2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Boundary Control of an Anti-Stable Wave Equation With Anti-Damping on the Uncontrolled Boundary

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***Abstract*— Much of the boundary control of wave equations in 1D is based on a single principle—passivity—under the assumption that control is applied through Neumann actuation on one boundary and the other boundary satisfies a homogeneous Dirichlet boundary condition. We have recently expanded the scope of tractable problems by allowing destabilizing anti-stiffness (a Robin type condition) on the uncontrolled boundary, where the uncontrolled system has a finite number of positive real eigenvalues. In this paper we go much further and develop a methodology for the case where the uncontrolled boundary condition has anti-damping, which makes the real parts of all the eigenvalues of the uncontrolled system positive and arbitrarily high, i.e., the plant is "anti-stable" (exponentially stable in negative time). Using a conceptually novel integral transformation, we obtain extremely simple, explicit formulae for the gain functions. For the case with only boundary sensing available (at the same end with actuation), we design backstepping observers which are dual to the backstepping controllers and have explicit output injection gains. We then combine the control and observer designs into an output-feedback compensator and prove exponential stability of the closed-loop system.**

I. INTRODUCTION

We consider the problem of stabilization of a one-dimensional wave equation which is controlled from one end and contains instability at the other (free) end (see [1], [2], [4], [7] for classical passivity-based designs for wave equations). The nature of instability (negative damping) is such that all of the open-loop eigenvalues are located on the right hand side of the complex plane, thus the plant is not just unstable, it is anti-stable.

Our control design is based on the method of "backstepping" [6], [5], [3], which results in explicit formulae for the gain functions. In a recent paper [3], backstepping method was used to design controllers and observers for an unstable wave equation with destabilizing boundary condition at the free end. However, in that paper the destabilizing term was proportional to displacement, while in this paper it is proportional to velocity, in the form of "anti-damper", which results in all eigenvalues being unstable instead of just a few. The concept of a boundary anti-damper is not of huge physical relevance, however, the design that we develop for this anti-stable system is a methodological breakthrough in boundary control of wave equations.

For the case when only boundary sensing is available (at the same end with actuation), we design backstepping observers which are dual to the backstepping controllers and have explicit output injection gains. Both setups are considered: Neumann actuation/Dirichlet sensing and Dirichlet actuation/Neumann sensing. We then combine the control and observer designs into dynamic compensators and prove exponential stability of the closed-loop system.

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II. PROBLEM FORMULATION

We consider the plant

$$u_{tt}(x,t) = u_{xx}(x,t) \tag{1}$$

$$u_x(0,t) = -q u_t(0,t) \tag{2}$$

$$u_x(1,t) = U(t)\,, \tag{3}$$

where $U(t)$ is the control input and q is a constant parameter. For $q = 0$, equations (1)–(3) model a string which is free at the end $x = 0$ and is actuated on the opposite end. For $q < 0$ the system (1)–(3) is stabilized trivially with the feedback law $U = -au(1,t)$, $a > 0$. In this paper we assume $q > 0$, so that the free end of the string is negatively damped, with all eigenvalues located on the right hand side of the complex plane (hence the open-loop plant is "anti-stable"). We also assume $q \neq 1$, since for $q = 1$ the plant is uncontrollable.

The objective is to exponentially stabilize the system (1)–(3) around the zero equilibrium. The case of Dirichlet actuation is considered in Section VI.

III. CONTROL DESIGN

Consider the transformation

$$\begin{aligned} w(x,t) = u(x,t) &- \int_0^x k(x,y) u(y,t)\, dy \\ &- \int_0^x s(x,y) u_t(y,t)\, dy \\ &- \int_0^x m(x,y) u_x(y,t)\, dy\,, \end{aligned} \tag{4}$$

where the gains $k(x,y)$, $s(x,y)$, and $m(x,y)$ are to be determined. We want to map the plant (1)–(3) into the following target system

$$w_{tt}(x,t) = w_{xx}(x,t) \tag{5}$$

$$w_x(0,t) = c w_t(0,t) \tag{6}$$

$$w_x(1,t) = -c_0 w(1,t)\,, \tag{7}$$

which is exponentially stable for $c > 0$ and $c_0 > 0$ (see, e.g., [3]). We also assume $c \neq 1$. As will be shown later, the transformation (4) is invertible in a certain norm, so that stability of the target system ensures stability of the closed-loop system.

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Compared with the backstepping transformations for parabolic PDEs, there are two additional terms in (4)—the second and the third integrals in (4). The term with u_t is natural because hyperbolic systems are second order in time and therefore the state variable is (u, u_t) instead of just u. The need for the term with u_x is less obvious and it is in fact the main conceptual novelty of the paper.

