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Robust Controller Order Reduction

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Abstract—**In this paper a method is proposed to reduce the order of a multi-input multi-output robust controller. The controller ensures system robust performance and reducing its order may result in loss of robust performance. A lower bound on the controller order is provided using balanced truncation technique which guarantees that the robust performance of the closed-loop system is maintained. Simulation results show the effectiveness of the proposed technique.**

I. Introduction

In the design of a feedback system, a model representing the dynamic characteristics of the plant is required. The controller designed for a model of the plant must be able to control the actual plant in spite of the differences between the two. The designed controller may not always guarantee robust stability under the new circumstances caused by the plant uncertainties [1]. Often, stability is not the only property of a closed-loop system that must be robust to plant perturbations. Tracking or regulation errors are caused by combined effect of exogenous disturbances acting on the system and plant perturbations can cause these errors to increase greatly. In such cases, the closed-loop performance can become unacceptable. Hence, it is necessary to check for both robust stability and performance of the system [2].

In common system order reduction problems, it is desired to approximate a system with a lower order model. Two important factors in doing so are to preserve the stability of the system and also come up with an error bound describing how close the reduced-order model is to the original system. In robust controller order reduction problem, it is also important to guarantee that closed-loop robust performance is preserved. One of the most powerful techniques of model order reduction is balanced truncation [3], [4] which is widely used for stable high-order systems.

In typical robust control problems, the generated controller can be of high order compared to other system components which is not desirable, specially in case of real-time implementation on slow industrial hardware which may limit the achievable sampling rate [5], [6]. The main objective of this work is to reduce the order of high-order robust controllers. Doing so can significantly reduce the complexity of the device required for realization of the controller while maintaining system robust performance. A numerically calculated tight bound is derived on controller uncertainty which, when combined with the error bound provided by balanced realization, provides a lower limit on the order of the reduced controller to maintain robust performance.

The paper is organized as follows. Section II contains the general problem formulation, explaining uncertainty modeling, $M-\Delta$ interconnection, structured singular value $\mu_\Delta(\bullet)$ and main loop theorem through Subsections A-D. The main results are presented in Section III where the numerical bound providing necessary and sufficient robust performance condition is derived in Subsection III-A. The controller order reduction error is then modeled as an additive uncertainty, leading to an application of the balanced truncation procedure to the stable part of the robust controller in Subsection III-B. Simulation results are given in Section IV which demonstrate the effectiveness of the proposed scheme on a modified HIMAT control problem [7]. Finally, concluding remarks are drawn in Section V.

II. Problem Formulation

A. Uncertainty Modeling

Consider the closed-loop control system in Fig. 1.

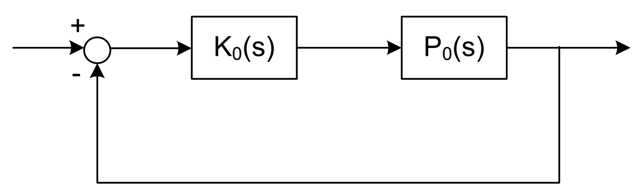

Fig. 1. Typical feedback control system with nominal models for the controller and plant.

The transfer functions $K_0(s)$ and $P_0(s)$ represent the nominal controller and plant models, respectively. Assume now that the plant is subject to a structured linear fractional perturbation $\Delta_P(s)$ with [7]:

$$\Delta_P(s) = W_P(s)\tilde{\Delta}_P(s), \quad \tilde{\Delta}_P(s) \in \mathcal{H}_\infty, \quad \left\|\tilde{\Delta}_P(s)\right\|_\infty < 1, \tag{1}$$

where $W_P(s)$ is the weighting function bounding the plant uncertainty. The control loop of Fig. 1 can now be modified

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as shown in Fig. 2 to account for uncertainties.

