representations but then also explicitly consider the model form (closed-loop form) as needed. We develop the results in three steps. We first give a sufficient condition for the maximum number of derivatives that can be used in feedback such that, for any set of gains, the system can be equivalenced to a retarded one (Theorems II.1). Second, we discuss the possibility of using higher derivatives of particular linear combinations of outputs while maintaining the retarded structure. A formal description of this general case would require us to develop the SCB in full intricacy (which detracts somewhat from the perspective put forth here), and so we only give a conceptual discussion.

Let us begin with the multiple-derivative-feedback model. Our condition for the maximum number of delayed-derivatives for which the dynamics is effectively retarded is easily phrased in terms of the *Markov parameters* of the plant (from which the special coordinate basis can be constructed, see [9]). We recall that the  $i$ th Markov parameter is given

by  $\mathcal{M}_i = CA^{i-1}B$ ,  $i = 1, 2, \dots$ . In terms of the Markov parameters, we recover the following upper bound on the order of the delayed derivative, such that any controller will yield a retarded delay system:

**Theorem II.1** *Consider the multiple-derivative-feedback model (1). If the first  $q$  Markov parameters are identically zero, then the delayed-derivative model for any  $M \leq q + 1$  can be rewritten as a retarded model.*

*Proof:* We claim that  $\mathbf{y}^{(i)}(t-h) = CA^{(i)}\mathbf{x}(t-h)$ ,  $i = 0, 1, 2, \dots, q$ . Let us verify this recursively. To do so, notice that the expression is clearly true for  $i = 0$ . Now say that the expression holds for arbitrary  $i \in 1, \dots, q-1$ , and consider  $\mathbf{y}^{(i+1)}(t-h)$ . However, noting that  $\mathbf{y}^{(i+1)}(t-h)$  equals  $\frac{d}{dt}\mathbf{y}^{(i)}(t-h)$ , we obtain that

$$\begin{aligned} \mathbf{y}^{(i+1)}(t-h) &= \frac{d}{dt}CA^{(i)}\mathbf{x}(t-h) \\ &= CA^{(i+1)}\mathbf{x}(t-h) + CA^{(i)}B\mathbf{u}(t-h). \end{aligned}$$

Noticing that the first  $q$  Markov parameters are nil, we recover the result for the first  $q$  output derivatives. From this result, we automatically find that

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} = A\mathbf{x} + B \sum_{i=0}^{M-1} K_i \mathbf{y}^{(i)}(t-h)$$

can in fact be written as

$$\dot{\mathbf{x}} = A\mathbf{x} + B \sum_{i=0}^{M-1} K_i CA^{(i)}\mathbf{x}(t-h),$$

for any  $M \leq q + 1$ . Hence, the system is of retarded type in this case. ■

We thus see that many feedback control systems that at first glance appear to be of neutral or even advanced type are in fact retarded systems. We notice that their spectra do not display any of the characteristics of neutral delay systems, including infinite root chains and hyper-sensitivity to parameter variations. This observation indicates that feedback of delayed derivatives of low enough order will not yield highly unstable/sensitive dynamics, and in fact may be of use in stabilization and other control tasks.

When the highest derivative  $M - 1$  in the multiple-derivative-feedback model is greater than or equal to the number of the first non-zero Markov parameter, it is easy to check that the dynamics will display the characteristics of neutral delay systems (e.g., infinite root-chains) for some feedback gains. However, certain linear combinations of the higher output derivatives may still be linear functions of the concurrent state, hence permitting a retarded representation of the closed-loop system for other gains. We exclude the details in the interest of space, but kindly ask the reader to see work on *special coordinate basis* for the relevant methodologies [9].