Substituting (4) into (5)–(7) we obtain

$$\begin{aligned} w_{tt}(x,t) = w_{xx}(x,t) &+ 2u(x,t)\frac{d}{dx}k(x,x) \\ &+ 2u_t(x,t)\frac{d}{dx}s(x,x) + 2u_x(x,t)\frac{d}{dx}m(x,x) \\ &+ \int_0^x (k_{xx}(x,y) - k_{yy}(x,y))u(y,t)\,dy \\ &+ \int_0^x (s_{xx}(x,y) - s_{yy}(x,y))u_t(y,t)\,dy \\ &+ \int_0^x (m_{xx}(x,y) - m_{yy}(x,y))u_x(y,t)\,dy \\ &- k_y(x,0)u(0,t) \\ &+ [qm_y(x,0) - s_y(x,0) - qk(x,0)]u_t(0,t) \\ &+ [m(x,0) - qs(x,0)]u_{xx}(0,t)\,. \end{aligned} \tag{8}$$

Matching all the terms we get the following PDE for $k(x, y)$:

$$k_{xx}(x,y) = k_{yy}(x,y), \tag{9}$$
$$k_y(x,0) = 0, \tag{10}$$
$$\frac{d}{dx}k(x,x) = 0, \tag{11}$$

and two coupled PDEs for $s(x, y)$ and $m(x, y)$:

$$s_{xx}(x,y) = s_{yy}(x,y), \tag{12}$$
$$s_y(x,0) = qm_y(x,0) - qk(x,0), \tag{13}$$
$$\frac{d}{dx}s(x,x) = 0, \tag{14}$$

and

$$m_{xx}(x,y) = m_{yy}(x,y), \tag{15}$$
$$m(x,0) = qs(x,0), \tag{16}$$
$$\frac{d}{dx}m(x,x) = 0. \tag{17}$$

Substituting (4) into the boundary condition (6) we get

$$\begin{aligned} 0 &= w_x(0,t) - cw_t(0,t) \\ &= [qm(0,0) - s(0,0) - q - c]u_t(0,t) \\ &\quad - k(0,0)u(0,t)\,, \end{aligned} \tag{18}$$

which gives two more conditions:

$$k(0,0) = 0, \tag{19}$$
$$qm(0,0) = s(0,0) + q + c\,. \tag{20}$$

The solution to (9)–(11), (19) is simply $k(x, y) \equiv 0$. This is a consequence of not changing stiffness of the system (i.e., if we were to add the term proportional to $w(0, t)$ to the boundary condition at $x = 0$, the gain $k(x, y)$ would not be zero). To solve the PDEs for s and m, we note that a general solution to (12) and (14) is $s(x, y) = \phi(x - y)$ and similarly for (15), (17) we have $m(x, y) = \psi(x - y)$ for arbitrary functions ϕ and ψ. Using (13) and (16) we obtain

$$\phi'(x) = q\psi'(x), \tag{21}$$
$$\psi(x) = q\phi(x)\,. \tag{22}$$

Integrating (21) from 0 to x and using (22), we obtain

$$(q^2 - 1)(\phi(x) - \phi(0)) = 0\,. \tag{23}$$

Since $q \neq 1$, we get that both $\phi(x)$ and $\psi(x)$ are constant functions of x. Finally, using the relationship (20) together with (22), we get

$$s(x,y) \equiv \frac{q+c}{q^2-1}, \tag{24}$$
$$m(x,y) \equiv \frac{q(q+c)}{q^2-1}\,. \tag{25}$$

The transformation (4) can therefore be written in one of the two forms, either as

$$\begin{aligned} w(x,t) = u(x,t) &- \frac{q(q+c)}{q^2-1}\int_0^x u_x(y,t)\,dy \\ &- \frac{q+c}{q^2-1}\int_0^x u_t(y,t)\,dy\,, \end{aligned} \tag{26}$$

or as

$$\begin{aligned} w(x,t) = -\frac{1+qc}{q^2-1}u(x,t) &+ \frac{q(q+c)}{q^2-1}u(0,t) \\ &- \frac{q+c}{q^2-1}\int_0^x u_t(y,t)\,dy\,. \end{aligned} \tag{27}$$

Differentiating (27) with respect to x, setting $x = 1$, and using the boundary condition (7), we obtain the following controller

$$\begin{aligned} U(t) = \frac{c_0 q(q+c)}{1+qc}u(0,t) &- c_0 u(1,t) \\ - \frac{q+c}{1+qc}u_t(1,t) &- \frac{c_0(q+c)}{1+qc}\int_0^1 u_t(y,t)\,dy\,. \end{aligned} \tag{28}$$

Our main result on stabilization is given by the following theorem.