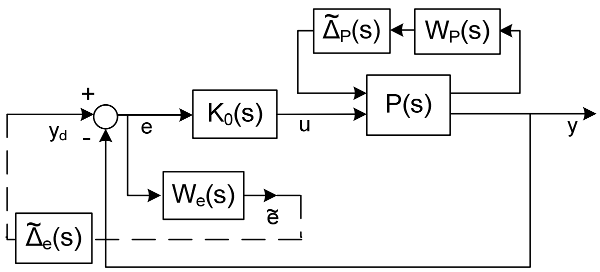

Fig. 2. Expanded control loop with linear fractional uncertainty and fictitious performance block.

Where $P(s)$ represents the generalized nominal plant in order to account for the perturbation. The block $\tilde{\Delta}_e(s)$ is a fictitious uncertainty which appears only in the control design stage to obtain robust performance using the main loop theorem, and is characterized as:

$$\begin{aligned} &\Delta_e(s) = W_e(s)\tilde{\Delta}_e(s), \quad \tilde{\Delta}_e(s) \in \mathcal{H}_\infty, \\ &\left\|\tilde{\Delta}_e(s)\right\|_\infty < 1. \end{aligned} \tag{2}$$

The uncertainty block $\tilde{\Delta}_e(s)$ and the corresponding weighting function $W_e(s)$ are considered in control design in order to ensure that the error e is maintained within acceptable limits in closed-loop operation.

Throughout the remainder of this paper, omitting the complex variable s means that unless specified, the system is sampled at frequency ω, e.g. $P = P(s)\big|_{s=j\omega}$.

B. M-Δ Interconnection

By partitioning the structure shown in Fig. 2, the perturbed closed-loop system at frequency ω can be represented as an $M-\Delta$ interconnection. Fig. 3 shows the LFT interconnection of the control loop with the controller and uncertainties separated from other system components [8].

The matrix Π contains the nominal plant as well as all weighting functions introduced earlier. The LFT interconnection leads further to the $M-\Delta$ interconnection as in Fig. 4, where M is equal to lower linear fractional transformation (LFT) of Π by K_0 [8]:

$$\begin{aligned} M &= F_l(\Pi, K_0) \\ &= \Pi_{11} + \Pi_{12}K_0\left(I - \Pi_{22}K_0\right)^{-1}\Pi_{21} \end{aligned} \tag{3}$$

and Δ is a structured block diagonal matrix composed of $\tilde{\Delta}_e$ and $\tilde{\Delta}_P$, the fictitious robust performance and the open loop system uncertainties:

$$\Delta = \begin{bmatrix} \tilde{\Delta}_e & 0 \\ 0 & \tilde{\Delta}_P \end{bmatrix}. \tag{4}$$

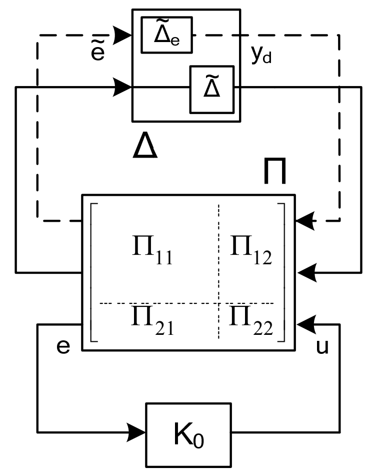

Fig. 3. The LFT interconnection of the expanded control loop.

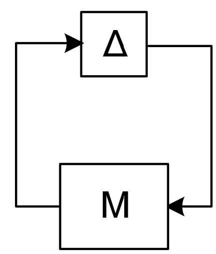

Fig. 4. The $M-\Delta$ interconnection isolating uncertainty blocks.

C. Structured Singular Value and Robust Stability

The structured singular value of $M \in \mathbb{C}^{n\times n}$ denoted by $\mu_\Delta(M)$, is defined as:

$$\mu_\Delta(M) := \frac{1}{\min\left\{\overline{\sigma}(\Delta_s) : \Delta_s \in \boldsymbol{\Delta}, \det\left(I - M\Delta_s\right) = 0\right\}} \tag{5}$$