### III. RETARDED EQUIVALENCE IN A MULTIPLY-DELAYED-DERIVATIVE MODEL

Delay differential equations with multiply-delayed first derivatives of the state vector are also prominently used in modeling systems subject to time delay (e.g., [1]). These neutral delay models originate from various control systems applications in which multiply-delayed observation derivatives are being used in feedback, as well as from modeling of systems in nature with response delays. Because these differential equations with multiply-delayed derivatives have traditionally been introduced in their differential equation form (rather than a control system form), we also progress from this *modeling* rather than controller design formulation. From this formulation, we study whether a state transformation can be used to transform the neutral differential equation into a retarded delay-differential equation (in an algebraic sense as well as in terms of the spectrum and sensitivity). We are able to obtain necessary and sufficient conditions for this equivalence to a retarded system through any state transformation. We first present this general case along with some motivational examples. We then remark on the development of delay-independent conditions, and illustrate our results in the simple but useful case that the model originates from a feedback control paradigm.

Formally, let us consider the following system:

$$\frac{d}{dt} \left( x(t) - \sum_{j=1}^M H_j x(t - \rho_j) \right) = Ax(t) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ , all matrices are real while  $\rho_j$  and  $\tau_j$  are positive constants for  $j = 1, \dots, M$ .

This is a classical model for neutral linear time-invariant delay systems, which we refer to as the **multiply-delayed-derivative model**. On the other hand we have the classical model for retarded delay systems:

$$\frac{d}{dt} x(t) = Ax(t) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) \quad (3)$$

We recall an important property of retarded delay systems:

**Lemma III.1** Consider a retarded system of the form (3) and the associated spectrum, i.e. the zeros of

$$g(s) = \det \left( sI - A - \sum_{j=1}^M \bar{H}_j e^{-s\tau_j} \right)$$

Then for any  $r \in \mathbb{R}$  there exists only a finite number of zeros of  $g(s)$  in the half plane  $\text{Re } s \geq r$ .

Let us first present an example, that makes clear that state transformation can achieve retarded equivalence in the multiply-delayed-derivative model:

**Example III.2** Consider the system

$$\begin{aligned} \frac{d}{dt} \left( x(t) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(t-1) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(t-2) \right) \\ = Ax(t) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) \end{aligned}$$

where  $A$  and  $\bar{H}_1, \dots, \bar{H}_M$  can be arbitrary. We define a state space transformation:

$$\tilde{x}(t) = x(t) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(t-1) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(t-2)$$

which is nicely invertible:

$$x(t) = \tilde{x}(t) - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{x}(t-1) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{x}(t-2)$$

This transformation results in a model in terms of  $\tilde{x}(t)$  which is of retarded type (3):

$$\begin{aligned} \frac{d}{dt} \tilde{x}(t) = A \left( \tilde{x}(t) - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{x}(t-1) \right. \\ \left. - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{x}(t-2) \right) \\ + \sum_{j=1}^M \bar{H}_j \left( \tilde{x}(t - \tau_j) - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{x}(t - \tau_j - 1) \right. \\ \left. - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{x}(t - \tau_j - 2) \right) \quad (4) \end{aligned}$$

**Example III.3** Consider the same example as in Example III.2. Consider this model in the frequency domain:

$$\begin{aligned} s \left( x(s) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{-s} x(s) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{-2s} x(s) \right) \\ = Ax(s) + \sum_{j=1}^M \bar{H}_j e^{-\tau_j s} x(s) \end{aligned}$$

Premultiply the above equation on both sides from the left by:

$$I - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-s} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-2s}$$

(which is invertible for all  $s \in \mathbb{C}$ ) We obtain, in the frequency domain:

$$\begin{aligned} sx(s) = \left( I - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-s} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-2s} \right) \\ \times \left( Ax(s) + \sum_{j=1}^M \bar{H}_j e^{-\tau_j s} x(s) \right) \end{aligned}$$

which in the time domain yields a model in terms of  $x(t)$

which is of retarded type (3):

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &\times \left( Ax(t-1) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j - 1) \right) \\ &- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left( Ax(t-2) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j - 2) \right) \end{aligned} \quad (5)$$

The interesting aspect is that this new model is of retarded type in the original state space coordinates without even using a basis transformation.