Theorem 1. *For any initial data* $(u(\cdot,0), u_t(\cdot,0)) \in H = H^2(0,1) \times H^1(0,1)$*, compatible with the boundary conditions, closed-loop system* (1)–(3), (28) *has a unique classical solution* $(u, u_t) \in C^1([0,\infty), H)$*, which is exponentially stable in the sense of the norm*

$$\left(\int_0^1 u_x(x,t)^2 dx + \int_0^1 u_t(x,t)^2 dx + u(1,t)^2\right)^{1/2}. \tag{29}$$

Proof. First, let us establish stability of the target system. With the Lyapunov function

$$\begin{aligned} V_1(t) = \frac{1}{2}\int_0^1 w_x(x,t)^2 dx &+ \frac{1}{2}\int_0^1 w_t(x,t)^2 dx + \frac{c_0}{2}w(1,t)^2 \\ &+ \delta\int_0^1 (x-2)w_x(x,t)w_t(x,t)\,dx\,, \end{aligned} \tag{30}$$

where δ is sufficiently small (so that V_1 is positive definite), we obtain

$$\begin{aligned}\dot{V}_1 = &-\frac{\delta}{2}\int_0^1 (w_x(x,t)^2 + w_t(x,t)^2)\,dx \\ &- \left[c - \delta(1+c^2)\right] w_t(0,t)^2 \\ &- \frac{\delta}{2}\left[w_t(1,t)^2 + c_0^2 w(1,t)^2\right] \\ \leq &-\omega V_1, \end{aligned} \tag{31}$$

where $\omega > 0$. Since there exist $a_1 > 0$, $a_2 > 0$ such that

$$a_1 V_2 \leq V_1 \leq a_2 V_2, \tag{32}$$

where

$$V_2(t) = \int_0^1 w_x(x,t)^2 dx + \int_0^1 w_t(x,t)^2 dx + w(1,t)^2\,, \tag{33}$$

we obtain

$$V_2(t) \leq \frac{a_2}{a_1} e^{-\omega t} V_2(0)\,. \tag{34}$$

Let us differentiate the transformation (27) with respect to t and x. We get

$$w_t(x,t) = -\frac{1+qc}{q^2-1} u_t(x,t) - \frac{q+c}{q^2-1} u_x(x,t)\,, \tag{35}$$

and

$$w_x(x,t) = -\frac{1+qc}{q^2-1} u_x(x,t) - \frac{q+c}{q^2-1} u_t(x,t)\,. \tag{36}$$

From (35) and (36) it easily follows that

$$\begin{aligned}&\int_0^1 (w_x(x,t)^2 + w_t(x,t)^2)\,dx \\ &\qquad \leq \frac{(c+1)^2}{(q-1)^2}\int_0^1 (u_x(x,t)^2 + u_t(x,t)^2)\,dx\,. \end{aligned} \tag{37}$$

Using the transformation (26) with $x = 1$, we get

$$\begin{aligned}w(1,t)^2 \leq\, & 2u(1,t)^2 \\ &+ \frac{4(q+c)^2}{(q-1)^2}\int_0^1 (u_x(x,t)^2 + u_t(x,t)^2)\,dx\,. \end{aligned} \tag{38}$$

From (37) and (38) we obtain

$$V_2 \leq 4\frac{(q+c)^2 + (c+1)^2}{(q-1)^2} V_3, \tag{39}$$

where

$$V_3(t) = \int_0^1 u_x(x,t)^2 dx + \int_0^1 u_t(x,t)^2 dx + u(1,t)^2\,. \tag{40}$$

One can easily show that the inverse transformation to (35)–(36) is

$$u_t(x,t) = -\frac{1+qc}{c^2-1} w_t(x,t) + \frac{q+c}{c^2-1} w_x(x,t)\,, \tag{41}$$

and

$$u_x(x,t) = -\frac{1+qc}{c^2-1} w_x(x,t) + \frac{q+c}{c^2-1} w_t(x,t)\,. \tag{42}$$

Setting $x = 1$ in (26) and using (41), (42) to express $u(1,t)$ in terms of $w(1,t)$, w_t, and w_x, we obtain

$$V_3 \leq 4\frac{(q+c)^2 + (q+1)^2}{(c-1)^2} V_2. \tag{43}$$

From (34), (39), and (43) one gets

$$\begin{aligned}V_3(t) \leq\, & \frac{16 a_2}{a_1}\frac{(q+c)^2 + (c+1)^2}{(c-1)^2(q-1)^2} \\ &\times \frac{(q+c)^2 + (q+1)^2}{(c-1)^2(q-1)^2} e^{-\omega t} V_3(0)\,. \end{aligned} \tag{44}$$

The existence and uniqueness of the solution follow by standard arguments as in [3]. First, the abstract operator describing the system is introduced, it is dissipative due to estimates above; then it is shown that it has a bounded inverse; the result follows from Lumer-Phillips theorem. □

IV. Observer Design

In this section, we design an observer for the plant (1)–(3) when only boundary measurements are available. We assume that displacement and velocity at the end $x = 1$ are measured (i.e., $u(1,t)$ and $u_t(1,t)$).