unless no $\Delta_s \in \boldsymbol{\Delta}$ makes $I - M\Delta_s$ singular, in which case $\mu_\Delta(M) = 0$. The set $\boldsymbol{\Delta}$ is composed of all complex uncertainty matrices with the same structure [7], [8], [9]. The robust stability theorem states that the $M-\Delta$ interconnection given in Fig. 4 is stable for all stable structured perturbations $\Delta(j\omega) \in \boldsymbol{\Delta}$, $\|\Delta\|_\infty < 1$, if and only if

$$\sup_\omega \mu_\Delta\left(M(j\omega)\right) \le 1. \tag{6}$$

Having $\tilde{\Delta}_e$ and W_e included in Δ and M, and assuming that (6) holds, implies that the gain from the desired output to the error is bounded, leading to robust performance of the perturbed system.

D. Main Loop Theorem

The main loop theorem states that for a general $M-\Delta$ interconnection of complex matrices:

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},\quad \Delta=\begin{bmatrix} \tilde{\Delta}_1 & 0 \\ 0 & \tilde{\Delta}_2 \end{bmatrix}, \tag{7}$$

the following statement holds [4]:

$$\mu_\Delta(M)\le 1 \Leftrightarrow \begin{cases} \mu_2(M_{22})\le 1, \text{ and} \\ \sup\limits_{\|\tilde{\Delta}_2\|<1} \mu_1\left(F_l\left(M,\tilde{\Delta}_2\right)\right)\le 1. \end{cases} \tag{8}$$

where the $\mu_1(\bullet)$ and $\mu_2(\bullet)$ operators are the structured singular values with respect to $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$.

III. Main Result

A. Condition for Robust Performance

So far, it is assumed that the robust controller K_0 maintains the robust performance of the control loop in Fig. 2. Assuming that K represents the perturbed controller with its uncertainty modeled as a stable multiplicative perturbation, i.e.:

$$K(s)=K_0(s)\left(I+\Delta_K(s)\right) \tag{9}$$

where

$$\Delta_K(s)=W_K(s)\tilde{\Delta}_K(s),\quad \tilde{\Delta}_K(s)\in\mathcal{H}_\infty, \\ \left\|\tilde{\Delta}_K(s)\right\|_\infty<1, \tag{10}$$

the interconnections in Fig. 3 and 4 can now be augmented in order to include controller uncertainty, as in Fig. 5 and 6. The matrix $\hat{\Pi}$ is directly formed using the sub-blocks of Π and W_K. The matrix $\hat{M}$ is calculated as:

$$\hat{M}\triangleq F_l\left(\hat{\Pi},K_0\right) = \begin{bmatrix} W_K K_0 L\Pi_{22} & W_K K_0 L\Pi_{21} \\ \Pi_{12}+\Pi_{12}K_0 L\Pi_{22} & \Pi_{11}+\Pi_{12}K_0 L\Pi_{21} \end{bmatrix}, \tag{11}$$

where $L\triangleq\left(I-\Pi_{22}K_0\right)^{-1}$, and the controller uncertainty is included in $\hat{\Delta}$:

$$\hat{\Delta}=\begin{bmatrix} \tilde{\Delta}_K & 0 \\ 0 & \Delta \end{bmatrix}. \tag{12}$$

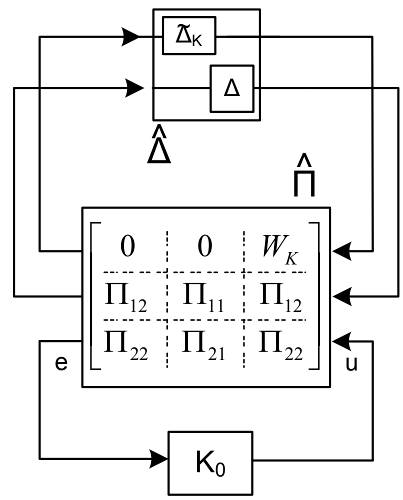

Fig. 5. The augmented LFT interconnection.