Based on these examples, we are motivated to determine conditions such that a neutral system of the form (2) can be transformed into a retarded system of the form (3). Next, we present our core mathematical result which will be needed to prove our main results. Due to page limitations the proof is omitted.

**Lemma III.4** Consider a function  $f$  of the form:

$$f(s) = 1 - \sum_{i=1}^R \alpha_i e^{-\beta_i s} \quad (6)$$

where we assume that  $0 < \beta_1 < \beta_2 < \dots < \beta_R$  and, without loss of generality, that  $\alpha_i \neq 0$  for  $i = 1, \dots, R$ . We have:

- The function  $f$  has all zeros in a strip:

$$\mathcal{L} := \{ s \in \mathbb{C} \mid \zeta_1 \leq \operatorname{Re} s \leq \zeta_2 \}.$$

for suitably chosen  $\zeta_1, \zeta_2 \in \mathbb{R}$ .

- $f$  has an infinite number of zeros
- Consider an analytical function  $g$  that is bounded on  $\bar{\mathcal{L}}$  defined by:

$$\bar{\mathcal{L}} := \{ s \in \mathbb{C} \mid \zeta_1 \leq \operatorname{Re} s \leq \zeta_2, |s| > 1 \}.$$

In that case, the function  $d(s)$  defined by:

$$d(s) = f(s) + \frac{1}{s}g(s)$$

has an infinite number of zeros in  $\bar{\mathcal{L}}$ .

Let us now present our main result regarding existence of a basis transformation for retarded equivalence of a neutral system:

**Theorem III.5** Consider a system of the form (2) and define

$$f(s) = \det \left( I - \sum_{j=1}^M H_j e^{-s\rho_j} \right).$$

There exists an invertible basis transformation of the form:

$$\tilde{x}(t) = x(t) - \sum_{j=1}^K W_j x(t - \rho_j) \quad (7)$$

such that  $\tilde{x}(t)$  satisfies a retarded delay model of the form (3) if and only if  $f(s) = 1$  for all  $s \in \mathbb{C}$ .

Moreover, in that case, we can choose  $W_j = H_j$  and  $K = M$  and the basis transformation (7) has the property that

$$\left( I - \sum_{j=1}^M H_j e^{-\rho_j s} \right)^{-1} = I - \sum_{j=1}^N \bar{V}_j e^{-\mu_j s}$$

for appropriately chosen  $\bar{V}_1, \dots, \bar{V}_N$  and  $\mu_1, \dots, \mu_N$ . Moreover, besides (7), we have that:

$$x(t) = \tilde{x}(t) - \sum_{j=1}^N \bar{V}_j \tilde{x}(t - \mu_j) \quad (8)$$

*Proof:* First, assume  $f(s) = 1$  for all  $s \in \mathbb{C}$ . In that case:

$$I - \sum_{j=1}^M H_j e^{-\rho_j s}$$

is invertible for all  $s \in \mathbb{C}$  and:

$$\left( I - \sum_{j=1}^M H_j e^{-\rho_j s} \right)^{-1} = \operatorname{adj} \left( I - \sum_{j=1}^M H_j e^{-\rho_j s} \right)$$

The adjoint matrix is determined by only using multiplication and addition and hence will be of the form:

$$I - \sum_{j=1}^N \bar{V}_j e^{-\mu_j s}$$

for appropriately chosen  $\bar{V}_1, \dots, \bar{V}_N$  and  $\mu_1, \dots, \mu_N$ . If we define:

$$\tilde{x}(t) = x(t) - \sum_{j=1}^M H_j x(t - \rho_j)$$

then (8) follows from the above arguments with the use of the Laplace transform. But then (2) is trivially transformed into a delay system of retarded type:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A \left[ \tilde{x}(t) - \sum_{j=1}^N \bar{V}_j \tilde{x}(t - \mu_j) \right] \\ &+ \sum_{j=1}^M \bar{H}_j \left[ \tilde{x}(t - \tau_j) - \sum_{j=1}^N \bar{V}_j \tilde{x}(t - \tau_j - \mu_j) \right] \end{aligned}$$

Conversely, assume  $f(s) \neq 1$  for some  $s \in \mathbb{C}$ . In that case, it is easily seen that  $f$  is of the form (6) which implies, by Lemma III.4, that  $f(s)$  has an infinite number of zeros in a strip  $\mathcal{L}$ .