Since we expect this observer to be dual to the controller designed in the previous section, it is natural to assume that the observer gains are also constant. We propose the following observer

$$\begin{aligned}\hat{u}_{tt}(x,t) &= \hat{u}_{xx}(x,t) + p_1[u(1,t) - \hat{u}(1,t)] \\ &\quad + p_2[u_t(1,t) - \hat{u}_t(1,t)] && (45)\\ \hat{u}_x(0,t) &= -q\hat{u}_t(0,t) + p_3[u(1,t) - \hat{u}(1,t)] \\ &\quad + p_4[u_t(1,t) - \hat{u}_t(1,t)] && (46)\\ \hat{u}_x(1,t) &= U(t) + p_5[u(1,t) - \hat{u}(1,t)] \\ &\quad + p_6[u_t(1,t) - \hat{u}_t(1,t)]\,. && (47)\end{aligned}$$

The observer error $\tilde{u} = u - \hat{u}$ satisfies

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - p_1\tilde{u}(1,t) - p_2\tilde{u}_t(1,t) \tag{48}$$

$$\tilde{u}_x(0,t) = -q\tilde{u}_t(0,t) - p_3\tilde{u}(1,t) - p_4\tilde{u}_t(1,t) \tag{49}$$

$$\tilde{u}_x(1,t) = -p_5\tilde{u}(1,t) - p_6\tilde{u}_t(1,t)\,. \tag{50}$$

Consider the transformation

$$\begin{aligned}\tilde{u}(x,t) = \tilde{w}(x,t) &+ \alpha\int_x^1 \tilde{w}(y,t)\,dy \\ &+ \beta\int_x^1 \tilde{w}_t(y,t)\,dy + \gamma\int_x^1 \tilde{w}_x(y,t)\,dy\,. \end{aligned} \tag{51}$$

Note that unlike in the control transformation (26), here the integrals run from x to 1. This is because the input and the output are collocated.

We map the observer error system into the system

$$\tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t) \tag{52}$$

$$\tilde{w}_x(0,t) = \tilde{c}\tilde{w}_t(0,t) \tag{53}$$

$$\tilde{w}_x(1,t) = -c_0\tilde{w}(1,t)\,, \tag{54}$$

which is exponentially stable for $\tilde{c} > 0$, $\tilde{c} \neq 1$ and $c_0 > 0$.

First, we differentiate the transformation (51) twice w.r.t. time (note that $\tilde{u}(1,t)=\tilde{w}(1,t)$),

$$\begin{aligned}\tilde{u}_{tt}(x,t) &= \tilde{u}_{xx}(x,t)+\alpha\tilde{w}_x(1,t)+\beta\tilde{w}_{xt}(1,t)+\gamma\tilde{w}_{tt}(1,t)\\ &= \tilde{u}_{xx}(x,t)-\alpha c_0\tilde{u}(1,t)\\ &\quad -\beta c_0\tilde{u}_t(1,t)+\gamma\tilde{u}_{tt}(1,t)\,. \end{aligned} \tag{55}$$

Comparing the above with (48) we get $\gamma=0$, $p_1=\alpha c_0$, $p_2=\beta c_0$. From (50) we get

$$\begin{aligned}\tilde{u}_x(1,t) &= \tilde{w}_x(1,t)-\alpha\tilde{w}(1,t)-\beta\tilde{w}_t(1,t)\\ &= -(c_0+\alpha)\tilde{u}(1,t)-\beta\tilde{u}_t(1,t)\,,\end{aligned} \tag{56}$$

which gives $p_5=c_0+\alpha$, $p_6=\beta$. Then, from (49)

$$\begin{aligned}\tilde{u}_x(0,t)+q\tilde{u}(0,t) &= -\alpha w(0,t)+q\alpha\int_0^1 w_t(y,t)\,dy\\ &\quad +\tilde{w}_t(0,t)[\tilde{c}-\beta+q-q\beta\tilde{c}]\\ &\quad -q\beta c_0\tilde{u}(1,t)\end{aligned} \tag{57}$$

and therefore $p_3=q\beta c_0$, $p_4=0$, $\alpha=0$, $\tilde{c}+q=\beta(1+q\tilde{c})$. Finally, we get the following observer

$$\hat{u}_{tt}(x,t)=\hat{u}_{xx}(x,t)+\frac{c_0(q+\tilde{c})}{1+q\tilde{c}}[u_t(1,t)-\hat{u}_t(1,t)] \tag{58}$$

$$\hat{u}_x(0,t)=-q\hat{u}_t(0,t)+\frac{c_0q(q+\tilde{c})}{1+q\tilde{c}}[u(1,t)-\hat{u}(1,t)] \tag{59}$$

$$\begin{aligned}\hat{u}_x(1,t) &= U(t)+c_0[u(1,t)-\hat{u}(1,t)]\\ &\quad +\frac{q+\tilde{c}}{1+q\tilde{c}}[u_t(1,t)-\hat{u}_t(1,t)]\,.\end{aligned} \tag{60}$$

and the transformation is

$$\tilde{u}(x,t)=\tilde{w}(x,t)+\frac{q+\tilde{c}}{1+q\tilde{c}}\int_x^1\tilde{w}_t(y,t)\,dy\,. \tag{61}$$

Note the duality of 4 observer gains in (58)–(60) to 4 control gains in (28) (for $\tilde{c}=c$), even though the control and observer transformations are different.