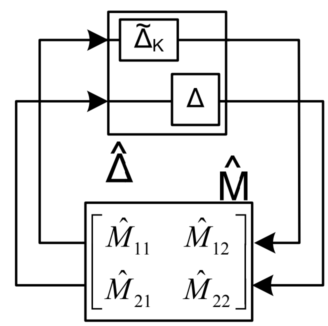

Fig. 6. The augmented $\hat{M}-\hat{\Delta}$ interconnection.

Note that the bottom right portion of $\hat{M}$ is equal to M as given in (3), which represents the system setup taking robust performance into account without considering controller uncertainty. It is now desired to find a condition on the maximum size of the controller uncertainty, $\bar{\sigma}(W_K)$, such that the augmented $\hat{M}(s)-\hat{\Delta}(s)$ system interconnection is robustly stable (i.e., the system with controller uncertainty has robust performance). It is assumed that $\tilde{\Delta}_K$ is a square full uncertainty matrix as well as W_K being square and acting like a single gain, i.e., $W_K\triangleq\alpha I$. As a result, the maximum size of the controller uncertainty is now α instead of $\bar{\sigma}(W_K)$.

Following the definition of $\hat{\Delta}$ and $\hat{M}$ in (11) and (12) with the representation given in (7) and (8), the main loop theorem for the system in Fig. 6 is written as:

$$\mu_{\hat{\Delta}}\left(\hat{M}\right)\le 1 \Leftrightarrow \begin{cases} \mu_{\Delta}\left(M\right)\le 1 \text{ and} \\ \sup_{\|\Delta\|<1} \mu_{\tilde{\Delta}_K}\left(F_l\left(\hat{M},\Delta\right)\right)\le 1. \end{cases} \quad (13)$$

The top inequality on the right-hand side of (13) neglects the effect of $\tilde{\Delta}_K$. In other words, it guarantees the robust performance of the perturbed system without controller uncertainty and as a result, does not include W_K. It is important to know that the mentioned inequality is already met as the robust controller K_0 already guarantees robust performance of the system. Therefore, it remains to deal with the second right-hand side inequality. The lower LFT $F_l\left(\hat{M},\Delta\right)$ is expanded as:

$$F_l\left(\hat{M},\Delta\right)=W_K K_0 L \times \left\{\Pi_{22}+\Pi_{21}\Delta\left(I-M\Delta\right)^{-1}\Pi_{12}\left(I+K_0 L\Pi_{22}\right)\right\}. \quad (14)$$

In other words, robust performance of the perturbed controller case is guaranteed if and only if α is bounded by a maximum value, which in turn satisfies the bottom right-hand side inequality of (13). However, having a tight inequality does not guarantee that $\mu_{\hat{\Delta}}\left(\hat{M}\right)$ is very close to 1. Also, searching for the aforementioned bound using (14) can impose a high computational cost as the size and complexity of Δ increases. Therefore, the search for the bounding value is performed using $\mu_{\hat{\Delta}}\left(\hat{M}\right)$ directly rather than trying to find the maximum gain of (14) as presented in the following proposition.

Proposition 1: The $\hat{M}\left(s\right)-\hat{\Delta}\left(s\right)$ system interconnection of Fig. 6 has robust performance if and only if the following inequality holds for every ω:

$$\alpha \le \alpha_{\max}, \text{ where}$$

$$\alpha_{\max} \triangleq \left\{\beta\in\mathbb{R}^+ : \mu_{\hat{\Delta}}\left(\begin{bmatrix}\beta\hat{M}^{\dagger}_{11} & \beta\hat{M}^{\dagger}_{12} \\ \hat{M}^{\dagger}_{21} & \hat{M}^{\dagger}_{22}\end{bmatrix}\right)=1\right\}, \quad (15)$$

$$\hat{M}^{\dagger}\triangleq\left\{\hat{M}:W_K=1\right\}.$$

Proof: According to (11) and (13) and the proposed definition of $\hat{M}^{\dagger}$, the following holds:

$$\sup_{\|\Delta\|<1}\mu_{\tilde{\Delta}_K}\left(F_l\left(\beta\hat{M}^{\dagger},\Delta\right)\right)=1$$
$$\Leftrightarrow \sup_{\|\Delta\|<1}\mu_{\tilde{\Delta}_K}\left(F_l\left(\hat{M}^{\dagger},\Delta\right)\right)=\beta^{-1} \quad (16)$$
$$\Leftrightarrow \mu_{\hat{\Delta}}\left(\begin{bmatrix}\beta\hat{M}^{\dagger}_{11} & \beta\hat{M}^{\dagger}_{12} \\ \hat{M}^{\dagger}_{21} & \hat{M}^{\dagger}_{22}\end{bmatrix}\right)=1.$$