Next consider  $h(s) = \det H(s)$  where:

$$H(s) = sI + \sum_{j=1}^M s H_j e^{-s\rho_j} - A - \sum_{j=1}^M \bar{H}_j e^{-s\tau_j}$$

It is not difficult to verify that:

$$h(s) = s^n f(s) + \sum_{i=0}^{n-1} s^i k_i(s)$$

where  $k_i(s)$  are exponential functions (i.e. a linear combination of exponentials). Then it is easily verified that:

$$g(s) = \sum_{i=0}^{n-1} s^{i-n+1} k_i(s)$$

is an analytic function which is bounded on the strip  $\bar{\mathcal{L}}$  as defined in Lemma III.4. Then, according to Lemma III.4 we find that  $f(s) + \frac{1}{s}g(s)$  has an infinite number of zeros in the strip  $\bar{\mathcal{L}}$  and hence also  $h(s)$  has an infinite number of zeros in the strip  $\bar{\mathcal{L}}$ . Recall that  $h(s)$  is the determinant of  $H(s)$  and hence there exists an infinite number of points  $s_k$  in  $\mathcal{L}$  for which  $H(s_k)$  is singular. In other words, there exists  $x_k \in \mathbb{C}^n$  with  $x_k \neq 0$  such that  $H(s_k)x_k = 0$ . Then  $x(t) = \text{Re}[e^{s_k t} x_k]$  satisfies the system dynamics. After all:

$$\begin{aligned} \dot{x}(t) &= \text{Re}[e^{s_k t} s_k x_k] \\ &= \text{Re} \left[ e^{s_k t} A x_k - e^{s_k t} \sum_{j=1}^M (s_k H_j e^{-s_k \rho_j} - \bar{H}_j e^{-s_k \tau_j}) x_k \right] \\ &= A x(t) - \sum_{j=1}^M H_j \dot{x}(t - \rho_j) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) \end{aligned}$$

Given the structure of our basis transformation, then (7) implies that  $\tilde{x}(t) = \text{Re}[e^{s_k t} \tilde{x}_k]$  where

$$\tilde{x}_k = x_0 - \sum_{j=1}^K W_j e^{-s_k \rho_j} x_k$$

This yields that  $s_k$  is also an element of the point spectrum for the system we obtain after the basis transformation. Therefore the system we obtain after a basis transformation has a point spectrum which has an infinite number of points in a strip  $\mathcal{L}$ . By Lemma III.1 this implies that this system cannot be a retarded delay system. ■

**Remark III.6** Note that just as in Example III.3, instead of a state space transformation we can also find a retarded model in terms of the original state  $x$  by premultiplying the model (after Laplace transformation) by:

$$I - \sum_{j=1}^N \bar{V}_j e^{-\mu_j s}$$

which is of course only well-defined in case  $f(s) = 1$ .

In the above theorem, we have given necessary and sufficient conditions such that basis transformations can be used to convert the multiply-delayed derivative model (2) into a neutral delay equation. If the condition of the above theorem is not satisfied, then in fact there does not exist even a more general state transformation to bring the system into retarded form. Since any reasonable basis transformation should preserve the spectrum, from the proof of the above theorem it is clear that if  $f(s) \neq 1$  for some  $s \in \mathbb{C}$  then the spectrum contains an infinite number of poles in a vertical strip in the complex plane. Hence the system does not satisfy the property outlined in Lemma III.1 that retarded systems

will always have only a finite number of poles in such a vertical strip.