Theorem 2. *For any initial data* $(\tilde{u}(\cdot,0),\tilde{u}_t(\cdot,0))\in H=H^2(0,1)\times H^1(0,1)$ *compatible with the boundary conditions, the observer error system* (48)–(50) *has a unique classical solution* $(\tilde{u},\tilde{u}_t)\in C^1([0,\infty),H)$*, which is exponentially stable in the sense of the norm*

$$\left(\int_0^1\tilde{u}_x(x,t)^2dx+\int_0^1\tilde{u}_t(x,t)^2dx+\tilde{u}(1,t)^2\right)^{1/2}\,. \tag{62}$$

Proof. With the Lyapunov function

$$\begin{aligned}V_4 &= \frac{1}{2}\int_0^1\tilde{w}_x(x,t)^2dx+\frac{1}{2}\int_0^1\tilde{w}_t(x,t)^2dx+\frac{c_0}{2}\tilde{w}(1,t)^2\\ &\quad +\delta_2\int_0^1(x-2)\tilde{w}_x(x,t)\tilde{w}_t(x,t)\,dx\,,\end{aligned} \tag{63}$$

where δ_2 is sufficiently small, exactly the same calculation as in (31) shows that

$$\dot{V}_4\le-\tilde{\omega}V_4, \tag{64}$$

where $\tilde{\omega}>0$.

Differentiating the transformation (61) we get

$$\tilde{u}_x(x,t)=\tilde{w}_x(x,t)-\frac{q+\tilde{c}}{1+q\tilde{c}}\tilde{w}_t(x,t), \tag{65}$$

and

$$\tilde{u}_t(x,t)=\tilde{w}_t(x,t)-\frac{q+\tilde{c}}{1+q\tilde{c}}\tilde{w}_x(x,t)-\frac{c_0(q+\tilde{c})}{1+q\tilde{c}}\tilde{w}(1,t)\,. \tag{66}$$

Note also that $\tilde{u}(1,t)=\tilde{w}(1,t)$. Therefore,

$$\|\tilde{u}_x\|^2+\|\tilde{u}_t\|^2+\tilde{u}(1)^2\le M_6(\|\tilde{w}_x\|^2+\|\tilde{w}_t\|^2+\tilde{w}(1)^2), \tag{67}$$

where $M_6=3+3\max\{c_0^2,1\}(q+\tilde{c})^2/(1+q\tilde{c})^2$.

The inverse to (65), (66) is

$$\begin{aligned}\tilde{w}_x(x,t) &= \frac{(1+q\tilde{c})^2\tilde{u}_x(x,t)+(q+\tilde{c})(1+q\tilde{c})\tilde{u}_t(x,t)}{(q^2-1)(c^2-1)}\\ &\quad +\frac{c_0(q+\tilde{c})^2}{(q^2-1)(c^2-1)}\tilde{u}(1,t)\end{aligned} \tag{68}$$

$$\begin{aligned}\tilde{w}_t(x,t) &= \frac{(1+q\tilde{c})^2\tilde{u}_t(x,t)+(q+\tilde{c})(1+q\tilde{c})\tilde{u}_x(x,t)}{(q^2-1)(c^2-1)}\\ &\quad +\frac{c_0(q+\tilde{c})(1+q\tilde{c})}{(q^2-1)(c^2-1)}\tilde{u}(1,t)\,.\end{aligned} \tag{69}$$

Therefore,

$$\|\tilde{w}_x\|^2+\|\tilde{w}_t\|^2+\tilde{w}(1)^2)\le M_7(\|\tilde{u}_x\|^2+\|\tilde{u}_t\|^2+\tilde{u}(1)^2), \tag{70}$$

where $M_7=3\max\{c_0^2,1\}((q+\tilde{c})^2+(1+q\tilde{c})^2)^2(q^2-1)^{-2}(\tilde{c}^2-1)^{-2}$. From (64), (67), and (70) we get that the norm (62) decays exponentially. The existence and uniqueness of the solution of the observer error system are obtained as in the proof of Theorem 1. □

V. Output Feedback

In this section we combine the controller and the observer designed in previous two sections to solve the output-feedback problem.