In other words, a search to find a bound on the controller uncertainty based on the bottom inequality on the right-hand side of (13) is equivalent to searching for the maximum β ($\alpha_{\max}$) in the last statement of (16). Note that in case of having $\beta=0$, the robust performance is already met as the LFT in (14) becomes equal to zero meaning no feedback through $\tilde{\Delta}_K$ in Fig. 6. This completes the proof.

As a result, a numerical search is performed in order to find $\alpha_{\max}$ at all frequencies. Normally, the robust performance test is performed over a frequency grid which can also be used here.

B. *Controller Order Reduction*

So far, a tight bound for the controller uncertainty gain α as a necessary and sufficient condition for robust performance has been derived. The reduced-order controller can be modeled as:

$$K_r\left(s\right)=K_0\left(s\right)+\Delta_r\left(s\right) \quad (17)$$

where $\Delta_r\left(s\right)$ is the order reduction error associated with $K_0\left(s\right)$, and $K_r\left(s\right)$ is the reduced-order controller. Recalling the controller perturbation definition in (9), $\Delta_r\left(s\right)$ can be considered as $K_0\left(s\right)\Delta_K\left(s\right)$ with the exception of having a known phase as it can be derived as a rational transfer function rather than the product of the robust controller, a weighting function and an uncertain bounded element. This discussion leads to the following proposition.

Proposition 2: Suppose that $\mathrm{A}\left(\omega\right)$ is a real function of ω equal to the upper bound derived in (15). The reduced-order controller $K_r\left(s\right)$ maintains robust performance of the $\hat{M}\left(s\right)-\hat{\Delta}\left(s\right)$ system interconnection if for every frequency ω:

$$\overline{\sigma}\left(\Delta_r\left(j\omega\right)\right)\le\overline{\sigma}\left(K_0\left(j\omega\right)\right)\cdot\mathrm{A}\left(\omega\right). \quad (18)$$

Proof: According to (10), the largest singular value of

$\Delta_r(j\omega)$ is bounded by $\bar{\sigma}(K_0(j\omega)W_K(j\omega))$, which in turn is bounded by $\bar{\sigma}(K_0(j\omega))\cdot \mathrm{A}(\omega)$ as in (15), a necessary and sufficient condition for robust performance. This completes the proof.

The balanced realization technique [3] provides an upper bound on the infinity norm of $\Delta_r(s)$. Therefore, after balancing the controller $K_0(s)$ or order N_0 and computing the associated Hankel singular values (HSV) [3], the smallest order of $K_r(s)$ guaranteeing robust performance using sufficient condition of (18), r_0, is:

$$r_0 \triangleq \min\left\{ r \middle| 2\sum_{m=r+1}^{N_0} \sigma_m \le \min_{\omega}\left(\mathrm{A}\cdot\bar{\sigma}(K_0)\right)\right\}. \tag{19}$$

In other words, it is safe to remove $N_0 - r_0$ last states of the balanced $K_0(s)$ while maintaining robust performance. Note that in case of having a strictly proper nominal controller, $\bar{\sigma}(K_0)$ gets close to zero at very high frequencies, but the product $\mathrm{A}\cdot\bar{\sigma}(K_0)$ does not necessarily tend to zero which would result in a very conservative bound in (19). Indeed, as $\bar{\sigma}(K_0)$ gets close to zero, $\hat{M}_{11}^{\dagger}$ and $\hat{M}_{12}^{\dagger}$ approach zero in (15), resulting in a large $\alpha_{\max}$. Also in the case of a robust controller with unstable poles, the controller is decomposed into the summation of stable and unstable components, $K_{0_S}(s)$ and $K_{0_U}(s)$. This decomposition can be done by applying a partial fraction expansion to $K_0(s)$ and then collecting stable and unstable parts separately. The model order reduction can then be applied to $K_{0_S}(s)$ using the same upper bound in (15). Fig. 7 shows how this decomposition does not require a new upper bound calculation.