Interestingly, the ability to transform the neutral differential equation into a retarded equation may be highly sensitive to changes in the delays:

**Example III.7** Consider the same system as in example (III.2) but with some uncertainty in the delay terms:

$$\begin{aligned} \frac{d}{dt} \left( x(t) - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(t - \rho_1) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(t - \rho_2) \right) \\ = A x(t) + \sum_{j=1}^M \bar{H}_j x(t - \tau_j) \end{aligned}$$

Applying Theorem III.5 we construct:

$$f(s) = 1 - e^{-2\rho_1 s} + e^{-\rho_2 s}$$

and note that the system is equivalent to a retarded system if and only if  $2\rho_1 = \rho_2$ , a property that is clearly trivially ruined by small perturbations in the delay.

Given the sensitivity to perturbations in the delays of the state space transformations, we can ask ourselves the question of whether we can find a characterization which is independent of the delays. The following theorem gives such a delay-independent characterization:

**Theorem III.8** Consider a system of the form (2). There exists for all  $\mu_1, \mu_2, \dots, \mu_M > 0$  an invertible basis transformation of the form:

$$\tilde{x}(t) = x(t) - \sum_{j=1}^N \bar{V}_j x(t - \mu_j) \quad (9)$$

such that  $\tilde{x}(t)$  satisfies a retarded delay model of the form (3) if and only if

$$\bar{f}(z_1, \dots, z_M) = \det \left( I - \sum_{j=1}^M H_j z_j \right)$$

has no zeros in  $\mathbb{C}^M$  or, equivalently, the function  $\bar{f}$  is equal to 1.

*Proof:* Note that for any value for  $\rho_1, \rho_2, \dots, \rho_M$  we have that such a basis transformation exists if and only if  $f(s)$  is a constant. We know

$$f(s) = 1 - \sum_{i=1}^R \alpha_i e^{-\beta_i s}$$

where  $\beta_1, \dots, \beta_R$  are a linear function of the  $\rho_1, \dots, \rho_M$  while the  $\alpha_i$  are independent of the  $\rho_j$ . Without loss of generality, we can exclude that  $\beta_i = \beta_j$  for all  $\rho_1, \rho_2, \dots, \rho_M$  (then we can simply combine both terms in one). But  $f(s)$  is then equal to a constant if either all  $\alpha_i$  are equal to zero or if  $\beta_i = \beta_j$  for some  $i$  and  $j$  and the corresponding  $\alpha_i$  cancel. In the first case, clearly  $f(s)$  is equal to a constant for all  $\rho_1, \dots, \rho_M$  and it is easily seen that  $\bar{f}(z_1, \dots, z_M)$

is equal to a constant (which, due to the structure of  $\bar{f}$  must be 1). Conversely if  $\beta_i = \beta_j$  then this is a nontrivial linear equation and the set of  $\rho_1, \dots, \rho_M$  that satisfy this form a hyperplane. Hence the points for which  $f(s)$  is a constant form the union of a finite set of hyperplanes and an arbitrary small perturbation brings you to a function  $f(s)$  which has a zero and then

$$\bar{f}(e^{\rho_1 s}, \dots, e^{\rho_M s}) = f(s) = 0 \quad \blacksquare$$

Note that the above condition on  $\bar{f}$  is still a necessary and sufficient condition, when only small perturbations of the delays (rather than arbitrary valuations of them) are possible. That is, if given  $\bar{\rho}_1, \dots, \bar{\rho}_M$ , we require existence of  $\varepsilon > 0$  such that for all  $\rho_1, \dots, \rho_M$  with  $|\rho_i - \bar{\rho}_i| < \varepsilon$  there is a basis transformation such that the new state satisfies a model of the form (3), then the condition is necessary and sufficient.