Theorem 3. *Consider the plant* (1)–(3) *with the observer* (58)–(60) *and the controller*

$$\begin{aligned}U(t) &= \frac{c_0q(q+c)}{1+qc}\hat{u}(0,t)-c_0u(1,t)\\ &\quad -\frac{q+c}{1+qc}u_t(1,t)-\frac{c_0(q+c)}{1+qc}\int_0^1\hat{u}_t(y,t)\,dy\,.\end{aligned} \tag{71}$$

For any initial data $(u(\cdot,0),u_t(\cdot,0),\hat{u}(\cdot,0),\hat{u}_t(\cdot,0))\in H=H^2(0,1)\times H^1(0,1)\times H^2(0,1)\times H^1(0,1)$ *compatible with the boundary conditions, the closed-loop system has a unique classical solution* $(u,u_t,\hat{u},\hat{u}_t)\in C^1([0,\infty);H)$*, which is exponentially stable in the sense of the norm*

$$\begin{aligned}&\big(\|u_x(t)\|^2+\|u_t(t)\|^2+u(1,t)^2\\ &\quad +\|\hat{u}_x(t)\|^2+\|\hat{u}_t(t)\|^2+\|\hat{u}(1,t)\|^2\big)^{1/2}\,.\end{aligned} \tag{72}$$

Proof. Consider two transformations: (61) and

$$\begin{aligned}\hat{w}(x,t) &= -\frac{1+qc}{q^2-1}\hat{u}(x,t)+\frac{q(q+c)}{q^2-1}\hat{u}(0,t)\\ &\quad -\frac{q+c}{q^2-1}\int_0^x\hat{u}_t(y,t)\,dy.\end{aligned} \tag{73}$$

It is straightforward to show that these transformations along with the control law (71) map the system $(\hat{u}, \tilde{u})$ into $(\hat{w}, \tilde{w})$-system, where $\tilde{w}$ part is given by (52)–(54) and $\hat{w}$ satisfies the following PDE:

$$\hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t) + A(q\tilde{w}_t(1,t) - x\tilde{w}_{tt}(1,t)) \quad (74)$$

$$\hat{w}_x(0,t) = c\hat{w}_t(0,t) - A\frac{q(1+qc)}{q+c}\tilde{w}(1,t) \quad (75)$$

$$\hat{w}_x(1,t) = -c_0\hat{w}(1,t) + \frac{\tilde{c}-c}{1+q\tilde{c}}\tilde{w}_t(1,t)\,, \quad (76)$$

where

$$A = \frac{c_0(q+\tilde{c})(q+c)}{(q^2-1)(1+q\tilde{c})}\,. \quad (77)$$

Note that the PDE (74)–(76) contains the terms proportional to $\tilde{w}_t(1,t)$ and $\tilde{w}_{tt}(1,t)$, which are in H^2 and H^3 respectively, while our Lyapunov functions used in control and observer designs are H^1 norms. To overcome this difficulty, we introduce a new variable $\breve{w}(x,t) = \hat{w}(x,t) + Ax\tilde{w}(1,t)$, which eliminates the term $\tilde{w}_{tt}(1,t)$. The remaining $\tilde{w}_t(1,t)$-terms are handled using the term $-\tilde{w}_t(1,t)^2$ in $\dot{V}_4$, which was simply discarded in the estimate (64) (see (31)).

The variable $\breve{w}(x,t)$ satisfies the following PDE:

$$\breve{w}_{tt}(x,t) = \breve{w}_{xx}(x,t) + Aq\tilde{w}_t(1,t) \quad (78)$$

$$\breve{w}_x(0,t) = c\breve{w}_t(0,t) - \frac{c_0c(q+\tilde{c})}{(1+q\tilde{c})}\tilde{w}(1,t) \quad (79)$$

$$\breve{w}_x(1,t) = -c_0\breve{w}(1,t) + \frac{\tilde{c}-c}{1+q\tilde{c}}\tilde{w}_t(1,t) + (c_0+1)A\tilde{w}(1,t)\,. \quad (80)$$

Using the Lyapunov function

$$V_5 = \frac{1}{2}\int_0^1 \breve{w}_x(x,t)^2 dx + \frac{1}{2}\int_0^1 \breve{w}_t(x,t)^2 dx + \frac{c_0}{2}\breve{w}(1,t)^2 + \delta_1 \int_0^1 (x-2)\breve{w}_x(x,t)\breve{w}_t(x,t)\,dx + KV_4 \quad (81)$$

one can show that for large enough K and small enough δ_1 and δ_2

$$\dot{V}_5 \le -\omega_1 V_5 - \alpha\tilde{w}_t(1,t)^2, \quad \omega_1 > 0,\ \alpha > 0\,. \quad (82)$$

Going back to the old variable $\hat{w}$, and using (82) we obtain exponential stability in $(\tilde{w}, \hat{w})$ variables. From the transformations (61) and (73) and their inverses we obtain exp. stability in $(\tilde{u}, \hat{u})$ variables, and therefore in $(u, \hat{u})$ variables. The existence and uniqueness of the solutions is proved as in Theorem 1. □

VI. Dirichlet Actuation and Neumann Sensing

In this section we use the control and observer transformations derived in Sections III and IV to design the controller and the observer for the case of Dirichlet actuation and Neumann sensing.