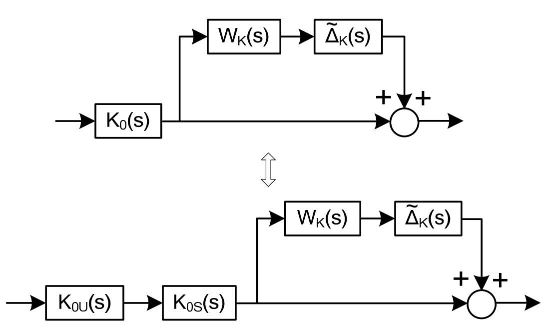

Fig. 7. Separation of stable and unstable controller components

The upper bound in (15) acts as a necessary and sufficient robust performance condition. At the same time, balanced realization technique provides an upper bound and not an exact measure of the order reduction error, e.g. its frequency domain behavior. Therefore, there might still be room for controller order reduction comparing to what (19) suggests. As a result, a future improvement would consist of a necessary and sufficient condition for robust performance which eventually will provide more accurate information about the maximum allowed difference between the original and reduced-order controller.

IV. Simulation Results

Consider a modified version of the HIMAT (highly maneuverable aircraft) control problem [7] where:

$$P_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right],$$

$$W_P(s) = \begin{bmatrix} \dfrac{50(s+100)}{s+10000} & 0 \\ 0 & \dfrac{50(s+100)}{s+10000} \end{bmatrix},$$

$$W_e(s) = \begin{bmatrix} \dfrac{0.1(s+0.3)}{s+0.03} & 0 \\ 0 & \dfrac{0.1(s+0.3)}{s+0.03} \end{bmatrix}.$$

The nominal plant $P_0(s)$ has additive uncertainty with corresponding weighting function $W_P(s)$. The robust controller $K_0(s)$ derived using DK iteration method is a 2-by-2 stable system with 24 states. Fig. 8 shows $\mu_\Delta(M)$ calculated across the frequency range of $[10^{-5}, 10^{3}]$ at 200 points. $\mu_\Delta(M)$ stays below 1 through the whole frequency range, therefore guaranteeing robust performance using the robust controller $K_0(s)$. As the next step, the maximum allowed gain associated with multiplicative controller uncertainty, $\alpha_{\max}$, is calculated at each frequency point using optimization techniques, forming the set $\mathrm{A}(\omega)$. The right-hand side of the inequality in (19) is then calculated using $\mathrm{A}(\omega)$ and $K_0(j\omega)$ which is compared against twice the cumulative sum of Hankel singular values associated with the balanced $K_0(s)$. This comparison suggests that the last 22 states of the balanced robust controller can be eliminated safely without making the system lose robust performance.

The reduced-order robust controller, $K_{red}(s)$, is:

$$K_{red} = \left[\begin{array}{cc|cc} -0.0059 & 0.0128 & -9.123\times10^{-6} & 0.0025 \\ -0.0128 & -0.0365 & -5.11\times10^{-6} & 0.0023 \\ \hline -4.082\times10^{-4} & 2.18\times10^{-4} & -8.532\times10^{-6} & 3.383\times10^{-6} \\ 0.0024 & -0.0023 & -0.905\times10^{-6} & 1.088\times10^{-6} \end{array}\right]$$

which has only 2 states. Fig. 9 shows $\mu_\Delta(M)$ calculated using the reduced-order robust controller and over the same frequency grid. Clearly, robust performance of the system is still maintained using the reduced-order robust controller. For comparison, the derived bound versus $\bar{\sigma}(K_0 - K_{red})$ is depicted in Fig. 10.