Of interest, the above delay-independent condition for retarded equivalence can be written explicitly in terms of the matrices  $H_i$ , rather than in terms of the existence of zeros of a function defined thereof. We note that the function  $\bar{f}$  is equal to 1 if and only if the polynomial matrix

$$F(z_1, \dots, z_M) = \sum_{j=1}^M H_j z_j$$

is nilpotent for all  $z_1, \dots, z_M$ . The latter implies that there exists  $m$  such

$$\left( \sum_{j=1}^M H_j z_j \right)^m = 0$$

Since the polynomial matrix is of dimension  $n \times n$ , we find that we can choose  $m \leq n$  and we can choose the same  $m$  for all  $z_1, \dots, z_M$ . A polynomial matrix is clearly zero only if all its coefficients are equal to zero.

Denote by  $\pi(k_1, \dots, k_M)$  with  $k_1 + \dots + k_M = m$ , all possible sequences  $(i_1, \dots, i_m)$  which contain  $k_j$  occurrences of the integer  $j$  for  $j = 1, \dots, M$ . In that case we define the **combinatorial sum**:

$$Q(k_1, \dots, k_M) = \sum_{(i_1, \dots, i_m) \in \pi(k_1, \dots, k_M)} H_{i_1} H_{i_2} \dots H_{i_m}$$

**Theorem III.9** Consider the multiply-delayed-feedback model (2)), where  $M$  is the number of delay terms and  $n$  is the dimension of  $\mathbf{x}(t)$ . If there exists an integer  $m$  such that the combinatorial sums  $Q(k_1, \dots, k_M)$  are zero for all  $k_1, \dots, k_M$  with  $k_1 + \dots + k_M = m$ , then the model is equivalent to a retarded model. Furthermore, if there is no such  $i$ , then the model cannot be viewed as retarded-equivalent for at least some sets of delays  $\rho_1, \dots, \rho_M$ .

This result follows algebraically from the above Theorem III.8. We omit the details.

Finally, let us briefly discuss an example where the multiply-delayed derivative model is obtained from a controls paradigm, to crystallize the connection between the special coordinate basis transformation (as used in the previous

section) and the transformation considered here. Precisely, let us consider an LTI plant  $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ ,  $\mathbf{y} = C\mathbf{x}$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$ ,  $\mathbf{y} \in \mathbb{R}^p$ , where the input  $\mathbf{u}$  is a linear combination of multiply-delayed outputs and output derivatives:

$$\mathbf{u}(t) = \sum_{i=1}^M \bar{K}_i \mathbf{y}(t - \tau_i) + K_i \dot{\mathbf{y}}(t - \rho_i), \quad t \geq 0,$$

where WLOG  $0 < \rho_1 < \rho_2 < \dots < \rho_M$ ,  $0 < \tau_1 < \tau_1 < \dots < \tau_M$ , and the gains  $\bar{K}_i$  and  $K_i$  may be arbitrary. We recover immediately from the special coordinate basis transformation (or from first principles) that the closed-loop dynamics of this neutral system is equivalent to a retarded system whenever  $CB = 0$ . However, we see that the condition is by no means necessary for retarded-equivalence. For instance, consider the system with state equation  $\dot{\mathbf{x}}(t) = \mathbf{u}(t)$

and observation  $\mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$ , with control law

$\mathbf{u}(t) = \dot{\mathbf{y}}(t - \rho)$ . This system's first Markov parameter  $CB$  is nonzero, and yet the closed-loop dynamics satisfy  $\dot{\mathbf{x}}(t) =$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 \mathbf{u}(t - 3\rho) = 0$ , or in other words the dynamics

are retarded-equivalent. This example makes evident that the special coordinate basis transformation is concerned with equivalencing delayed output derivatives with the *concurrent* state, and so is a special case of the transformation developed in this section for the multiply-delayed-derivative model. We leave it to future work to check the whether the broader transformation can be given a structural control-theoretic interpretation, and whether such a transformation can be applied to the multiple derivative feedback model.

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