Consider the plant

$$u_{tt}(x,t) = u_{xx}(x,t) \quad (83)$$

$$u_x(0,t) = -qu_t(0,t) \quad (84)$$

$$u(1,t) = U(t)\,. \quad (85)$$

Using the transformation (27), we map this plant into the target system

$$w_{tt}(x,t) = w_{xx}(x,t) \quad (86)$$

$$w_x(0,t) = cw_t(0,t) \quad (87)$$

$$w(1,t) = 0\,, \quad (88)$$

which is exponentially stable for $c > 0$, $c \neq 1$. The controller is obtained by setting $x = 1$ in (27):

$$U(t) = \frac{q(q+c)}{1+qc}u(0,t) - \frac{q+c}{1+qc}\int_0^1 u_t(y,t)\,dy\,. \quad (89)$$

Theorem 4. *For any initial data $(u(\cdot,0), u_t(\cdot,0)) \in H = H^2(0,1) \times H^1(0,1)$ compatible with the boundary conditions, the closed-loop system* (83)–(85), (89) *has a unique classical solution $(u, u_t) \in C^1([0,\infty), H)$, which is exponentially stable in the sense of the norm*

$$\left(\|u_x(t)\|^2 + \|u_t(t)\|\right)^{1/2}\,. \quad (90)$$

Proof. Starting with the Lyapunov function

$$V_6(t) = \frac{1}{2}\int_0^1 w_x(x,t)^2 dx + \frac{1}{2}\int_0^1 w_t(x,t)^2 dx + \delta\int_0^1 (x-2)w_x(x,t)w_t(x,t)\,dx\,, \quad (91)$$

where δ is sufficiently small, we obtain

$$\dot{V}_6 = -\frac{\delta}{2}\int_0^1 (w_x(x,t)^2 + w_t(x,t)^2)\,dx - \left[c - \delta(1+c^2)\right]w_t(0,t)^2 - \frac{\delta}{2}w_x(1,t)^2 \le -\omega V_6, \quad \omega > 0 \quad (92)$$

The rest of the proof is very similar to the proof of Theorem 1. □

When only measurements of $u_x(1,t)$ and $u_{xt}(1,t)$ are available, we design the observer

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + p_1[u_x(1,t) - \hat{u}_x(1,t)] + p_2[u_{xt}(1,t) - \hat{u}_{xt}(1,t)] \quad (93)$$

$$\hat{u}_x(0,t) = -q\hat{u}_t(0,t) + p_3[u_x(1,t) - \hat{u}_x(1,t)] + p_4[u_{xt}(1,t) - \hat{u}_{xt}(1,t)] \quad (94)$$

$$\hat{u}(1,t) = U(t)\,, \quad (95)$$

where p_1, p_2, p_3, p_4 are constants to be chosen.

The observer error system is

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - p_1\tilde{u}_x(1,t) - p_2\tilde{u}_{xt}(1,t) \quad (96)$$

$$\tilde{u}_x(0,t) = -q\tilde{u}_t(0,t) - p_3\tilde{u}_x(1,t) - p_4\tilde{u}_{xt}(1,t) \quad (97)$$

$$\tilde{u}(1,t) = 0\,. \quad (98)$$

We use the observer transformation (61), derived for the case of the Dirichlet sensing to map the observer error system into the following system:

$$\tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t) \quad (99)$$

$$\tilde{w}_x(0,t) = \tilde{c}\tilde{w}_t(0,t) \quad (100)$$

$$\tilde{w}(1,t) = 0\,. \quad (101)$$

Substituting (61) into (96)–(98), we get the following conditions: $p_1 = p_2 = 0$, $p_3(1+q\tilde{c}) = -q(q+\tilde{c})$, $p_4(1+q\tilde{c}) = -q-\tilde{c}$. Therefore, the observer is

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) - \frac{q+\tilde{c}}{1+q\tilde{c}}[u_{xt}(1,t) - \hat{u}_{xt}(1,t)] \quad (102)$$

$$\hat{u}_x(0,t) = -q\hat{u}_t(0,t) - \frac{q(q+\tilde{c})}{1+q\tilde{c}}[u_x(1,t) - \hat{u}_x(1,t)] \quad (103)$$

$$\hat{u}(1,t) = U(t)\,. \quad (104)$$

Note the duality of two observer gains in (102), (103) to the two control gains in (89) (for $\tilde{c} = c$).