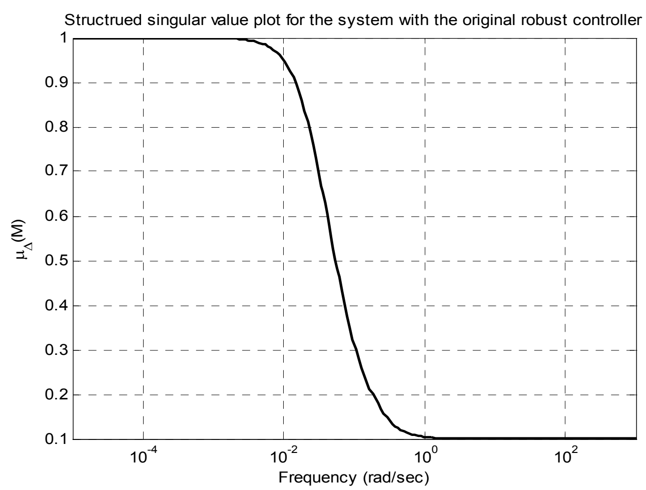

Fig. 8. $\mu_\Delta(M)$ plot using the controller $K_0(s)$.

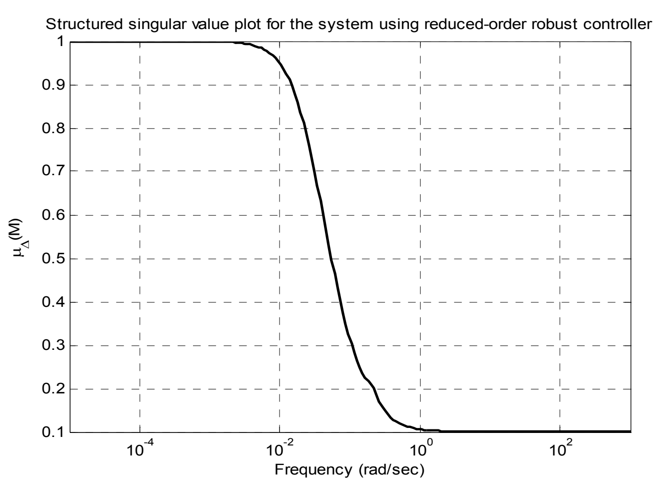

Fig. 9. $\mu_\Delta(M)$ plot using the reduced-order controller $K_{red}(s)$.

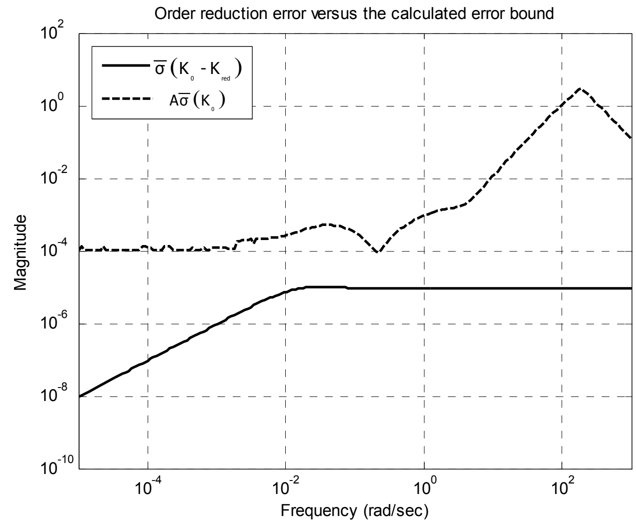

Fig 10. Order reduction error compared against the derived tight bound.

V. Conclusions

A method is proposed to reduce the order of a multi-input multi-output robust controller. A tight bound on the magnitude of the controller uncertainty is derived based on the structured singular value of a system matrix composed of original system components which provides a necessary and sufficient condition for maintaining system robust performance. The bound is then used to provide a lower bound on the controller order which guarantees the robust performance of the closed-loop system. The balanced truncation technique is used for controller order reduction as a powerful and well-developed order reduction method. Simulation results show the efficiency of the proposed method.

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