Theorem 5. *Consider the plant* (83)–(85) *with the observer* (102)–(104) *and the controller*

$$U(t) = \frac{q(q+c)}{1+qc}\hat{u}(0,t) - \frac{q+c}{1+qc}\int_0^1 \hat{u}_t(y,t)\,dy\,. \quad (105)$$

For any initial data $(u(\cdot,0), u_t(\cdot,0), \hat{u}(\cdot,0), \hat{u}_t(\cdot,0)) \in H = H^2(0,1)\times H^1(0,1)\times H^2(0,1)\times H^1(0,1)$ *compatible with the boundary conditions, the closed-loop system has a unique classical solution* $(u, u_t, \hat{u}, \hat{u}_t) \in C^1([0,\infty); H)$*, which is exponentially stable in the sense of the norm*

$$\big(\|u_x(t)\|^2 + \|u_t(t)\|^2 + \|\hat{u}_x(t)\|^2 + \|\hat{u}_t(t)\|^2 + \|u_{xx}(t) - \hat{u}_{xx}(t)\|^2 + \|u_{xt}(t) - \hat{u}_{xt}(t)\|^2\big)^{1/2}. \quad (106)$$

Proof. The transformations (61) and

$$\hat{w}(x,t) = -\frac{1+qc}{q^2-1}\hat{u}(x,t) + \frac{q(q+c)}{q^2-1}\hat{u}(0,t) - \frac{q+c}{q^2-1}\int_0^x \hat{u}_t(y,t)\,dy \quad (107)$$

map (96)–(98), (102)–(104) into (99)–(101) and the following system

$$\hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t) - \frac{q+\tilde{c}}{1+q\tilde{c}}\tilde{w}_{xt}(1,t) + \frac{(q+c)(q+\tilde{c})}{(q^2-1)(1+q\tilde{c})}x\tilde{w}_{xtt}(1,t) \quad (108)$$

$$\hat{w}_x(0,t) = c\hat{w}_t(0,t) + \frac{q(1+qc)(q+\tilde{c})}{(q^2-1)(1+q\tilde{c})}\tilde{w}_x(1,t) \quad (109)$$

$$\hat{w}(1,t) = 0\,. \quad (110)$$

First, we establish exponential stability of the system (99)–(101) with the Lyapunov function

$$V_7(t) = \frac{1}{2}\int_0^1 \tilde{w}_x(x,t)^2\,dx + \frac{1}{2}\int_0^1 \tilde{w}_t(x,t)^2\,dx + \frac{1}{2}\int_0^1 \tilde{w}_{xx}(x,t)^2\,dx + \frac{1}{2}\int_0^1 \tilde{w}_{xt}(x,t)^2\,dx + \delta_1\int_0^1 (x-2)\tilde{w}_x(x,t)\tilde{w}_t(x,t)\,dx + \delta_2\int_0^1 (x-2)\tilde{w}_{xx}(x,t)\tilde{w}_{xt}(x,t)\,dx\,. \quad (111)$$

It is straightforward to show that

$$\dot{V}_7 \le -\omega V_7 - \alpha\tilde{w}_{xt}(1,t)^2, \quad \omega > 0, \alpha > 0\,. \quad (112)$$

Note that, unlike in the case of Dirichlet sensing, here the Lyapunov function has to contain H^2 norms for us to be able to show exponential stability of the observer error system. This is due to the H^2 nature of the terms $\tilde{u}_x(1,t)$ appearing in (96)–(98).

To eliminate the term proportional to $\tilde{w}_{xtt}(1,t)$ in (108) we introduce a new variable

$$\breve{w}(x,t) = \hat{w}(x,t) - \frac{(q+c)(q+\tilde{c})}{(q^2-1)(1+q\tilde{c})}x\tilde{w}_x(1,t)\,. \quad (113)$$

With the Lyapunov function

$$V_8(t) = \frac{1}{2}\int_0^1 \breve{w}_x(x,t)^2\,dx + \frac{1}{2}\int_0^1 \breve{w}_t(x,t)^2\,dx + \delta\int_0^1 (x-2)\breve{w}_x(x,t)\breve{w}_t(x,t)\,dx + KV_7(t),$$

we obtain

$$\dot{V}_8 \le -\omega_2 V_8, \quad \omega_2 > 0 \quad (114)$$

for sufficiently large K and sufficiently small δ, δ_1, and δ_2. The rest of the proof is similar to the proof of Theorem 3. □

VII. Conclusions

In this paper we introduced a new integral transformation for wave equations and used it to obtain explicit controllers and observers for a wave equation with negative damping at the boundary. The application of the presented approach to other hyperbolic systems is very promising and will be the subject of future work.

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